class # 20618

MATH 6702, SPRING 2024
M-W-F 12:40 p.m., MW 154
A DAY-BY-DAY LIST OF TOPICS

The following references are accessible through the course homepage at
`https://people.math.osu.edu/derdzinski.1/courses/6702/6702.html`

[**DG**]: *Differential Geometry*, [**DF**]: *Distributions and the Frobenius Theorem*
[**PS**]: *Projective Spaces and Grassmannians*, [**TC**]: *Tractor Connections*
[**AC**]: *Algebraic Curvature Tensors*, [**IP**]: *Inner Products up to a Factor*
[**MC**]: *Metrics of Constant Curvature*, [**CF**]: *Conformal Flatness*
[**MT**]: *Milnor's Tetrahedron*, [**FR**]: *Further References*

January 8: Definition of a (smooth, Hausdorff) manifold [**DG**, pp. 1–2] including the countability axiom [**DG**, p. 53]. The resulting topology [**DG**, pp. 1]. Continuous and differentiable mappings between manifolds, homeomorphisms, diffeomorphisms [**DG**, pp. 5–6]. Continuity in terms of convergent sequences on the one hand, and of pre-images of open sets on the other. Tangent vectors and spaces [**DG**, p. 18]. Differentials of mappings [**DG**, p. 20]. The rank theorem [**DG**, p. 33]. the inverse mapping theorem [**DG**, p. 198]. Submersions, immersions, submanifolds, with or without the subset topology [**DG**, p. 34], examples of the latter being provided by a “figure eight” in $\mathbb{R}^2$ [**DG**, the missing Fig. 5 on p. 35]. Geometric properties, including openness of sets [**DG**, p.4]. Vector bundles over a set M, sections. Local sections and local trivializations (when M is a manifold), transition functions, compatibility of local trivializations [**DG**, pp. 57–58].

January 10: Revisited topics: charts and compatibility, the definition of an n-dimensional C^∞ atlas, maximal atlases and the existence of a unique maximal atlas containing a given one [**DG**, p.1]. Definition of a C^∞ (Hausdorff) manifold [**DG**, pp.1–2] including countability axiom (that is, the existence of a countable subatlas of the maximal atlas [**DG**, p.53]: a manifold is thus a set with a fixed maximal atlas satisfying the Hausdorff and countability axioms). Examples of manifolds: zero-dimensional ones (that is, nonempty countable sets); vector spaces [**DG**, pp.2]; affine spaces [DG, pp. 3 and 191-192]; open submanifolds [**DG**, p.5]. Cartesian products of manifolds [**DG**, p.4]. Differentials of mappings, their coordinate description, and the chain rule [**DG**, p. 20]. Tangent spaces of open submanifolds [**DG**, p.4]. Index notation with the summing convention [**DG**, p. 17]. Partial derivatives and components of tangent vectors relative to a coordinate system [**DG**, p. 17–19]. The transformation rule [**DG**, p. 18]. Arbitrary C^∞ mappings $M \times N \to P$ written as multiplications. Multiplications of tangent vectors by points and the Leibniz rule [**DG**, pp.38–39]. Tangent spaces of Cartesian products [**DG**, p.38].

January 12: Tangent spaces in vector and affine spaces [**DG**, p. 19]. The fact that a subset of a manifold can carry at most one structure of a submanifold with the subset topology [**DG**, Corollary 9.5 on p. 35]. Submanifolds with the subset topology obtained as (nonempty) preimages, under smooth mappings, of their regular values, the dimensions and tangent spaces of such preimages [**DG**, Theorem 9.6 on p. 35]. Example: Euclidean spheres. Closed, compact, (pathwise) connected substes of a manifold, connected components of a set [**DG**, pp. 2, 4, 6]. Homeomorphicity of bijective continuous mappings from compact sets [**DG**, Problem 4 on p. 9]. Examples: tori, real/complex projective spaces, real/complex Grassmannians [**DG**, pp. 3–4], [**PS**], and their compactness [**DG**, Problem 15 on p. 10], [**PS**]. Whitney's embedding theorem for compact manifolds [**DG**, pp. 54–55]. Spherical embeddings of projective spaces.

January 17: Characterization of immersions [submersions] by the existence of smooth local left [right] inverses. The implicit mapping theorem [**DG**, p. 199]. Spherical embeddings of projective spaces (continued) and Grassmannians. Codimension-zero embeddings (that is, locally diffeomorphic injective mappings) from compact manifolds into connected manifolds. Diffeomorphic identfications $\mathrm{I\!RP}^1 \approx S^1$ and $\mathbb{C}\mathrm{P}^1 \approx S^2$. The submersion property of $\pi : V \smallsetminus \{0\} \to \mathrm{P}V$ and $\pi : \mathrm{St}_q V \to \mathrm{Gr}_q V$ verified using local right inverses. Continued from Jan. 8th: vector bundles over a set M, sections, sections and trivializations over a subset K of the base M. Transition functions. The total space η, the projections $\pi : \eta \to M$ and sections over K treated as mappings $K \to \eta$ [**DG**, pp. 66–67]. Local sections and local trivializations (when M is a manifold), compatibility, C^∞ atlases of local trivializations, maximal ones, C^∞ vector bundles over manifolds. Smooth local sections [**DG**, pp. 57–58]. The total space as a manifold and the submersion property of the projection $\pi : \eta \to M$, the fibres as submanifolds with the subset topology [**DG**, p. 67].

January 19: The general case of a (locally trivial) fibre bundle, involving three manifolds: the total space η, the base M, the model fibre $\mathrm{I\!F}$, and a C^∞ projection $\pi : \eta \to M$ such that M is covered by open subsets U with diffeomorphic identifications $\pi^{-1}(U) \approx U \times \mathrm{I\!F}$ (called local trivializations) which make π appear as the factor projection $U \times \mathrm{I\!F} \to U$. The observation that π is then a surjective submersion, and hence the fibres $\eta_x = \pi^{-1}(x)$, for all $y \in M$, are submanifolds of η with the subset topology, diffeomorphic to $\mathrm{I\!F}$. Vector bundles of fibre dimension q over the scalar field $\mathrm{I\!K} = \mathrm{I\!R}$ or $\mathrm{I\!K} = \mathbb{C}$ as a special case, having $\mathrm{I\!F} = \mathrm{I\!K}^q$. Examples of vector bundles: first, product bundles $M \times \mathrm{I\!F}$ [**DG**, p.58]. Then, tangent bundles TM, with the local trivializations consisting of coordinate vector fields p_j [**DG**, p.59]. Finally, the tautological line bundles over projective spaces [**DG**, p. 59] and tautological vector bundles over Grassmannians [**DG**, p.60]. Natural operations on real/complex vector bundles η, ζ over a manifold M, resulting in the direct sum $\eta \oplus \zeta$ [**DG**, p. 68], the bundle $\mathrm{Hom}(\eta, \zeta)$ [**DG**, p. 73], the sections of which may be identified with vector-bundle morphisms $\varPhi : \eta \to \zeta$ [**DG**, p. 69], its special case $\eta^* = \mathrm{Hom}(\eta, M \times \mathrm{I\!K})$, the dual of η, where $\mathrm{I\!K}$ is the scalar field [**DG**, p. 66]. Smooth subbundles of vector bundles [**DG**, p.71]. The image and kernel of a constant-rank morphism [**DG**, p.72].

January 22: The image and kernel of a constant-rank morphism, continued [**DG**, p. 72]. Spherical embeddings of projective spaces, revisited. Equivalence of open-covering compactness to sequential compactness [**DG**, p.4]: the Heine-Borel theorem [**DG**, p.54]. Compactness in Euclidean spaces, equivalent to being closed and bounded [**DG**, p.9]. Euclidean embeddings resulting from spherical ones via stereographic projections [**DG**, p. 3]. The conjugate of a complex vector space [**DG**, p. 68 and Problems 5–6 on p. 69]. Further natural operations on vector bundles: the conjugate $\bar{E}$ of a complex vector bundle E [**DG**, p. 68], with the convention that $\bar{E} = E$ for real vector bundles E, and the pullback [**DG**, p.68]. The restriction of a vector bundle to a submanifold [**DG**, p. 72]. Quotient bundles [**DG**, p. 72]. The tangent and normal bundles of an immersion [**DG**, p. 72]. Vector-bundle isomorphisms and trivial bundles [**DG**, p. 70]. The directional derivative $d_w f$ of a function f along a vector field w, and the fact that the operator d_w uniquely determines w [**DG**, p. 22].

January 24: Functional calculus of diagonalizable linear endomorphisms A. The case where A is self-adjoint: smoothness of the square root $A \mapsto \sqrt{A}$ when $A > 0$ (derived from the fact that $A \mapsto A^2$ is both bijective and – due to the inverse mapping theorem – locally diffeomorphic), and of the absolute value $|A| = \sqrt{A^2}$, as well as signum $\mathrm{sgn}\, A = |A|^{-1} A$ when $\det A \neq 0$. Smoothness of the mapping sending A with $A^* = A$ and

$\det A \neq 0$ to the direct sum of its eigenspaces corresponding to positive eigenvalues (since the latter space is the kernel of $B = \operatorname{sgn} A - 1$, while a basis of $\operatorname{Ker} B$ depending smoothly on B can be selected as we saw when we discussed the image and kernel of a constant-rank vector-bundle morphism (Jan. 19-22). Spherical embeddings of projective spaces and Grassmannians, revisited. Germs of C^∞ functions/sections at a point [**DG**, pp.19, 75]. Abundance of cut-off functions and global extensibility of germs [**DG**, p. 217]. Continued: the fact that the operator d_w uniquely determines w [**DG**, p. 22]. The Lie bracket of vector fields and the relation $d_{[v,w]} = d_v d_w - d_w d_v$ [**DG**, p. 23]. The Jacobi identity [**DG**, p. 24]. The Lie algebra $\mathcal{X}M$ of smooth vector fields on M [**DG**, p. 26]. Vector fields projectable under mappings [**DG**, p.23]. Push-forwards of vector fields under diffeomorphisms. Projectability of Lie brackets [**DG**, Theorem 6.1 on p. 24]. Distributions on a manifold M, defined to be (smooth) vector subbundles of the tangent bundle TM [**DF**, p. 1]. Integral manifolds of a distribution [**DF**, p. 1]. The normal bundle $D^{\mathrm{nrm}} = (TM)/D$ of a distribution D on M, with the quotient projection morphism $\pi : TM \to D^{\mathrm{nrm}}$ [**DF**, p. 1], and the curvature (tensor, form) of D, which assigns to each $x \in M$ the skew-symmetric bilinear mapping $\varOmega : D_x \times D_x \to D_x^{\mathrm{nrm}}$ such that $\varOmega(v,w) = \pi[v,w]$ for local sections v, w of D [**DF**, p. 1].

January 26: Well-definedness of $\varOmega$ [**DF**, Exercise 2 on p. 8]. Three operations on vector spaces: $L(V, V', W) = \operatorname{Hom}(V, \operatorname{Hom}(V', W))$, $S(V, V, W)$, $A(V, V, W)$, and their conterparts for smooth vector bundles. The curvature $\varOmega$ of D as a smooth section of $A(D, D, D^{\mathrm{nrm}})$. The observation that a smooth vector field on M, tangent to a submanifold P, is smooth when restricted to a vector field on P, which does not follow directly from [**DG**, Theorem 6.1 on p. 24], but requires a (simple) use of rank-theorem coordinates; thus, if two vector fields are tangent to a submanifold, so is their Lie bracket. The conclusion that the curvature $\varOmega$ of D must vanish along every integral manifold of D. Integrability of a distribution [**DF**, p. 1], meaning that every point lies in an integral manifold [**DF**, p. 1], which – according to the preceding sentence – implies vanishing of $\varOmega$. The Frobenius theorem: integrability of a distribution is equivalent both to vanishing of its curvature, and to its being locally diffeomorphically equivalent to a constant distribution on a vector space. The decomposition of M into the leaves (maximal connected integral manifolds) of an integrable distribution, also referred to as a foliation (which is more than M just being a disjoint union of submanifolds of a fixed dimension). The fact that a smooth mapping $N \to M$ taking values in a leaf of an integrable distribution is also smooth as a mapping into the leaf [**DF**, p. 6]. Lie groups. Examples: countable groups, vector spaces [**DG**, p. 12]; the set G of invertible elements in a finite-dimensional real/complex associative algebra $\mathcal{A}$ with unit. (A one-sided inverse is also necessarily two-sided: $[xy = 1$ for some $y] \Leftrightarrow [L_x$ surjective$] \Leftrightarrow [L_x$ injective$] \Rightarrow [yx = 1$ since $xyx = x]$, while the first two equivalences show that G is open in $\mathcal{A}$, being the preimage, under $x \mapsto L_x$, of the open set $\mathrm{GL}(\mathcal{A}) \subseteq \operatorname{End}(\mathcal{A})$ formed by all linear automorphisms of $\mathcal{A}$). The special case $G = \mathrm{GL}(V)$ when $\mathcal{A} = \operatorname{End}(V)$, including $\mathrm{GL}(n, \mathbb{K})$ [**DG**, p. 12]. A Lie subgroup of a Lie group G, defined to be a submanifold H of G which is also a subgroup, without requiring – unlike in [**DG**, p. 43] – that H, with these two structures, be a Lie group.

January 29: Uniqueness of the manifold structure of a leaf of an integrable distribution [**DF**, p. 7]. Left-invariant vector fields on a Lie group, their smoothness, and the Lie algebra $\mathfrak{g}$ of a Lie group G [**DG**, pp. 27-28]. Left-invariant distributions D on a Lie group G [**DF**, p. 7], their smoothness, and integrability of such D when $D_1 = T_1 H$ for a Lie subgroup H, with integral manifolds provided by left cosets of H. The conclusion that a Lie subgroup is a Lie group in its own right [**DF**, bottom of p. 7]. Smooth left actions of a Lie group G on a manifold M, with examples: $\mathrm{GL}(V)$ on V and $\mathrm{GL}(V)$ on $\mathrm{End}(V)$ (or, $L(V,V,\mathbb{K})$) via conjugation (or, inverse pullback), G on G via inner automorphisms, $G\times G$ on G [**DG**, p. 44-45]. The isotropy groups H_x of an action [**DG**, p. 46]. Submanifolds with the subset topology arising as (nonempty) preimages of points under constant-rank smooth mappings, their dimensions and tangent spaces [**DG**, p. 41]. The conclusion that isotropy groups of an action by G are Lie subgroups of G [**DG**, p. 46]. Examples: orthogonal and unitary groups $\mathrm{O}(V,\langle\,,\rangle)$, $\mathrm{U}(V,\langle\,,\rangle)$, including $\mathrm{O}(n)$ and $\mathrm{U}(n)$ [**DG**, p. 47]. Lie-group homomorphisms $F:G\to H$ [**DG**, p. 13], constancy of their rank [**DG**, p. 42]. , the resulting Lie-algebra homomorphism $F_*:\mathfrak{g}\to\mathfrak{h}$, and the fact that $\mathrm{Ker}\,F$ is a normal Lie subgroup of G with the subset topology. Examples: $\mathrm{SL}(V)$, $\mathrm{SO}(V,\langle\,,\rangle)$, $\mathrm{SU}(V,\langle\,,\rangle)$, with the special cases $\mathrm{SL}(n,\mathbb{K})$, $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ [**DG**, p. 47], the spheres S^0, S^1, S^3.

January 31: The identity component G^{o} of a Lie group G, which is an open-and-closed normal subgroup of G [**DG**, p. 43], generated by any connected neighborhood of 1. The natural bijective correspondence between left-invariant distributions D on a Lie group G and vector subspaces $\mathfrak{h}$ of $\mathfrak{g}$, given by $D_x=\{w_x:w\in\mathfrak{g}\}$ for all $x\in G$ [**DF**, p. 7]. The observation that such D is integrable if an only if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. [**DF**, p. 7]. The resulting bijective correspondence between connected Lie subgroups H of G and a Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$, where H is the leaf of D (see the last sentence) through 1 [**DF**, pp. 7-8]. The algebra $\mathbb{H}$ of quaternions [**DG**, p. 13]. Incidental isomorphisms and (two-to-one) almost-isomorphisms: $\mathrm{SO}(1)=\{1\}$, $\mathrm{SO}(2)=S^1$, $\mathrm{SU}(2)=S^3\approx\mathbb{R}\mathrm{P}^3$, $S^3\to\mathrm{SO}(3)$, $S^3\times S^3\to\mathrm{SO}(4)$ [**DG**, p. 48]. The set $\mathcal{B}(V)$ of ordered bases of a finite-dimensional vector space V over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Diffeomorphic identifications $\mathrm{GL}(V)\approx\mathcal{B}(V)$. An "upper triangular basis" for any $A\in\mathrm{End}(V)$ when $\mathbb{K}=\mathbb{C}$, and the resulting curve joining Id to A in $\mathrm{End}(V)$, which implies connectedness of $\mathrm{GL}(V)$ and $\mathcal{B}(V)$ [**DG**, pp. 49, 51]. Two connected components of $\mathrm{GL}(V)$ or $\mathcal{B}(V)$, called the orientations of V, if $\mathbb{K}=\mathbb{R}$ and $V\neq\{0\}$, with two bases lying in the same component precisely when their transition determinant is positive [**DG**, pp. 192-193].

February 2: Orientations, continued. The canonical orientation of the underlying real space of a nonzero finite-dimensional complex vector space, and its compatibility with the direct-sum operation. The Frobenius theorem rephrased as the statement that "complete integrability" of a system of first-order partial differential equations, "solved" for the partial deerivatives, is equivalent to equality of mized second-order partial derivatives being a formal (or "algebraic") consequence of the system. More details follow: a distribution D of dimension p on an m-dimensional manifold M, and the replacement of M by a coordinate domain identified via the coordinate mapping $(x^1,\dots,x^p,y^{p+1},\dots,y^m)$ with a Euclidean rectangle so that D_x, at any point x, intersects trivially the span of the last $m-p$ coordinate vectors $\partial_\lambda(x)$. The projection π acting as $(x^1,\dots,x^p,y^{p+1},\dots,y^m)\mapsto(x^1,\dots,x^p)$ between Euclidean rectangles, the first identifed with M, the second referred to as the "base" B. The convention $1\le j,k,l\le p<p+1\le\lambda,\mu,\nu\le m$ about the ranges of the indices j,k,l,λ,μ,ν. The unique smooth functions H_j^μ with the property that $e_j=\partial_j+H_j^\mu\partial_\mu$ and the images $\hat{\partial}_\lambda$ of ∂_λ under the projection morphism $TM\to D^{\mathrm{nrm}}$ are local trivializing sections for D and, respectively, D^{nrm}. (These H_j^μ exist: $e_j(x)$ are, at every $x\in M$, the preimages, under the isomorphism $d\pi_x:D_x\to T_{\pi(x)}B$, of the coordinate vectors at $\pi(x)$ in the base rectangle B. The local trivializations e_j,∂_λ in TM and $dy^j,dy^\lambda-H_k^\lambda dx^k$ in T^*M, dual to each other [**DF**, formula (5) on p. 2]. The equations $dy^\lambda=H_j^\lambda dx^j$ of the distribution D, in the sense that a vector (field) v is tangent to D if

and only if both sides yield the same value on it: $v^\lambda = H_j^\lambda v^j$ [**DF**, formula (6) on p. 2]. The component functions Ω_{jk}^λ of the curvature Ω of D, characterized by $\Omega(e_j, e_k) = \Omega_{jk}^\lambda \hat{\partial}_\lambda$, and the equality $\Omega_{jk}^\lambda = \partial_j H_k^\lambda - \partial_k H_j^\lambda + H_j^\mu \partial_\mu H_k^\lambda - H_k^\mu \partial_\mu H_j^\lambda$. Mappings $\varphi : Q \to M$ into our coordinate domain M that are tangent to the distribution D, and their characterization: $\partial_a y^\lambda = (\partial_a x^j) H_j^\lambda(x^1, \dots, x^p, y^{p+1}, \dots, y^m)$ whenever z^a are local coordinates in the manifold Q, with x^j and y^λ standing for $x^j \circ \varphi$ and $y^\lambda \circ \varphi$ [**DF**, formula (9) on p. 3]. Two special cases, where Q is an open interval $I \subseteq \mathbb{R}$ (or, $Q = B$ and φ happens to be a mapping of the form $(x^1, \dots, x^p) \mapsto (x^1, \dots, x^p, y^{p+1}, \dots, y^m)$, with z^a constituting the standard coordinate t (or, the coordinates x^j), so that $\dot{y}^\lambda = \dot{x}^j H_j^\lambda(x^1, \dots, x^p, y^{p+1}, \dots, y^m)$ (the condition describing curves tangent to D), the notational convention that $(\,)^\cdot = d/dt$ being used from now on, or, respectively, $\partial_j y^\lambda = H_j^\lambda(x^1, \dots, x^p, y^{p+1}, \dots, y^m)$ [**DF**, formulae (10) and (12) on p. 3]. The observation that integral manifolds P of D are, locally, graphs of mappings $(x^1, \dots, x^p) \mapsto (x^1, \dots, x^p, y^{p+1}, \dots, y^m)$, as the bundle projection π restricted to P must be locally diffeomorphic, P being transverse to the fibres, and so complete integrability of a system of partial differential equations having the form $\partial_j y^\lambda = H_j^\lambda(x^1, \dots, x^p, y^{p+1}, \dots, y^m)$ is equivalent to vanishing of Ω, that is, to symmetry of $\partial_j H_k^\lambda + H_j^\mu \partial_\mu H_k^\lambda$ in j, k. A condition necessary and sufficient for such a system to be completely integrable thus amounts to the system's internal consistency: to the symmetry relation $\partial_j \partial_k y^\lambda = \partial_k \partial_j y^\lambda$ being a formal consequence of the system itself [**DF**, p. 4]. An outline of the proof of the Frobenius theorem via induction on the dimensions q of "partial" integral manifolds, with both the case $q = 1$ and the induction step arising from solvability of ordinary differential equations: the required first-order partial differential equations on a rectangle of the next dimension is established via the fact that zero is the only solution, assuming the value 0 somewhere, of a system of linear homogeneous ordinary differential equations [**DF**, pp. 4-5].

Homework: [**DF**, Exercises 5, 7, 12 on pp. 8-9].

February 5: The Koszul definition of a linear connection ∇ in a real/complex vector bundle E over a manifold M [**FR**, item BM, pp. 22-23]. Example: the standard flat connection, denoted here by D, in a product vector bundle $M \times \mathbb{F}$ over M, with $\mathrm{D}_v \psi = d_v \psi$ for smooth sections ψ treated as functions $M \to \mathbb{F}$, so that D has the zero components in a costant global trivialization of $M \times \mathbb{F}$ [**DG**, Example 20.2 on p. 77]. The local character of ∇ resulting from abundance of cut-off functions [**DG**, p. 217], meaning that the restriction of $\nabla_w \psi$ to an open set U depends only on the restrictions to U of the smooth vector field w on M and the smooth section ψ of E. Restrictibility of ∇ to a connection in the restriction of E to any open submanifold U of M, that is, well-definedness of $\nabla_w \psi$ when w and ψ are only smooth sections of TM and E with the same domain U. The component functions Γ_{ja}^b of ∇ relative to a local coordinate system x^j in M and local trivializing sections e_a of E, with $\nabla_{\partial_j} e_a = \Gamma_{ja}^b e_b$. The formula $[\nabla_w \psi]^a = w^j(\partial_j \psi^a + \Gamma_{jb}^a \psi^b)$, that is, $\nabla_w \psi = w^j(\partial_j \psi^a + \Gamma_{jb}^a \psi^b) e_a$ for smooth local sections w of TM and ψ of E. The conclusion that the dependence of $\nabla_w \psi$ on w is pointwise (and not just local): at any $x \in M$, the value $[\nabla_w \psi]_x \in E_x$ is uniquely determined by w_x and by the restriction of ψ to any given neighborhood of x. Well-definedness of $\nabla_v \psi \in E_x$ for $v \in T_x M$ and a smooth local section of E defined on a neighborhood of x. The resulting linear operator $[\nabla \psi]_x : T_x M \to E_x$, for any such ψ, and the existence of such ψ with $[\nabla \psi]_x = 0$ having any prescribed value $\psi_x \in E_x$, immediate from the above formula for $[\nabla_w \psi]^a$. The interpretation of $\nabla \psi$ as a section of $\mathrm{Hom}(TM, E)$ (that is, a vector-bundle morphism

$TM \to E$, cf. [**DG**, pp. 69 and 73]) whenever ψ is a smooth section of a vector bundle E over M with a fixed linear connection ∇. The linear operator $T_xM \ni v \mapsto v_\phi^{\mathrm{hrz}} \in T_{(x,\phi)}E$ of horizontal lift associated with ∇, any $x \in M$, and any $\phi \in E_x$, given by $v_\phi^{\mathrm{hrz}} = d\psi_x v$, where ψ is any smooth local section of E defined on a neighborhood U of x such that $\psi_x = \phi$ and $[\nabla\psi]_x = 0$. (Local sections ψ of E defined on U are identified here – in the usual fashion – with mappings $\psi : U \to E$ having $\pi \circ \psi = \mathrm{id}$, so that $\psi(x) = (x, \psi_x)$ whenever $x \in U$). Correctness of the definition of v_ϕ^{hrz} (that is, its independence of the choice of ψ): as $\psi^a(x) = \phi^a$ and $(\partial_j\psi^a)(x) = -\Gamma_{jb}^a(x)\phi^b$, in the local coordinates x^j, ϕ^a for E arising from our local coordinate system x^j in M and local trivializing sections e_a of E [**DG**, p. 67], the components of $d\psi_x v$ consist of v^j (the components of v relative to x^j) and $v^j(\partial_j\psi^a)(x) = -v^j\Gamma_{jb}^a(x)\phi^b$. The relation $d\pi_{(x,\phi)}v_\phi^{\mathrm{hrz}} = v$, immediate from the chain rule [**DG**, p. 20], applied to the equality $\pi \circ \psi = \mathrm{id}$ (and reflected by the just-mentioned fact that the initial components of v_ϕ^{hrz} are v^j, the components of v). The horizontal distribution H on E corresponding to ∇, with $H_y = \{v_\phi^{\mathrm{hrz}} : v \in T_xM\}$ for $y = (x,\phi) \in E$, so that the horizontal-lift operator $T_xM \ni v \mapsto v_\phi^{\mathrm{hrz}} \in H_y$ (equal, by the way, to $d\psi_x$ for any smooth local section of E, defined on a neighborhood of x, with $\psi_x = \phi$ and $[\nabla\psi]_x = 0$) is an isomorphism with the inverse $d\pi_y : H_y \to T_xM$. The horizontal lift of a (smooth, local) vector field w tangent to M, relative to a linear connection ∇ in a vector bundle $\pi : E \to M$, defined to be the (smooth, local) vector field w^{hrz} tangent to E given by $w_y^{\mathrm{hrz}} = v_\phi^{\mathrm{hrz}}$ for $v = w_x$ and $y = (x,\phi) \in E$, so that $x = \pi(y)$. The components $v^j, -v^k\Gamma_{kb}^a\phi^b$ of v^{hrz} in local coordinates x^j, ϕ^a for E discussed earlier, which we will also informally express as $v^{\mathrm{hrz}} \sim (v^j, -v^k\Gamma_{kb}^a\phi^b)$. Smoothness of H. The vertical and horizontal projection bundle morphisms $TE = H \oplus V \to V$ and $TE \to H$, depending on ∇ via H, and written as $[\,]^{\mathrm{vrt}}$, $[\,]^{\mathrm{hrz}}$. The equalities $\xi^{\mathrm{hrz}} \sim (\xi^j, -\Gamma_{kb}^a\phi^b\xi^k)$, $\xi^{\mathrm{vrt}} \sim (0, \xi^a + \Gamma_{kb}^a\phi^b\xi^k)$ whenever ξ is a vector (field) tangent to the total space E, with $\xi \sim (\xi^j, \xi^a)$, and $\xi^{\mathrm{vrt}}, \xi^{\mathrm{hrz}}$ denoting its V and H components (projections). The equality $[\nabla_v\psi]_x = [d\psi_x v]^{\mathrm{vrt}}$, for any $x \in M$, any $v \in T_xM$ and any smooth local section ψ of E defined on a neighborhood of x, showing that H uniquely determines ∇. Proof of this equality based on noting that, in local coordinates x^j, ϕ^a for E mentioned above (January 14), $d\psi_x v \sim (v^j, v^k(\partial_k\psi^a)(x))$, which equals the sum of the horizontal vector $v_\phi^{\mathrm{hrz}} \sim (v^j, -v^k\Gamma_{kb}^a(x)\psi^b(x))$ and the vertical vector with the components $(0, v^k[(\partial_k\psi^a)(x) + \Gamma_{kb}^a(x)\psi^b(x)])$, that is, the vertical vector corresponding to $(\nabla_v\psi)_x$ under the canonical isomorphic identication between a finite-dimensional real vector space (treated as a manifold) and its tangent space at any point [**DG**, Example 5.1 on p. 19].

February 7: The curvature tensor $R = R^\nabla$ of a linear connection ∇ in a real/complex vector bundle E over M, assigning to $x \in M$ the skew-symmetric bilinear mapping $R_x = T_xM \times T_xM \to \mathrm{End}\,E_x$ into the space $\mathrm{End}\,E_x$ of real/complex endomorphisms of E_x, and characterized by $R(v,w)\psi = \nabla_w\nabla_v\psi - \nabla_v\nabla_w\psi + \nabla_{[v,w]}\psi$ for smooth local sections v, w of TM and ψ of E. The pointwise dependence of $R(v,w)\psi$ on v, w and ψ, due to *the easily-verified formula* $[R(v,w)\psi]^a = R_{jkb}{}^a v^j w^k \psi^b$, *where the component functions* $R_{jkb}{}^a$ *of* R *relative to any* x^j *and* e_a *as above are given by* $R_{jkb}{}^a = \partial_k\Gamma_{jb}^a - \partial_j\Gamma_{kb}^a + \Gamma_{kc}^a\Gamma_{jb}^c - \Gamma_{jc}^a\Gamma_{kb}^c$ **(Problem 1)**. The curvature operators $R_x(v,w) \in \mathrm{End}(E_x)$ corresponding to a linear connection ∇

with the curvature tensor $R = R^\nabla$ in a real/complex vector bundle E over a manifold M and vectors v, w tangent to M at a point x. The "universal shortcut" to be used for proving equalities known to be pointwise, but phrased in terms of smooth local sections of vector bundles (including vector fields): one may always assume that the first-order partial derivatives of all the sections involved vanish at the point in question. The equality $([v^{\mathrm{hrz}}, w^{\mathrm{hrz}}]^{\mathrm{vrt}})_y = R_x(v_x, w_x)\phi$ for $y = (x, \phi) \in E$ and vector fields v, w tangent to M, showing that $R = R^\nabla$ essentially becomes the curvature Ω of the horizontal distribution H on E corresponding to ∇, as long as one identifies H_y with T_xM via the isomorphism $d\pi_y : H_y \to T_xM$, the normal bundle H^{nrm} with the vertical distribution V, and the quotient projection $\pi : TE \to H^{\mathrm{nrm}}$ with the vertical projection $TE \to V$. *A proof of this last equality, based on combining the "universal shortcut" with the coordinate expression of the Lie bracket* [**DG**, formula (6.7) on p. 23], *the fact that* $[v^{\mathrm{hrz}}, w^{\mathrm{hrz}}]$ *projects under the bundle projection* $\pi : E \to M$ *onto* $[v, w]$, *cf.* [**DG**, Theorem 6.1 on p. 24], *plus the following relations* (February 5): $\xi^{\mathrm{vrt}} \sim (0, \xi^a + \Gamma^a_{kb}\phi^b\xi^k)$ *applied to* $\xi = [v^{\mathrm{hrz}}, w^{\mathrm{hrz}}]$) *and* $v^{\mathrm{hrz}} \sim (v^j, -v^k\Gamma^a_{kb}\phi^b)$, *along with the analog of the latter for* w^{hrz} **(Problem 2)**. Another proof: the relation $v^{\mathrm{hrz}} \sim (v^j, -v^k\Gamma^a_{kb}\phi^b)$ rewritten as $v^a = H^a_j v^j$ with $H^a_j = \Gamma^a_{jb}\phi^b$ gives $\Omega^a_{jk} = R_{jkb}{}^a\phi^b$ for the component functions $R_{jkb}{}^a$ of R (see the sixth line of February 7) and Ω^a_{jk} associated with the horizontal distribution H of ∇, cf. February 2. The standard flat connection D in a product vector bundle (February 5) and the *flatness of* D*, that is, vanishing of its curvature* **(Problem 3)**. The effect on connections of the natural operations on real/complex vector bundles E, E' over a manifold M, namely, the direct sum $E \oplus E'$, the Hom bundle $\mathrm{Hom}(E, , E')$, and its special case, the dual $E^* = \mathrm{Hom}(E, M \times \mathbb{K})$, presented using the "comma" notation $\psi^a{}_{,j}$ instead of $[\nabla\psi]^a_j$ (which one also writes as $\nabla_j\psi^a$) for smooth local sections ψ of a vector bundle E over manifold M, a linear connection ∇ in E, local coordinates x^j in M, and local trivializing sections e_a in E, so that $\psi^a{}_{,j} = \partial_j\psi^a + \Gamma^a_{jb}\psi^b$ [**DG**, p. 85], and nstarting from the direct sum [**DG**, p. 97]. Next, the functor Hom applied to linear connections in vector bundles [**DG**, p. 87], uniquely characterized by the Leibniz rule $\nabla_w(\Phi\psi) = (\nabla_w\Phi)\psi + \Phi\nabla_w\psi$ for (local) smooth sections Φ and ψ of $\mathrm{Hom}(E, E')$ and E, with the component formula $\Phi^\lambda_{a,j} = \partial_j\Phi^\lambda_a + \Gamma^\lambda_{j\mu}\Phi^\mu_a - \Gamma^b_{ja}\Phi^\lambda_b$ and the local definition $\nabla_{\partial_j}(e^a \otimes e_\lambda) = \Gamma^\mu_{j\lambda}e^a \otimes e_\mu - \Gamma^a_{jb}e^b \otimes e_\lambda$ for the resulting connection in $\mathrm{Hom}(E, E')$ (all connections involved being denoted by ∇), where x^j, e_a, e_λ are, respectively, a local coordinate system in the base manifold M, and local trivializing sections for the original vector bundles E, E' [**DG**, p. 87]. The special case of the dual connection in $E^* = \mathrm{Hom}(E, M \times \mathbb{K})$, the dual bundle of E, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, with $\xi_{a,j} = \partial_j\xi_a - \Gamma^b_{ja}\xi_b$ and $\nabla_{\partial_j}e^a = -\Gamma^a_{jb}e^b$ [**DG**, pp. 87-88].

Homework: Problems 1, 2 and 3, italicized above.

February 9: The pullback $\overset{*}{\nabla} = F^*\nabla$ of a linear connection ∇ in a real/complex vector bundle E over a manifold M under a smooth mapping $F : N \to M$ [**DG**, p. 96], which is a connection in the pullback vector bundle F^*E, characterized by the condition $\overset{*}{\nabla}_v[F^*\psi] = \nabla_w\psi \in E_x = (F^*E)_y$ whenever $y \in N$ and $w = dF_y v$, while ψ is a smooth local section of E defined on a neighborhood of $x = F(y)$. The components $\overset{*}{\Gamma}{}^a_{\lambda b} = (\partial_\lambda F^j)\Gamma^a_{jb}$ of $\overset{*}{\nabla}$ in a local trivialization of the form F^*e_a. Existence and uniqueness of $\overset{*}{\nabla}$ due to the fact that the above characterization of $\overset{*}{\nabla}_v[F^*\psi]$ implies the formula $\overset{*}{\Gamma}{}^a_{\lambda b} = (\partial_\lambda F^j)\Gamma^a_{jb}$ (thus ensuring uniqueness), while *the latter – used, locally, to define a con-*

nection – is easily seen to yield the former **(Problem 1)**. The easy consequence of this characterization of $\overset{*}{\nabla}_v[F^*\psi]$, stating that, *if a smooth local vector field v on N is projectable under F onto a smooth local vector field w on M* [**DG**, p. 23], *then, for any smooth local section ψ of E, one has $\overset{*}{\nabla}_v[F^*\psi] = F^*[\nabla_w\psi]$* **(Problem 2)**. The conjugate of a connection in a complex vector bundle (linear connections in $\overline{E}$ being naturally identified with those in E). The vector space $L(V_1,\dots,V_r;V')$ consisting of all r-linear mappings $B : V_1\times\ldots\times V_r \to V'$ for vector spaces $V_1,\dots,V_r,V'$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The isomorphism $L(V_1,\dots,V_r;V') \to \mathrm{Hom}(V_j, L(V_1,\dots,\widehat{V_j},\dots,V_r;V')$ – where $\widehat{\ }$ means 'delete' – sending $B \in L(V_1,\dots,V_r;V')$ to the operator associating with $v_j \in V_j$ the $(r-1)$-linear mapping B' given by $B'(v_1,\dots,\widehat{v}_j,\dots,v_r) = B(v_1,\dots,v_r)$. The resulting operation $L(E_1,\dots,E_r;E')$ applied fibrewise to smooth vector bundles $E_1,\dots,E_r,E'$ over a manifold M, with the equivalent recursive definition $L(E_1,\dots,E_r;E') = \mathrm{Hom}(E_1, L(E_2,\dots,E_r;E'))$ for $r > 1$ and $L(E_1;E') = \mathrm{Hom}(E_1,E')$, which turns $L(E_1,\dots,E_r;E')$ into a smooth vector bundle over M, and its effect on connections. *The Leibniz rule $[\nabla_w B](\psi_1,\dots,\psi_r) = \nabla_w[B(\psi_1,\dots,\psi_r)] - B(\nabla_w\psi_1,\psi_2,\dots,\psi_r) - \ldots - B(\psi_1,\dots,\psi_{r-1},\nabla_w\psi_r)$ whenever $\psi_1,\dots,\psi_r$ and B are smooth local sections of $E_1,\dots,E_r$ and $L(E_1,\dots,E_r;E')$, while w is any smooth vector field on the base manifold, and ∇ denotes the linear connection in $L(E_1,\dots,E_r;E')$ induced by linear connections in $E_1,\dots,E_r,E'$* **(Problem 3)**. The observation that, *for ∇, $L(E_1,\dots,E_r;E')$ and w as in the last sentence, in the case where $E_j = E_k$ both carry the same connection, j,k being distinct, the operator ∇_w acting on smooth sections of $L(E_1,\dots,E_r;E')$ commutes with the switch of the jth and kth arguments* **(Problem 4)**. Smooth subbundles of $L(E_1,\dots,E_r;E')$ defined by requiring symmetry or skew-symmetry with respect to a subset K of $\{1,\dots,r\}$ having the same bundle E_k for all $k \in K$, and the conclusion that, due to the preceding sentence, for ∇ as above, such subbundles are ∇-parallel. Smooth vector subbundles that are ∇-parallel for a linear connection ∇ in a given real/complex vector bundle E. Example: *the summand subbundles in a direct-sum vector bundle are parallel relative to any direct-sum connection* **(Problem 5)**. Bilinear symmetric and sesquilinear Hermitian forms in real/complex vector spaces. The fact that *each of them is uniquely determined by the corresponding quadratic function* **(Problem 6)**. Spacelike, timelike and null vectors. Spacelike and timelike subspaces. Null subspaces of a real/complex vector space with a fixed bilinear symmetric (or, sesquilinear Hermitian) form, defined by requiring the form to vanish on them or, equivalently (see Problem 6), consisting of null vectors. The existence of orthogonal and orthonormal bases for a biilinear symmetric or sesquilinear Hermitian form in a finite-dimensional real/complex vector spaces, 'orthonormal' meaning that the "squares" (diagonal entries) equal ±1 or 0. The positive/negative index and defect (nullity) of a form as above (the numbers $i_\pm$ and i_0 of positive/negative and zero diagonal entries), and their independence of the basis used (Sylvester's law of inertia), due to their characterization in terms of the maximum dimensions of spacelike/timelike subspaces, or spacelike-or-null and timelike-or-null subspaces – that characterization being obvious since a spacelike (or, timelike) subspace must have a trivial intersection with any timelike-or-null (or, spacelike-or-null) subspace. The maximum dimension of a null subspace $\mathcal{V}$ in the finite-dimensional case, equal to $i_0 + \min(i_-, i_+)$ for the positive/negative index $i_\pm$ and nullity i_0, as one sees fixing an orthonormal basis: according to the trivial-intersections conclusion three lines before, $\mathcal{V}$ has trivial intersections with the spans of all timelike-or-null, and of all spacelike-or-null vectors of the basis, which gives $\dim\mathcal{V} \le i_0 + i_\pm$, while the maximum dimension is realized by the span of all u_j and $v_k + w_k$, where u_j (or v_k,

or w_k) are mutually distinct null (or spacelike, or timelike) vectors from the basis, and j (or, k) ranges from 1 to i_0 (or, to $\min(i_-,i_+)$). The orthogonal complement $\mathcal{V}^\perp$ of a vector subspace $\mathcal{V}$ in a finite-dimensional real vector space $\mathcal{T}$, relative to a fixed symmetric bilinear form on $\mathcal{T}$, and the relation $\dim\mathcal{V}+\dim\mathcal{V}^\perp=\dim\mathcal{T}$, valid whenever $\langle\,,\rangle$ or $\mathcal{V}$ is nondegenerate (where the latter means that $\mathcal{V}\cap\mathcal{V}^\perp=\{0\}$, and hence yields $\mathcal{V}\oplus\mathcal{V}^\perp=\mathcal{T}$). Proof: for $k=\dim\mathcal{V}$ and $m=\mathcal{T}$, and a basis $e_1,\dots,e_m$ of $\mathcal{T}$ such that $e_1,\dots,e_k$ lie in $\mathcal{V}$ (where they thus form a basis), $x\mapsto(\langle x,e_1\rangle,\dots,\langle x,e_q\rangle)\in\mathrm{I\!R}^q$ is injective (and hence an isomorphism): on $\mathcal{T}$ in the former case if $q=m$, and on $\mathcal{V}$ in the latter when $q=k$. In both cases, $\mathcal{T}\ni x\mapsto(\langle x,e_1\rangle,\dots,\langle x,e_k\rangle)\in\mathrm{I\!R}^k$ is then surjective, with the kernel $\mathcal{V}^\perp$. Pseudo-Euclidean and pseudo-Hermitian inner products. Their sign patterns (metric signatures). Pseudo-Riemannian and pseudo-Hermitian fibre metrics in vector bundles E, including the Riemannian and Hermitian ones, always required to be smooth as sections of $\mathrm{Hom}(E,\bar{E}^*)$, where $\bar{E}^*$ is simultaneously the dual of the conjugate of E and the conjugate of the dual [**DG**, Section 17, Problems 5–6]. The special case of tangent bundles: pseudo-Riemannian and Riemannian metrics/manifolds.
Homework: Problems 1, 2, 3, 4, 5 and 6, italicized above.
February 12: Covariant differentiation, relative to a linear connection ∇, of smooth sections $I\ni t\mapsto\phi(t)\in E_{x(t)}$ of the vector bundle E along a curve $I\ni t\mapsto x(t)$ the base manifold M, written as $\nabla_{\dot{x}}$, and defined to be a special case of a pullback connection, for $N=I$ and $F(t)=x(t)$, where $I\subseteq\mathrm{I\!R}$ is an open interval with the fixed constant tangent vector field $w=1$, that is, the standard coordinate vector field $w=d/dt$ on I (and $y^\lambda=t$, the index λ having a one-element range). The component formula $[\nabla_{\dot{x}}\phi]^a=\dot{\phi}^a+\Gamma_{jb}^a\dot{x}^j\phi^b$, immediate since $[\overset{*}{\nabla}_w\phi]^a=v^\lambda[\partial_\lambda\phi^a+(\partial_\lambda F^j)\Gamma_{jb}^a\psi^b]$. The equality $[\nabla_{\dot{x}}\phi](t)=\nabla_{\dot{x}(t)}\psi$, due to the chain rule, in the case where $\phi(t)=\psi_{x(t)}$ for a (local) smooth section ψ of E, which implies that *for a linear connection ∇ in a vector bundle E over a manifold M, a point $x\in M$, a tangent vector $v\in T_xM$, and a smooth local section ψ of E defined on a neighborhood of x, the covariant derivative $\nabla_v\psi\in E_x$ depends only on the restriction of ψ to an arbitrarily chosen smooth curve in M passing through x with the velocity v* **(Problem 1)**. Sections $I\ni t\mapsto\phi(t)\in E_{x(t)}$ along a curve $I\ni t\mapsto x(t)$ that are ∇-parallel: $\nabla_{\dot{x}}\phi=0$ or, equivalently, for which the curve $t\mapsto(x(t),\phi(t))$ in E is horizontal (tangent to H), the equivalence being clear from the relation $0=\xi^{\mathrm{vrt}}\sim(0,\,\xi^a+\Gamma_{kb}^a\phi^b\xi^k)$ (see January 14) characterizing horizontal vectors among all vectors ξ tangent to E. The conclusion that *one has $\dot{y}^{\mathrm{vrt}}=\nabla_{\dot{x}}\phi$, at every $t\in I$, for any smooth curve $t\mapsto y(t)=(x(t),\phi(t))$ in the total space E of a vector bundle with a linear connection ∇* **(Problem 2)**. Parallel transport along a curve in the base [**DG**, p. 82]. The effect of the pullback of connections on covariant differentiation of smooth sections along curves: $\overset{*}{\nabla}_{\dot{y}}\psi=\nabla_{\dot{x}}\psi$ for a section ψ of F^*E along a curve $t\mapsto y(t)\in N$ (which is simultaneously a section of E along the curve $t\mapsto x(t)=F(y(t))\in M$), obvious if one writes $[\nabla_{\dot{x}}\psi]^a=\dot{\psi}^a+\Gamma_{jb}^a\dot{x}^j\psi^b$ and $[\overset{*}{\nabla}_{\dot{y}}\psi]^a=\dot{\psi}^a+\overset{*}{\Gamma}{}_{\lambda b}^a\dot{y}^\lambda\psi^b$, noting that $\dot{x}^j=y^\lambda\partial_\lambda F^j$ and $\overset{*}{\Gamma}{}_{\lambda b}^a=(\partial_\lambda F^j)\Gamma_{jb}^a$. Parallel transport along a curve in the base, relative to a linear connection in a vector bundle. The parallel transport as "conjugation" in the case of Hom connections. Linear connections on a manifold, that is, in its tangent bundle. The torsion tensor field $\Theta=\Theta^\nabla$ of such a connection ∇ and its component functions $\Theta_{jk}^l=\Gamma_{jk}^l-\Gamma_{kj}^l$ [**DG**, pp. 79-80]. Torsion-free connections, also referred to as symmetric.
Homework: Problems 1 and 2, italicized above.

February 14: Sylvester's law of inertia implying that, if a bilinear symmetric (or, sesquilinear Hermitian) form g is nondegenerate, all such nearby forms have the same sign pattern as g (since, on the unit sphere S in a maximal g-spacelike or g-timelike subspace, the quadratic function of g is positive/negative, which implies the same for nearby forms due to compactness of S). Constancy od the sign pattern of a fibre metric, required by definition (but automatically satisfied when the base manifold is connected). The observation that *a mapping symmetric in the first two and skew-symmetric in the last two arguments, defined on $K \times K \times K$, where the set K has more than one element, and taking values in an Abelian group G with no elements of order 2, must be identically zero* **(Problem 1)**, which can also be justified by the fact that, in a permutation group, any two transpositions are conjugate to each other and hence have the same value under any homomorphism into an Abelian group such as $\{1,-1\}$ with multiplication. A generalization: injectivity of the assignment $\alpha \mapsto (\beta,\gamma)$ with $\beta(p,q,r) = \alpha(p,q,r) - \alpha(q,p,r)$ and $\gamma(p,q,r) = \alpha(p,q,r) + \alpha(p,r,q)$, where $\alpha : K \times K \times K \to G$, for K,G as above. (Namely, $\gamma(p,q,r) + \gamma(q,p,r) - \gamma(r,p,q) + \beta(r,p,q) + \beta(r,q,p) + \beta(p,q,r) = 2\alpha(p,q,r)$.) The h-modified components $\Gamma_{jab} = \Gamma_{ja}^c h_{cb}$ of a linear connection ∇ in a real vector bundle with a fibre metric h, so that $\Gamma_{jab} = h(\nabla_{\partial_j} e_a, e_b)$ and $\Gamma_{ja}^b = \Gamma_{jac} h^{cb}$. Here $h_{ab} = h(e_a, e_b)$ are the component functions of h [**DG**, p. 101], and h^{ab} are its '"reciprocal" components, with the matrix relation $[h^{ab}] = [h_{ab}]^{-1}$ [**DG**, p. 105]. Compatibility of the fibre metric h with ∇, meaning that $\nabla h = 0$ and, due to the Leibniz rule, equivalent to $\partial_j h_{ab} = \Gamma_{jab} + \Gamma_{jba}$. The Levi-Civita connection of a pseudo-Riemannian manifold (M,g), which is the unique torsion-free connection ∇ in TM, compatible with g. Its components, the Christoffel symbols Γ_{jk}^l given by $\Gamma_{jk}^l = \Gamma_{jkq} g^{ql}$ with $2\Gamma_{jkl} = \partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}$. The last equality, along with the existence of uniqueness of ∇, as obvious consequences of the formula for 2α eight lines earlier, applied to the equations $\Gamma_{jkl} - \Gamma_{kjl} = 0$ and $\Gamma_{jkl} + \Gamma_{jlk} = \partial_j g_{kl}$. The velocity vector field $t \mapsto \dot x(t)$ of a smooth curve $t \mapsto x(t)$ a manifold M, giving rise to the geodesic curvature (acceleration) vector field $\nabla_{\dot x}\dot x$, relative to any connection ∇ in TM, with the components $[\nabla_{\dot x}\dot x]^k = \ddot x^k + \Gamma_{lj}^k \dot x^l \dot x^j$. The geodesics of a connection ∇ on M, that is, the smooth curves $t \mapsto x(t) \in M$ with $\nabla_{\dot x}\dot x = 0$ or, in coordinates, $\ddot x^k + \Gamma_{lj}^k \dot x^l \dot x^j = 0$. Existence and uniqueness of the ∇-geodesic $t \mapsto x(t)$ with any initial data $(x(a), \dot x(a)) = (z,v) \in TM$, defined on a maximal open interval containing the given parameter value $a \in \mathrm{I\!R}$. The observation that, for a geodesic, the analog of $\ddot x^k + \Gamma_{lj}^k \dot x^l \dot x^j$ obtained by replacing t with a new parameter s equals $(dt/ds)^2 \dot x$, and so affine parameter changes are the only ones leading from a nonconstant geodesic to a geodesic. Example: constant-speed line segments as geodesics of the standard flat connection in an open subset of a finite-dimensional affine space. The affine space of linear connections in a vector bundle E, with the space of smooth sections) of $\mathrm{Hom}(TM, \mathrm{Hom}(E,E))$ serving as its translation vector space. The assignment $\nabla \mapsto \nabla - \Theta^\nabla/2$ *constituting a projection from the set of all linear connections in TM onto the subset formed by torsion-free ones, meaning that it is valued in the latter set an equal to the identity on it* **(Problem 2)**. The observation that ∇ has the same geodesics as its torsion-free part $\nabla - \Theta^\nabla/2$. The exponential mapping $\exp_z : U_z \to M$, at a point z of a manifold M carrying a fixed connection ∇ in TM, having the domain $U_z \subseteq T_z M$ formed by all $v \in T_z M$ such that there exists a ∇-geodesic $[0,1] \ni t \mapsto x(t)$ with $(x(0), \dot x(0)) = (z,v)$, and $\exp_z v$ then equals $x(1)$. (Obviously, $0 \in U_z$ and $\exp_z 0 = z$.) Two consequences of invariance of the geodesic equation under affine parameter changes (five lines eearlier). First, U_z is star-shaped in the

sense of being closed under multiplications by all scalars $c \in (0,1)$. In fact, if $v \in U_z$ and $c \in (0,1)$, using the geodesic $[0,1] \ni t \mapsto x(t)$ chosen as above we obtain a new geodesic, $[0,1] \ni t \mapsto y(t) = x(ct)$, for which $(y(0), \dot{y}(0)) = (z, cv)$, and so $cv \in U_z$. Secondly, given any geodesic $t \mapsto x(t)$ with $x(0) = z$ defined on a maximal open interval I containing 0, setting $v = \dot{x}(0) \in T_zM$ one gets $tv \in U_z$ and $x(t) = \exp_z tv$ for all $t \in I$ (which is obvious when $t = 0$, and otherwise follows if one notes that the geodesic $t^{-1}I \ni t' \mapsto x(tt')$ has the initial velocity tv, while $[0,1] \subseteq t^{-1}I$). Openness of U_z, immediate from the open-domain assertion in the regularity theorem for ordinary differential equations with parameters, the initial conditions being considered a part of the parameters [**DG**, Theorem 80.3 on p. 213], which also implies smoothness of $\exp_z : U_z \to M$.
Homework: Problems 1 and 2, italicized above.
February 16: The open-domain assertion in the regularity theorem for ordinary differential equations with parameters, the initial conditions being considered a part of the parameters [**DG**, Theorem 80.3 on p. 213], which implies not only openness of U_z and smoothness of $\exp_z : U_z \to M$, but also smoothness of the "at-large" exponential mapping $\exp : U^{\mathrm{Exp}} \to M$ given by $\mathrm{Exp}(x,v) = \exp_x v$, along with openness of its domain $U^{\mathrm{Exp}} = \{(x,v) : x \in M \text{ and } v \in U_x\}$ as a subset of TM. Thus, U^{Exp} is a neighborhood of the zero section in TM. Here we speak of a "neighborhood" of Y when Y is a subset of a manifold M, referring to any open set in M which contains Y. The zero section of a vector bundle E over a manifold M is the submaniifold $\{(x,v) \in E : v = 0 \text{ in } E_x\}$, identified with M. (Generally, smooth sections of E are naturally identified with submaniifolds of the total space which the bundle projection π maps diffeomorphically onto the base M.) The observation that $T_{(x,0)}E = T_xM \oplus E_x$. The fact that *the differential of* $\exp_z$ *at* 0 *equals the identity operator* $T_zM \to T_zM$ (**Problem 1**), and so, by the inverse mapping theorem, $\exp_z$ restricted to a possibly smaller star-shaped neighborhood of 0 in T_zM maps it diffeomorphically onto a neighborhood of z in M. By forming a composite in which the inverse of the latter diffeomorphism is followed by a linear isomorphism $T_zM \to \mathrm{I\!R}^m$, with $m = \dim M$, one gets a coordinate chart, referred to as a geodesic, or normal, coordinate system for ∇ at z. The augmented exponential mapping $\mathrm{Exp} : U^{\mathrm{Exp}} \to M \times M$ with $\mathrm{Exp}(x,v) = (x, \exp_z v)$ and its local diffeomorphicity, at any $(x,0)$ in the zero section M, its *differential at* $(x,0)$ *being the identity automorphism of* $T_{(x,0)}(TM) = T_xM \oplus T_xM = T_{(x,x)}(M \times M)$ (**Problem 1**); cf. [**DG**, p.38] for the last equality. Weak local convexity: given a manifold M and a connection ∇ in TM, every point $z \in M$ has arbitrarily small neighborhoods U such that any two points $x, y \in U$ can be joined by a ∇-geodesic *depending smoothly on the pair* (x,y). The meaning of the last italicized clause: for all $x, y \in U$ one has $\exp_x v = y$, where $v = v(x,y)$, with some smooth mapping $U \times U \ni (x,y) \mapsto (x, v(x,y)) \in TM$. (Namely, the latter mapping is nothing else that a locai inverse of Exp restricted to a suitable neighborhood of $(z,0)$ in TM.) Linear involutions in vector spaces and the resulting ± 1 "eigenspace" decompositions [**DG**, p. 145]. Examples: *the linear/antilinear decomposition of real-linear operators between complex vector spaces, and the Hermitian/skewH-ermitian decomposition of a sesquilinear form on such a space* (**Problem 3**). The second "universal shortcut" in addition to the one from February 7: given a linear connection ∇ in a real/complex vector bundle E over a manifold M and a point $z \in M$, one can find local trivializing sections e_a defined on a neighborhood of z such that $\Gamma_{ja}^b(z) = 0$ for all j, a, b and every local coordinate system x^j at z, immediate from the existence of smooth local sections ψ of E at z having any prescribed value $\psi_x \in E_x$ and $[\nabla\psi]_x = 0$ (see February 5). Equivalence between vanish-

ing at $z \in M$ of the torsion tensor field of a given connection in TM and the existence of local coordinates at z having $\Gamma^l_{jk}(z) = 0$ [**DG**, Problem 2 on p. 95], which is based on observing that, for the given coodinates x^j and new coodinates $x^{j'}$ at z, all $\partial_{j'}$ are parallel at z (or, equivalently, $\partial_j \partial_{j'} x^k = -\Gamma^k_{jl}\partial_{j'}x^l$) if and only if $\partial_j \partial_k x^{k'} = \Gamma^l_{jk}\partial_l x^{k'}$ at z [**DG**, formula (26.9) on p. 95], the equivalence being immediate if one multiplies the last equality by $\partial_{j'}x^k$, differentiates by parts obtaining $(\partial_k x^{k'})\partial_j\partial_{j'}x^k = -\Gamma^l_{jk}(\partial_{j'}x^k)\partial_l x^{k'}$, and then further multiplies by $\partial_{k'}x^p$, repeatedly using the fact that – due to the chain rule – the matrices $[\partial_k x^{j'}]$ and $[\partial_{j'}x^k]$ are each other's inverses. The ensuing third "universal shortcut" allowing us to assume that $\Gamma^l_{jk} = 0$ at any chosen point z, as long as the given connection in TM is torsion-free. The first Bianchi identity [**DG**, pp. 94]: $R(u,v)w + R(v,w)u + R(w,u)v = 0$ for torsion-free connections and any vector fields u, v, w, with the coordinate version $R_{jkl}{}^q + R_{klj}{}^q + R_{ljk}{}^q = 0$ easily derived from the third "universal shortcut" just mentioned. Another observation: *if $m = \dim M$, a ∇-geodesic coordinate system at z may equivalently be characterized as any coordinate chart $x^1, \dots, x^m$ on a neighborhood U of z which identifies z and U with 0 and, respectively, with a star-shaped neighborhood U' of 0 in $\mathbb{R}^m$, so that $\Gamma^l_{jk}(x^1, \dots, x^m)x^k x^l = 0$ identically on U', which also gives $\Gamma^l_{jk} + \Gamma^l_{kj} = 0$ at the origin* **(Problem 4)**. The possibility of *using this very last conclusion for an alternative proof of third "universal shortcut"* **(Problem 5)**. The covariant derivative ∇R of the curvature tensor $R = R^\nabla$ of a connection ∇ in a real/complex vector bundle E over a manifold M, depending, since R is a section of $A(TM, TM, \mathrm{Hom}(E, E))$, also on a fixed connection – usually assumed torsion-free – in TM, and arising via repeated application of the Hom functor [**DG**, pp. 94-95]. The second Bianchi identity [**DG**, pp. 95]: $R_{jkb}{}^a{}_{,l} + R_{klb}{}^a{}_{,j} + R_{ljb}{}^a{}_{,k} = 0$, for the connection in TM again assumed torsion-free, *easily derived using the second and third "universal shortcuts"* **(Problem 6)**, and *having the coordinate-free form $[\nabla_u R](v,w) + [\nabla_v R](w,u) + [\nabla_w R](u,v) = 0$* **(Problem 7)**.
Homework: Problems 1, 2, 3, 4, 5, 6 and 7, italicized above.
February 19: The second covariant derivative $\nabla(\nabla\psi)$ of a smooth local section ψ of a real/complex vector bundle E over a manifold M [**DG**, p. 88], formed with the aid of fixed connections in E and TM, both denoted by ∇, and obviously – due to the Leibniz rule of February 7 – characterized by $[\nabla_w(\nabla\psi)]v = \nabla_w\nabla_v\psi - \nabla_u\psi$, where $u = \nabla_w v$, for smooth local vector fields v on M, *the component version of which reads $[\nabla_w(\nabla\psi)]v = \psi^a{}_{,jk}v^j w^k e_a$, where $\psi^a{}_{,jk} = \partial_k\psi^a{}_{,j} + \Gamma^a_{kb}\psi^b{}_{,j} - \Gamma^l_{kj}\psi^a{}_{,l}$* **(Problem 1)**. The Ricci identity $\psi^a{}_{,jk} - \psi^a{}_{,kj} = R_{jkb}{}^a\psi^b$ valid when the connection in TM is torsion-free [**DG**, p. 88], immediate if one uses the second and third "universal shortcuts" (February 16) but, *once rewritten as $[\nabla_w(\nabla\psi)]v - [\nabla_v(\nabla\psi)]w = R(v,w)\psi$, also easily derived from the above coordinate-free description of $\nabla(\nabla\psi)$ combined with the original expression for $R(v,w)\psi$, cf. February 7* **(Problem 2)**. The Ricci tensor r of a connection on a manifold [**DG**, p. 80], with the components traditionally denoted by R_{jk}. The tensor product $V_1 \otimes \dots \otimes V_r = L(V_1^*, \dots, V_r^*; \mathbb{K})$ of finite-dimensional vector spaces $V_1, \dots, V_r$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and the r-linear mapping $V_1 \times \dots \times V_r \to V_1 \otimes \dots \otimes V_r$ of tensor multiplication, written as $(v_1, \dots, v_r) \mapsto v_1 \otimes \dots \otimes v_r$ [**DG**, p. 143]. The tensor-product basis of $V_1 \otimes \dots \otimes V_r$ arising from bases of $V_1, \dots, V_r$ [**DG**, p. 146]. The universal factorization property [**DG**, p. 147]. The canonical isomorphisms $V_1 \otimes \dots \otimes V_r \cong V_{\sigma(1)} \otimes \dots \otimes V_{\sigma(r)}$ for any permutation σ of $\{1, \dots, r\}$, as well as $V \otimes \mathbb{K} \cong V$ and $\mathrm{Hom}(V, V') \cong V^* \otimes V'$ [**DG**, p. 148]. The (fibrewise

defined) tensor product $E_1 \otimes \ldots \otimes E_r$ of smooth vector bundles $E_1, \ldots, E_r$ over a manifold M, carrying a linear connection naturally induced by linear connections in $E_1, \ldots, E_r$. *The Leibniz rule* $\nabla_w(\psi_1 \otimes \ldots \otimes \psi_r) = [\nabla_w \psi_1] \otimes \ldots \otimes \psi_r + \ldots + \psi_1 \otimes \ldots \otimes \nabla_w \psi_r$ *for smooth local sections* $\psi_1, \ldots, \psi_r$ *of* $E_1, \ldots, E_r$ **(Problem 3)**. The rth symmetric and exterior powers, $V^{\odot r}$ and $V^{\wedge r}$, of a finite-dimensional vector space V, the r-linear symmetric and exterior multiplications of vectors, the bases of $V^{\odot r}$ and $V^{\wedge r}$ resulting from a basis of V, so that $\dim V^{\odot r} = \binom{n}{r}$ if $n = \dim V$, and the universal factorization properties [**DG**, pp. 143-144]. The space $\hat{V} = V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^*$ of (p, q) tensors in a finite-dimensional real/complex vector space V, with p factors equal to V and q to V^*. One also also refers to these tensors of type (p, q) as being p times contravariant, q times covariant. Thus, vectors are once contravariant; linear functionals once covariant; bilinear forms twice covariant; endomorphisms once contravariant, once covariant. Tensor fields of type (p, q) on a manifold. The Ricci symbol $\varepsilon_{j_1 \cdots j_n}$ equal to the signum of $(j_1, \ldots j_n)$ when it is a permutation of $\{1, \ldots, n\}$ and to zero otherwise [**DG**, p. 32], so that $\det B = \varepsilon_{j_1 \cdots j_n} B_1^{j_1} \ldots B_n^{j_n}$ if B is an $n \times n$ matrix. Volume forms in an n-dimensional real/complex vector space V, that is, nonzero elements ω of $[V^*]^{\wedge n}$ which, cvonsequently, span $[V^*]^{\wedge n}$. The formula $\omega(\varPhi e_1, \ldots, \varPhi e_n) = (\det B) \omega(e_1, \ldots, e_n)$ for any linear endomorphism $\varPhi$ of V, any basis $e_1, \ldots, e_n$ of V, and any any volume form ω, where B is the matrix of $\varPhi$ relative to the basis $e_1, \ldots, e_n$ [**DG**, p. 31], implying well-definedness of the determiant $\det \varPhi$ (as the $\varPhi$-pullback $\varPhi^* : [V^*]^{\wedge n} \to [V^*]^{\wedge n}$ in the line $[V^*]^{\wedge n}$ equals the multiplication by $\det \varPhi$).

Homework: Problems 1, 2 and 3, italicized above.

February 21: The immediate conclusion that $\det(\varPhi \varPsi) = (\det \varPhi) \det \varPsi$ for endomorphisms $\varPhi, \varPsi : V \to V$, since $(\varPhi \varPsi)^* = \varPsi^* \varPhi^* : [V^*]^{\wedge n} \to [V^*]^{\wedge n}$. The derivation associated with a linear endomorphism $\varPhi : V \to V$, which itself is a linear endomorphism of the space $[V^*]^{\otimes r} \otimes W = L(V, \ldots, V, W)$, for any vector space W and any integer $r \geq 1$, and sends any α to β given by $\beta(v_1, \ldots, v_r) = \alpha(\varPhi v_1, v_2, \ldots, v_r) + \ldots + \alpha(v_1, \ldots, v_{r-1}, \varPhi v_r)$. The observation that *if* α *is symmetric, or skew-symmetric, so is* β **(Problem 1)**. The formula $(\det A)^\cdot = (\det A) \operatorname{tr}(A^{-1} \dot{A})$ for any C^1 curve $t \mapsto A = A(t) \in \mathrm{GL}(V)$ [**DG**, p. 31]. The conclusion that, for h_{ab} and h^{ab} introduced on February 14, $h^{ab} \partial_j h_{ab} = \partial_j \log |\det h|$, with $\det h$ depending on the local trivialization e_a [**DG**, p. 125], and so $\varGamma_{kj}^k = \partial_j \log \sqrt{|\det g|}$ in the case of the Levi-Civita connection of a pseudo-Riemannian metric g [**DG**, p. 125]. The relation $\alpha(B_1^{i_1} v_{i_1}, \ldots, B_r^{i_r} v_{i_r}) = (\det B) \alpha(v_1, \ldots, v_r)$, if α is r-linear, skew-symmetric, and B is an $r \times r$ matrix [**DG**, p. 32]. The fact that $\det B = \pm 1$ for the transition matrix B between two orthonormal bases of a pseudo-Euclidean space [**DG**, Problem 18 on p. 50]. Densities in finite-dimensional real vector spaces [**DG**, p. 123]. Example: the absolute value of a top-degree exterior form. (By degree r exterior forms in V one means elements of $[V^*]^{\wedge r}$.) The volume element of a pseudo-Euclidean space. The oriented smooth real-line bundle of densities in a smooth real vector bundle over a manifold M [**DG**, p. 123], and the connection induced in it by any given connection in TM, via the connection in $[T^*M]^{\wedge n}$, for $n = \dim M$. The volume element $\mu = dg$ of an n-dimensional pseudo-Riemannian manifold (M, g), with the essential component $\mu_{1 \ldots n} = \sqrt{|\det g|}$ [**DG**, p. 124]. The divergence operator δ associated with a smooth positive density μ on an n-dimensional manifold, sending any smooth vector field w to the function δw characterized by the local-coordinate formula $\mu_{1 \ldots n} \delta w = \partial_j (w^j \mu_{1 \ldots n})$ [**DG**, p. 125]. Coordinate-inde-

pendence of this definition, verified by simultaneously showing that $\delta w = \mathrm{tr}\nabla w$ when μ is the volume element dg of a pseudo-Riemannian metric g and ∇ is the Levi-Civita connection of g [**DG**, p. 125].
Homework: Problem 1, italicized above.
February 23: The volume form of an oriented nonzero pseudo-Euclidean vector space. Orientability and orientations of a real vector bundle E of positive fibre dimension, and the volume form of a fibre metric in E when E is oriented (meaning: orientable, with a fixed orientation). Tracelessness of the curvature operators $R(v,w)$ of a connection ∇ in a real vector bundle of fibre dimension q admitting, locally, a nonzero parallel density (or, volume form ω): as $\omega_{1\ldots q,j} = \partial_j\omega_{1\ldots q} - \Gamma_{ja}^a\omega_{1\ldots q}$, assuming that $\nabla\omega = 0$ we get symmetry of $\partial_k\Gamma_{ja}^a$ in j,k, which amounts to $R_{jka}{}^a = 0$, as one sees using the second "universal shortcut" (February 16) to get $\Gamma_{ja}^b = 0$ at any given point z. The special case, cf. [**DG**, Problem 9 on p. 105], of a connection compatible with a fibre metric h, with the stronger conclusion: $R(v,w)$ are h-skew-adjoint, both in the real and the complex case [**DG**, Proposition 28.2 on p. 103], the complex case being immediate from the real one since *a complex-line-ar is self/skew adjoint relative to a pseudoo-Hermitian inner product if and only if it is self/skew adjoint relative to its real part* **(Problem 1)**. The h-modified curvature tensor of a connection, where h is a fibre metric, and its components R_{jkab}. 'Parallel' meaning the same as 'invariant under all parallel transports' due to the equality $[\nabla_{\dot{x}}\phi](t) = \nabla_{\dot{x}(t)}\psi$ (February 12, seventh line). Symmetry of the Ricci tensor when the connection ∇ in TM is torsion-free and, locally, admits a nonzero parallel density (or, volume form), immediate from the first Bianchi identity (example: Levi-Civita connections). The space $\mathcal{R}(\mathcal{T})$ of algebraic curvature tensors R in a real vector space $\mathcal{T}$, defined by the requirements that $R(u,u',v,v') = -R(u',u,v,v') = -R(u,u',v',v)$, as well as $R(u,v,w,\,\cdot\,)+R(v,w,u,\,\cdot\,)+R(w,u,v,\,\cdot\,) = 0$ for all $u,v,w,u',v' \in \mathcal{T}$, imposed on a quadrilinear form $R : \mathcal{T}\times\mathcal{T}\times\mathcal{T}\times\mathcal{T} \to \mathrm{I\!R}$ [**AC**]. The tetrahedron version of Milnor's octahedron argument [**MT**] showing that one then also has $R(u,u',v,v') = R(v,v',u,u')$, which, with the ad hoc notation $abcd$ for $R(a,b,c,d)$, consists of the following steps: $2abcd = abcd + badc = -(bcad + cabd) - (adbc + dbac) = -(cbda + acdb) - (dacb + bdca) = -(cbda + bdca) - (acdb + dacb) = dcba + cdab = 2cdab$ (summary: place c,d at the end, use Bianchi, place a,b at the end, again use Bianchi).
Homework: Problem 1, italicized above.
February 26: General reference: [**AC**]. The space $\mathcal{S}(\mathcal{T}) = [\mathcal{T}^*]^{\odot 2} = S(\mathcal{T},\mathcal{T},\mathrm{I\!R})$ of symmetric $(0,2)$ tensors in $\mathcal{T}$, that is, symmetric bilinear forms $\mathcal{T}\times\mathcal{T}\to\mathrm{I\!R}$. The g-trace functional $\mathrm{tr} : \mathcal{S}(\mathcal{T}) \to \mathrm{I\!R}$, under the assumption – made from now on – that $\mathcal{T}$ is finite-dimensional and carries a fixed pseudo-Euclidean inner product $g \in \mathcal{S}(\mathcal{T})$, where tr assigns to any bilinear form $b : \mathcal{T}\times\mathcal{T}\to\mathrm{I\!K}$ the trace of the linear operator $A : \mathcal{T}\to\mathcal{T}$ characterized by $g(Au,\,\cdot\,) = b(u,\,\cdot\,)$ for all $u \in \mathcal{T}$. *The formula* $\mathrm{tr}\, b = \sum_i \varepsilon_i b(e_i,e_i)$ *whenever* e_i *is an orthonormal basis of* $\mathcal{T}$ *and* $\varepsilon_i = g(e_i,e_i) \in \{1,-1\}$ **(Problem 1)**. More on the space $\mathcal{R}(\mathcal{T})$ of algebraic curvature tensors in a real vector space $\mathcal{T}$ of dimension $n \geq 1$ carrying a fixed pseudo-Euclidean inner product $g \in \mathcal{S}(\mathcal{T})$. The Ricci contraction operator $\mathrm{Ric} : \mathcal{R}(\mathcal{T}) \to \mathcal{S}(\mathcal{T})$, with $[\mathrm{Ric}(R)](u,v) = \mathrm{tr}\, R(u,\,\cdot\,,v,\,\cdot\,)$, Another proof of symmetry of r for Levi-Civita connections. The Kulkarni-Nomizu product $b\wedge d \in \mathcal{R}(\mathcal{T})$ of symmetric $(0,2)$ tensors $b,d \in \mathcal{S}(\mathcal{T})$, with $2(b\wedge d)(u,u',v,v') = b(u,v)d(u',v') + b(u',v')d(u,v) - b(u',v)d(u,v') - b(u,v')d(u',v)$. the easily verified formula $2\,\mathrm{Ric}(g\wedge d) = (n-2)d + (\mathrm{tr}\, d)g$, with the special case $\mathrm{Ric}(g\wedge g) = 2(n-1)g$. One-dimensionality of $\mathcal{R}(\mathcal{T})$ when $n = 2$ and the resulting equality $R = Kg\wedge g$, with the Gaussian curvature k of R, both immediate

since $\mathrm{Ric}(g\wedge g)=2(n-1)g\neq 0$, and so $g\wedge g\neq 0$ unless $n=1$, while $R\mapsto R_{1212}$, in any basis, is clearly injective. The conclusion that $\mathrm{Ric}(R)=Kg$ if $n=2$, and its version for pseudo-Riemannian surfaces. Three subpaces of $\mathcal{R}(\mathcal{T})$, namely, $\mathcal{W}(\mathcal{T})=\mathrm{Ker\,Ric}$, $\mathcal{B}(\mathcal{T})=\{g\wedge b:b\in\mathcal{S}(\mathcal{T})\}$ and $\mathcal{E}(\mathcal{T})=\{g\wedge b:b\in\mathcal{S}(\mathcal{T})$ and $\mathrm{tr}\,b=0\}$. The resulting direct-sum decompositions $\mathcal{R}(\mathcal{T})=[\mathrm{I\!R}g\wedge g]\oplus\mathcal{E}(\mathcal{T})\oplus\mathcal{W}(\mathcal{T})$ and and $\mathcal{B}(\mathcal{T})=[\mathrm{I\!R}g\wedge g]\oplus\mathcal{E}(\mathcal{T})$. The fact that $\mathcal{W}(\mathcal{T})=\{0\}$ if $n\leq 3$. For $n=3$ we get $W=0$ as follows. If $R\in\mathcal{R}(\mathcal{T})$ and $r=\mathrm{Ric}(R)$, in any orthonormal basis e_i of $\mathcal{T}$ with $\varepsilon_i=g(e_i,e_i)\in\{1,-1\}$, if i,j,k are distinct, $\varepsilon_iR_{ijik}=r_{jk}$, so, setting $a_{ij}=a_{ji}=\varepsilon_i\varepsilon_jR_{ijij}$, we get $a_{ij}+a_{ik}=\varepsilon_ir_{ii}$. Thus, $R=0$ whenever $r=0$ since the three numbers a_{ij} then are mutually opposite (and hence all zero). The scalar curvature functional $\mathcal{R}(\mathcal{T})\ni R\mapsto s=\mathrm{tr}\,[\mathrm{Ric}(R)]\in\mathrm{I\!R}$. The explicit decomposition $R=S+E+W$ (scalar plus Einstein plus Weyl) of any $R\in\mathcal{R}(\mathcal{T})$, corresponding to the above equality $\mathcal{R}(\mathcal{T})=[\mathrm{I\!R}g\wedge g]\oplus\mathcal{E}(\mathcal{T})\oplus\mathcal{W}(\mathcal{T})$, which uses the Einstein, Schouten and Weyl tensors e,h and W of any $R\in\mathcal{R}(\mathcal{T})$, given by $e=r-sg/n$, $h=r-sg/(2n-2)$, if $n\geq 2$ and $W=R-2(g\wedge h)/(n-2)$, if $n\geq 3$. Specifically, $n(n-1)S=sg\wedge g$ and $(n-2)E=2g\wedge e$. (For $n=1$ and $n=2$ we already know that $R=S=E=W=0$ and, respectively, $E=W=0$.) Proof that S,E,W have the form just described: W differs from R by an element of $\mathcal{B}(\mathcal{T})=\{g\wedge b:b\in\mathcal{S}(\mathcal{T})\}$ while, obviously, $\mathrm{tr}\,h=(n-2)s/(2n-2)$, and so thd formula $2\,\mathrm{Ric}(g\wedge d)=(n-2)d+(\mathrm{tr}\,d)g$ (see above) yields $\mathrm{Ric}(W)=0$, as required. Easily derived *explicit expressions of* R *as a linear combination of* $sg\wedge g$, $g\wedge e$ *and* W, *or* $sg\wedge g$, $g\wedge e$ *and* W **(Problem 2)**. The scalar curvature function $s=\mathrm{tr}\,r$ of a pseudo-Riemannian metric in any dimension $n\geq 1$, along with its Einstein, Schouten and Weyl tensors e,h and W, expressed as above, with n assumed greater than 1 of 2 when necessary. The resulting classes of metrics: scalar-flat ($S=0$), Einstein ($E=0$), conformally flat ($W=0$), of constant (sectional) curvature ($E=W=0$), conformally flat scalar-flat ($S=W=0$), Ricci-flat ($S=E=0$), flat ($R=0$), where the lowest dimensions $n\leq 3$ require additional provisions. The fact that, for the Lie group G of invertible elements in a finite-dimensional real/complex associative algebra $\mathcal{A}$ with unit 1, under the standard identification $\mathfrak{g}=T_1G=\mathcal{A}$, the Lie bracket in $\mathfrak{g}$ becomes the commutator: $[v,w]=vw-wv$ (since $[v,w]=d_vw-d_wv$ for vector fields v,w on an open set ... and the differntial of a lineat operator A at any point equals A)

Homework: Problems 1 and 2, italicized above.

February 29: The sectional curvature and Ricci curvature functions, $K:\mathrm{Gr}_2^+\mathcal{T}\to\mathrm{I\!R}$ and $\mathrm{Rc}:\mathrm{P}^+\mathcal{T}G^1\to\mathrm{I\!R}$, associated with any $R\in\mathcal{R}(\mathcal{T})$, and the fact that R is uniquely detdermined by the former $\mathrm{Gr}_2^+\mathcal{T}$ and $\mathrm{P}^+\mathcal{T}$ being the subsets of $\mathrm{Gr}_2\mathcal{T}$ and $\mathrm{P}\mathcal{T}$ consisting of nondegenerate planes/lines in $\mathcal{T}$. Constant sectional curvature K meaning precisely that $R=Kg\wedge g$. Integral curves of smooth vector fields on a manifold [**DG**, p. 219]. The local flow e^{tw} of a smooth vector field w on a manifold M [**DG**, pp. 219-221], constituting a smooth mapping $Y_w\ni(z,t)\mapsto e^{tw}z\in M$ from the open subset Y_w of $M\times\mathrm{I\!R}$, given by $Y_w=\{(z,t)\in M\times\mathrm{I\!R}:t\in I_z\}$, into M, with I_z denoting the maximal open interval containing 0 on which one can define an integral curve $t\mapsto x(t)$ of w having $x(0)=z$, and then $x(t)=e^{tw}z$ for all $t\in I_x$. (Openness of Y_w and smoothness of $(z,t)\mapsto e^{tw}z$ are both immediate from the regularity theorem mentioned near the end on February 14.) The phrase "$e^{tw}z$ exists" expressing the relation $(z,t)\in Y_w$, that is, $t\in I_z$. Correctness of our notation: $e^{tw}z$ depends only on z and the product vector field tw, rather than t and w separately (which is obvious when $tw=0$, and otherwise, follows since, given $I_z\ni t\mapsto x(t)$ as above and any $\lambda\in\mathrm{I\!R}\smallsetminus\{0\}$, the maximal integral curve $\lambda^{-1}I_z\ni t'\mapsto x(\lambda t')$ of $w'=\lambda w$ shows that $e^{tw}z$ exists, for $t=\lambda t'$, if and only if so does $e^{t'w'}z$, and then $e^{tw}z = e^{t'w'}z$). The homomorphic property of local flows: if $e^{sw}z$ and $e^{tw}e^{sw}z$ both exist, then $e^{(t+s)w}z$ exists as well, and $e^{tw}e^{sw}z = e^{(t+s)w}z$ [**DG**, p. 220]. The flow transformations e^{tw} given by $x \mapsto e^{tw}x$, each with the (open, possibly empty) domain U_t consisting of all x such that $e^{tw}z$ exists, which constitute – unless U_t is empty – diffeomorphisms $e^{tw} : U_t \to U_{-t}$ having the inverses e^{-tw} [**DG**, p. 220]. The effect of projectability of vector fields under mappings $F : M \to N$ on integral curves: if $(dF)w = u$ on $F(M)$, then F sends integral curves of w to those of u. The special cases of push-forwards of vector fields under diffeomorphisms, and of left-invariant vector fields on a Lie group.

March 1: Completeness of a smooth vector field w on M, defined as the equality $Y_w = M \times \mathrm{I\!R}$, so that it follows if Y_w contains $M \times [-\varepsilon, \varepsilon]$ for some $\varepsilon \in (0,\infty)$ and, consequently, smooth vector fields with compact supports, including smooth vector fields on compact manifolds, as well as left-invariant and right-invariant vector fields on Lie groups, are all complete [**DG**, p. 224]. Proof: given the maximal integral curve $I_z \ni t \mapsto x(t)$ of w with $x(0) = z$, and any $c \in I_z$, using the integral curve $[-\varepsilon, \varepsilon] \ni t \mapsto y(t)$ of w for which $y(0) = x(c)$ we get $y(t-c) = x(t)$ whenever $t \in [-\varepsilon, \varepsilon] + c$ from uniqueness of solutions applied to $t = c$, so, as $c \in I_z$ was arbitrary, $[-\varepsilon, \varepsilon] + I_z \subseteq I_z$, clearly implying that $I_z = \mathrm{I\!R}$. One-parameter subgroups of a Lie group G, meaning: Lie-group homomorphisms $\mathrm{I\!R} \to G$, and the observation that they are precisely the maximal integral curves $t \mapsto x(t)$ of both left-invariant and right-invariant vector fields on G having $x(0) = 1$. Namely, if $s, t \in \mathrm{I\!R}$, then $\mathrm{I\!R} \ni t \mapsto [x(s)]^{-1}x(s+t)$ is an integral curve of the same vector field, and hence it equals $t \mapsto x(t)$ due to uniqueness of solutions, proving that $x(s+t) = x(s)x(t)$. The exponential mapping $\exp : \mathfrak{g} \to G$ given by $\exp v = e^v 1$. The flow of $w \in \mathfrak{g}$, with $e^{tw} = R_{\exp tw}$. The exponential-series formula for $\exp$ when G is the Lie group of invertible elements in a finite-dimensional associative algebra $\mathcal{A}$ with unit. Push-forwards of tensor fields and connections under diffeomorphisms. Left-invariant connections ∇ on a Lie group G identified with arbitrary bilinear mappings $\mathfrak{g} \times \mathfrak{g} \ni (v, w) \mapsto \nabla_v w \in \mathfrak{g}$. The left-invariant connections ∇^c, for any $c \in \mathrm{I\!R}$, sending $(v, w) \in \mathfrak{g}$ to $c[v, w]$, and their obvious bi-invariance. The special cases: $\mathrm{L} = \nabla^0$, and the standard bi-invariant torsion-free connection $\mathrm{D} = \nabla^{1/2}$. The bi-invariant connection R making all right-invariant vector fields parallel (the version of L for G with the reversed multiplication). Affine combinations of connections. The Lie derivative $\pounds_w \Theta$ of a tensor field (or a connection) Θ in the direction of a vector field w, both assumed smooth, defined to be the derivative with respect to the real variable t, at $t = 0$, of $(de^{-tw})\Theta$. The interpretation of $\pounds_w \Theta$ at a point, in the case of tensor fields, as the limit of a difference quotient, and its analog for covariant derivatives (with parallel transports replacing flow transformations). The formulae $\pounds_w f = d_w f$ for functions f, immediate from the definition, and $\pounds_w u = [w, u]$ for vector fields u, to be proved later.

March 4: Proof of the formula $\pounds_w u = [w, u]$ for smooth vector fields w, u, obtained as a trivial consequence of the identity $d[(de^{tw})u]/dt = (de^{tw})[u, w] = [(de^{tw})u, w]$ on U_{-t} [**DG**, p. 222], the second equality in which is obvious as the local flow of w preserves w (by shifting the parameter of its integral curves), while push-forwards under diffeomorphisms are Lie-bracket homomorphisms. The mutually equivalent conditions imposed on vector fields w, u: vanishing of $[w, u]$, or of $\pounds_w u$, or of $\pounds_u w$, the local flow of w preserving u, vice versa, the two flows commuting (locally or "wherever defined"). Corollary: a connected Lie group G is Abelian if and only if so is its Lie algebra $\mathfrak{g}$ (meaning that $[\,,\,] = 0$ on $\mathfrak{g}$). Proof based on the relation $e^{tw} = R_{\exp tw}$ for $w \in \mathfrak{g}$ (March 1). The

fourth "universal shortcut" allowing us, when we prove an equality involving a smooth section ψ of a vector bundle, to assume that either $\psi = 0$ identically or $\psi \neq 0$ everywhere in the domain under consideration, based on *denseness, in the base manifold, of the union of the set defined by $\psi \neq 0$ and the interior of the zero set of ψ* **(Problem 1)**. The formula $[\pounds_w\nabla]^l_{jk} = w^l{}_{,jk} - w^q R_{qkj}{}^l$ for functions f and torsion-free connections ∇, the former obvious, the latter, also written as $[\pounds_w\nabla]_u v = [\nabla_v(\nabla w)]u - R(w,v)u$ with arbitrary vector fields u, v, justified – via the fourth "universal shortcut" – as follows. If $w = 0$ identically, both sides vanish. *At points where $w \neq 0$, locally, $w = \partial_1$ for some local coordinate system x^j* **(Problem 2)**, and we have $[\pounds_w\nabla]^l_{jk} = \partial_1 \Gamma^l_{jk}$ (from the definition of Lie derivative) as well as $w^l{}_{,j} = \Gamma^l_{j1}$ and $w^l{}_{,jk} = \partial_k \Gamma^l_{j1} + \Gamma^l_{kq}\Gamma^q_{j1} - \Gamma^q_{kj}\Gamma^l_{q1}$ (see Febrary 19), so that $[\pounds_w\nabla]^l_{jk} - w^l{}_{,jk} = R_{1kj}{}^l$ due to the component description of R (February 7) combined with symmetry of Γ^l_{jk} in j,k (February 12). Tensor, symmetric and exterior multiplications between tensor-product and symmetric/exterior power spaces [**DG**, p. 148]. The contraction in the jth contravariant and kth covariant argument/index, which is a linear operator assigning to a (p,q) tensor Θ in V the $(p-1,q-1)$ tensor $\hat\Theta$ defined, when $1 \le j \le p$ and $1 \le k \le q$, by setting $\hat\Theta(\xi^1,\dots,\xi^{j-1},\xi^{j+1},\dots,\xi^p,v_1,\dots,v_{k-1},v_{k+1},\dots v_q) = \mathrm{tr}\,[\Theta(\xi^1,\dots,\xi^{j-1},\,\cdot\,,\xi^{j+1},\dots,\xi^p,v_1,\dots,v_{k-1},\,\cdot\,,v_{k+1},\dots v_q)$ for $\mathrm{tr} : L(V^*,V;\mathbb{K}) \to \mathbb{K}$ that sends (ξ,v) to ξv, $\mathbb{K}$ being the scalar field. The component expression $\hat\Theta^{a_1\dots a_{j-1}a_{j+1}\dots a_p}_{b_1\dots b_{k-1}b_{k+1}\dots b_q} = \delta^{b_k}_{a_j}\Theta^{a_1\dots a_p}_{b_1\dots b_q}$, and *the equivalent characterization of $\Theta \mapsto \hat\Theta$ via $v_1\otimes\ldots\otimes v_p\otimes\xi^1\otimes\ldots\otimes\xi^q \mapsto (\xi^j v_k)v_1\otimes\ldots\otimes v_{k-1}\otimes v_{k+1}\otimes\ldots\otimes v_q\otimes\xi^1\otimes\ldots\otimes\xi^{j-1}\otimes\xi^{j+1}\otimes\ldots\otimes\xi^p$* **(Problem 2)**. General properties of the operator $\pounds_w$ acting on smooth tensor fields: due to the analogous facts concerning push-forwards, $\pounds_w$ obeys the Leibniz rule relative to tensor multiplication, and commutes with contractions. The fact that $d[(de^{tw})\Theta]/dt = (de^{-tw})\pounds_w\Theta$ whenever Θ is a smooth tensor field (or a connection), immediate from the definition of $\pounds_w\Theta$ since $e^{tw}e^{sw}z = e^{(t+s)w}z$ [**DG**, p. 220]. The obvious corollary that $\pounds_w\Theta = 0$ if and only if the local flow of w preserves Θ. A system of derivations for vector spaces V, V', V'' and a bilinear "multiplication" $V \times V' \ni (x,y) \mapsto xy \in V''$, defined to be a triple of linear endomorphisms of V, V' and V'', all denoted by a common symbol such as δ, and satisfying the condition $\delta(xy) = (\delta x)y + x(\delta y)$ whenever $x \in V$ and $y \in V'$, an example of which is $\delta = \pounds_w$ and the spaces V, V', V'' of smooth tensor fields of types $(p,q), (p',q')$ and $(p+p', q+q')$ with the tensor multiplication (see the above Leibniz rule). The trivial observation that the commutator of two such derivation systems (carried out separately in each of V, V', V'') is again a derivation system. The property $\pounds_{[w,u]} = [\pounds_w, \pounds_u]$ of Lie derivatives acting on smooth tensor fields of any type (p,q), immediate since it holds when applied to functions and vector fields (January 24).
Homework: Problems 1 and 2, italicized above.
March 6: Variations $(t,s) \mapsto x(t,s)$ of smooth curves in a manifold M and smooth sections $(t,s) \mapsto \psi(t,s) \in E_{x(t,s)}$, along such a variation, of a vector bundle E over M [**DG**, p. 90]. The partial derivatives x_t, x_s and partial covariant derivatives ψ_t, ψ_s, the former being sections of TM along the variation, the latter, depending on a fixed connection in E, constituting, again, sections of E along the variation [**DG**, p. 91]. The second and higher order partial covariant derivatives, written without parentheses: $\psi_{tt}, \psi_{ts}, \psi_{st}, \psi_{ss}, \psi_{ttt}$, etc., as well as $x_{tt}, x_{ts}, x_{st}, x_{ss}$, all requiring the presence of a fixed connection in the tangent bundle, always assumed torsion-free [**DG**, p. 91]. The equality

$x_{ts} = x_{st}$, easily verified using the third "universal shortcut" of February 16, and *the Ricci identity* $\psi_{ts} - \psi_{st} = R(x_t, x_s)\psi$ **(Problem 1)**. The holonomy group $\mathrm{Hol}_x \subseteq \mathrm{GL}(E_x)$, at a point $x \in M$, of a linear connection ∇ in a real/complex vector bundle E over a manifold M, formed by all parallel transports along piecewise smooth loops at x (curves in M joining x to itself). The word 'piecewise' may actually be dropped since *a suitable monotone reparametrization has the derivatives of all orders vanishing at the finitely many exceptional parameter values* **(Problem 2)**, and – as we saw – reparametrizations leave parallel transports unaffected. The observation that the parallel transport along any piecewise smooth curve in M joining x to y is an isomorprism $E_x \to E_y$, conjugating $\mathrm{GL}(E_x)$ onto $\mathrm{GL}(E_y)$ and Hol_x onto Hol_y. Thus, when M is connected, Hol_x has a unique isomorphism type (and an isomorphically unique action on E_x). The holonomy theorem: if $\mathrm{Hol}_z \subseteq G$ for a Lie subgroup G of $\mathrm{GL}(E_z)$with the Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(E_z)$, then all the curvature operators $R_z(v, w)$, for $v, w \in T_zM$, lie in $\mathfrak{g}$. First step in the proof: the variation $[-\pi, \pi] \times [0, 1] \ni (t, s) \mapsto x(t, s) = (1+e^{it})s$ of circles in $\mathbb{C}$, with $x_t = ise^{it}$ and $x_s = 1+e^{it}$. Second step: the observation that, if a smooth curve $t \mapsto x(t)$ in a manifold M lies in a submanifold P, and the derivatives of x^j of orders $1, \ldots, r-1$ all vanish at some given t, for some local coordinates, then the vector u with the components $d^r x^j/dt^r$, evaluated at t, is tangent to P at $x(t)$. In addition, *this vector* u *does not depend on local coordinates used, as* $d_u f$ *equals* $d^r[f(x(t))]/dt^r$ *at the given* t, for all functions f **(Problem 3)**. Third step: with a fixed torsion-free connection in TM, and some neighborhood of the disk $|z-1| = 1$ in $\mathbb{C}$ treated as embedded in M so that $0 \in \mathbb{C}$ becomes our z and the vectors $1, i$ tangent to $\mathbb{C}$ at 0 are identified with any two prescribed linearly independent vectors v, w in T_zM, we consider a section ψ of E along our variation, with $\psi(-\pi, s) = \phi \in E_z$ constant (independent of s) and $\psi_t = 0$, so that $\psi(\pi, s) = A_s\phi$, for A_s denoting the parallel transport along $[-\pi, \pi] \ni t \mapsto x(t, s)$. Fourth step: the Ricci identity gives $\psi_{st} = R(x_s, x_t)\psi$ and $\psi_{sst} = R(x_s, x_t)\psi_s + [R(x_s, x_t)\psi]_s$ which, whenever $s = 0$ (for the remainder of this long sentence) gives $\psi_{st} = 0$ (as $x_t = 0$), and so $\psi_s(\pi, 0) = 0$, that is, $A_0 = \mathrm{Id}$ and $[dA_s/ds]_{s=0} = 0$, while $\psi_{sst} = R_z(x_s, x_{ts})\psi$ (remember, $s = 0$ and $x_t = 0$). Fifth step: at $s = 0$ (still), with $x(t, 0) = z$, the covariant derivative equals the ordinary derivative, and so, since $x_s = 1 + e^{it}$ and $x_{ts} = x_{st} = ie^{it}$, our ψ_{ss} starts from the value 0 at $t = -\pi$ and has the t-derivative $R_z(x_s, x_{ts})\psi = R_z(1 + e^{it}, ie^{it})\phi = R_z(1, ie^{it})\phi + R_z(e^{it}, ie^{it})\phi$, where $\psi = \phi$ (a constant) along the curve $t \mapsto x(t, 0)$. Now d^2A_s/ds^2 at $s = 0$ equals the integral of the sum $R_z(1, ie^{it}) + R_z(e^{it}, ie^{it})$ over $t \in [-\pi, \pi]$. The first integrates to 0, having the periodic antiderivative $R_z(1, i\sin t)$. (We identify $\mathbb{C}$ with a plane in T_zM.) The second term is constant, and equal – along with the integral – to $R_z(v, w)$, due to skew-symmetry of R in the first two arguments. The holonomy theorem follows from the second step, with $r = 2$. The Gauss lemma: in a pseudo-Riemannian manifold, with the metric also denoted by $\langle\,,\rangle$, one has $\langle x_t, x_s\rangle_t = 0$ for any variation $(t, s) \mapsto x(t, s)$ such that $x_{tt} = 0$ and $\langle x_t, x_t\rangle$ is constant (a variation of geodesics, all with the same "speed"), and its proof via trivial Leibniz-rule calculation using the fact that $x_{ts} = x_{st}$ (see above). The simplest geometric application, to $x(t, s) = \exp_z tv(s)$, where $s \mapsto v(s)$ is a smooth curve in the domain $U_z \subseteq T_zM$ of the exponential mapping $\exp_z$ at the point z of a pseudo-Riemannian manifold (February 14) having a constant value of $g_z(v(s), v(s))$, with the conclusion that $\langle x_t, x_s\rangle = 0$. An interpretation of this last conclusion: if we fix $z \in M$ along with $v \in U_z$, and let $(\ ,\ )$ be the symmetric bilinear form on T_zM given by $(u, w) = g_y(Hu, Hw)$, where $H : T_zM \to T_yM$ is the differential of $\exp_z$ at v, for

$y = \exp_z v$, so that $(\ ,\ )$ equals the H-pullback of g_y to T_zM, the tangent space of U_z at v, then $(\ ,\ )$ and $\langle\ ,\rangle = g_z$ agree on the span $\mathrm{I\!R}v$ of v, while v is $(\ ,\ )$-orthogonal to the $\langle\ ,\rangle$-orthogonal complement of v [**DG**, Lemma 32.2 on p. 111]. A physical interpretation of the curvature as tidal effect, preventing (when nonzero) a truly inertial motion of rigid macroscopic objects [**DG**, p. 92].
Homework: Problems 1, 2 and 3, italicized above.
March 20: Consequences, for tensor fields ξ of type $(0,1)$ and Θ of type $(1,1)$, of the formula $\pounds_w u = [w,u]$ for vector fields u, namely, $(\pounds_w\xi)u = d_w(\xi u) - \xi[w,u]$ and $(\pounds_w\Theta)u = [w,\Theta u] - \Theta[w,u]$, as well as the versions of these relations involving a fixed torsion-free connection ∇, which read $\pounds_w\xi = \nabla_{\!w}\xi + \xi\nabla w$ and $\pounds_w\Theta = \nabla_{\!w}\Theta + [\Theta,\nabla w]$. The definition of an almost-complex structure on a manifold: a $(1,1)$ smooth tensor field J with $J^2 = -\mathrm{Id}$, *which amounts to realizing the tangent bundle as the underlying real bundle of a complex vector bundle* **(Problem 1)**. Kähler connections on manifolds with a fixed almost-complex structure J, meaning: torsion-free connections ∇ such that $\nabla J = 0$. The observation that *for any connection ∇ in TM and any almost-complex structure J on M, one has $\nabla J = 0$ if and only if ∇ is a connection in TM treated as a complex vector bundle* **(Problem 2)**. Holomorphic mappings between almost-complex manifolds (that is, manifolds carrying fixed almost-complex structures), defined by requiring smoothness and complex-linearity of the differential at each point. Holomorphic vector fields w on an almost-complex manifold, that is, w with the local flow which consists of holomorphic mappings, which is *equivalent to the condition $\pounds_w J = 0$* **(Problem 3)**. Corollary: due to the general formula $\pounds_w\Theta = \nabla_{\!w}\Theta + [\Theta,\nabla w]$ (see above), a smooth vector field w on a manifold with a Kähler connection is holomorphic if and only if J and ∇w commute when viewed as endomorphisms of the tangent bundle. The relation $(\pounds_w b)(u,v) = d_w[b(u,v)] - b([w,u],v) - b(u,[w,v])$ involving a tensor field b of type $(0,2)$ and any vector fields w,u,v, all smooth, along with its the version $(\pounds_w b)(u,v) = [\nabla_{\!w} b](u,v) + b(\nabla_{\!u} w, v) + b(u, \nabla_{\!v} w)$ in the presence of a torsion-free connection ∇. The special case $\pounds_w g = g(A\cdot\,,\,\cdot)$ arising when ∇ is the Levi-Civita connection of a pseudo-Riemannian metric g and we set $A = B + B^*$ for $B = \nabla w$. Killing vector fields w (infinitesimal isometries) on a pseudo-Riemannian manifold (M,g), defined by requiring that $\pounds_w g = 0$ (or, in other words, the local flow of w consist of isometries), which is also equivalent to skew-adjointness of ∇w at every point. Affine diffeomorphisms between manifolds with fixed connections in their tangent bundles, that is, diffeomorphisms sending one connection, via push-forward, onto the other. Affine vector fields w on a manifold with a fixed connection ∇ (also called infinitesimal affine transformations), defined by requiring the flow mappings e^{tw} to be affine, which is nothing else than the condition $\pounds_w\nabla = 0$ and hence, for torsion-free connections, amounts to $w^k{}_{,jl} = w^s R_{slj}{}^k$ (see March 4). The last condition rewritten, with a lowered index, $w_{,jkl} = R_{jkl}{}^q w_q$, thus holds for every Killing field.
Homework: Problems 1, 2 and 3, italicized above.
March 22: The length of the image curve $[a,b] \ni t \mapsto x(t) = \exp_z v(t)$, with the step $|H\dot v| \ge |\langle \dot v, v\rangle|/|v|$ in its proof [**DG**, p. 111] immediate since $|Hu| \ge |\langle u,v\rangle|/|v|$, where H is the differential of $\exp_z$ at $v = v(t)$. The further observation that, under the additional assumption of injectivity of the differential of $\exp_z$ at $v(t)$ for every $t \in [a,b]$, the above inequality $\boldsymbol{L} \ge |r(b) - r(a)|$ is strict unless all $v(t)$ lie in a single line segment emanating from 0 in T_zM and the function $t \mapsto r(t)$ is weakly monotone [**DG**, p. 112]. The injectivity radius $r_{\mathrm{inj}}(z) \in (0,\infty]$ of a Riemannian manifold at a point z [**DG**, p. 112]. Lemma 32.3 in [**DG**, p. 114].

March 25: Proposition 32.4, Corollary 32.5 and Theorem 32.6 in [**DG**, pp. 112-113]. Lemma 33.1 in [**DG**, p. 114].

March 27: Proposition 32.4, Corollary 32.5, Lemma 33.1 and Theorem 32.6 in [**DG**, pp. 112-114]. Inequalities between symmetric bilinear forms. A fourth equivalent condition, not listed explicitly in Theorem 32.6 (but appearing in its proof): any finite-length piecewise C^1 curve $[a,b] \ni t \mapsto x(t) \in M$, with $a < b < \infty$, has a linit as $t \to b$. Completeness of g implying completeness of h when $g \le Ch$ with a constant C. Completenss of a metric on an affine space bounded from below by a positive constant. Finite partitions of unity for a compact set Y in a manifold, and the special case of those subordinate to a given open covering of Y [**DG**, p. 121].

March 29: Reminder from February 21: the divergence operator associated with a smooth positive density on a manifold [**DG**, p. 125], and the relation $\operatorname{div} w = \operatorname{tr} \nabla w$ in the case of the volume element of a metric. The integral of a compactly-supported smooth density and the volume of a compact pseudo-Riemannian manifold [**DG**, p. 124]. The divergence theorem [**DG**, p. 127]. Integration by parts.

April 1: The relation $\operatorname{div} w = \operatorname{tr} \nabla w$ of February 21 generalized to the case of a torsion-free connection ∇ admitting a ∇-parallel smooth positive density μ. Proof of the integration-by-parts formula, based on this last relation. The L^2 inner product $(\,,\,)$ and norm $\|\;\|$. The second covariant derivative ∇df, also known as the Hessian, of a smooth local function in a manifold with a connection in TM [**DG**, p. 89] *having the components $f_{,kj} = \partial_j \partial_k f - \Gamma_{kj}^l \partial_j$, which implies symmetry of the Hessian of f for torsion-free connections* **(Problem 1)**. The gradient vector field $w = \nabla f$ of a smooth function f on a pseudo-Riemannian manifold (M, g), characterized by $g(w, \cdot\,) = df$, its components $f_{,}^{\,k} = g^{kj} f_{,j}$, also written as $f^{,k}$ The Laplacian Δ with $\Delta\phi = \delta\nabla\phi$ applied to smooth functions ϕ on a pseudo-Riemannian manifold and *the equality $(\Delta\phi, \psi) = -(\nabla\phi, \nabla\psi)$ between L^2 inner products when one of the functions has a compact support* **(Problem 2)**, establishing both symmetry of Δ and, via the special case $(\phi, \Delta\phi) = -\|\nabla\phi\|^2$, its nonpositivity if the metric is positive definite. ***Bochner's lemma, stating that in the latter case, unders the connectedness assumption, the condition $\Delta f \ge 0$ implies constancy of f*** **(Problem 3)**. The Bochner identity $r(\,\cdot\,, v) = \operatorname{div} \nabla v - d(\operatorname{div} v)$, in coordinates: $R_{jk} v^k = v^k{}_{,jk} - v^k{}_{,kj}$, immediate, via contraction in $l = p$, from the Ricci identity $R_{jpk}{}^l v^k = v^l{}_{,jp} - v^l{}_{,pj}$ (see February 19) and the resulting Bochner integral formula $\int r(v, v)\,\mathrm{d}g = \|\delta v\|^2 \mathrm{d}g - \int \operatorname{tr}(\nabla v)^2 \mathrm{d}g$ for a compactly-supported smooth vector field v on a pseudo-Riemannian manifold (M, g) with the Levi-Civita connection ∇ and the volume element $\mathrm{d}g$. Successively narrower special cases: $(rv, v) = \|\delta v\|^2 - \int \operatorname{tr}(\nabla v)^2 \mathrm{d}g$ when g is positive definite, and $(rv, v) = \|\Delta f\|^2 - \|\nabla df\|^2$ if, in addition, $v = \nabla f$ for a compactly-supported smooth function $f : M \to \mathbb{R}$, with $v \mapsto rv$ referring to r treated as a bundle endomorphism of TM via g-index-raising. The L^2 norm $\|\nabla df\|$ comes here from inner product $\langle b, b' \rangle$ of $(0,2)$ tensors [**DG**, p. 107] related, again via g-index-raising, to the inner product of endomorphisms A, B of a pseudo-Euclidean space given by $\langle A, B \rangle = \operatorname{tr} AB^*$. The Lichnerowicz theorem, stating that on a compact Riemannian manifold of dimension $n \ge 2$ having $r \ge (n-1)Kg$, where $K \in \mathbb{R}$, any nonzero eigenvalue λ of $-\Delta$ must satisfy the condition $\lambda \ge nK$.
Homework: Problems 1, 2 and 3, italicized above.

April 3: The Schwarz inequality $|\Delta f|^2 \le n|\nabla df|^2$ for the $(0,2)$ tensors g and ∇df, the Hessian of a smooth function f on an n-dimensional Riemannian manifold, which is obvious since $\langle g, g\rangle = n$, and combined with the relations $(f, \Delta f) = -\|\nabla f\|^2$ and $(rv, v) = \|\Delta f\|^2 - \|\nabla df\|^2$ if $v = \nabla f$ (see April 1) has the immediate consequence

$$nK\lambda\|f\|^2 = nK(f, -\Delta f) = nK\|\nabla f\|^2$$
$$\le \frac{n}{n-1}(r\nabla f, \nabla f) = \frac{n}{n-1}[\|\Delta f\|^2 - \|\nabla df\|^2] \le \|\Delta f\|^2 = \lambda^2\|f\|^2$$

in the case where $K, \lambda \in \mathrm{I\!R}$, while $r \ge (n-1)Kg$ and $\Delta f = -\lambda f$ for a smooth function $f : M \to \mathrm{I\!R}$ on a compact Riemannian manifold (M, g) of dimension $n \ge 2$. Proof of the Lichnerowicz theorem, obvious from the above chain of inequalities. The projected linear connections $\nabla^\pm$ in $E^\pm$, or ∇^r in E^r, for $r = 1, \dots, q$, arising when ∇ is a linear connection in a vector bundle $E = E^+ \oplus E^-$, or $E = E^1 \oplus \ldots \oplus E^q$, over a manifold M, given by $\nabla_v^\pm \psi = [\nabla_v \psi]^\pm$ or, respectively, $\nabla_v^r \psi = [\nabla_v \psi]^r$, with $[\ ]^\pm$ (or $[\ ]^r$) denoting the projection morphism from E onto the summand $E^\pm$ (or E^r) treated as a subbundle of E. Compatibility of $\nabla^\pm$ with $\theta^\pm$ whenever $\theta^\pm$ are fibre metrics in $E^\pm$ and ∇ is compatible with their orthogonal direct sum θ, since $d_v[\theta^\pm(\psi, \phi)] = d_v[\theta(\psi, \phi)] = \theta(\nabla_v \psi, \phi) + \theta(\psi, \nabla_v \phi) = \theta([\nabla_v \psi]^\pm, \phi) + \theta(\psi, [\nabla_v \phi]^\pm)$ for sections ψ, ϕ of $E^\pm$. The fact that $[F^*\nabla]^\pm = F^*\nabla^\pm$ for pullbacks under a mapping F, immediate when the following characterization (February 9): $[F^*\nabla]_v[F^*\psi] = \nabla_w \psi \in E_y = [F^*E]_x$ with $y = F(x)$ and $w = dF_x v$ is applied to $\nabla^\pm$ as well. The special case $\nabla_{\dot x}^\pm \phi = [\nabla_{\dot x} \phi]^\pm$ for smooth sections along curves. The pullback $\varPhi^*\varTheta$ of a $(0, q)$ tensor $\varTheta$ in a vector space V' under a linear operator $\varPhi : V \to V'$, which is the $(0, q)$ tensor in V with $[\varPhi^*\varTheta](v_1, \dots, v_q) = \varTheta(\varPhi v_1, \dots, \varPhi v_q)$ whenever $v_1, \dots, v_q \in V$. The pullback $F^*\varTheta$ of a $(0, q)$ tensor field $\varTheta$ on a manifold N under a smooth mapping $F : M \to N$, defined to be the $(0, q)$ tensor field $\hat\varTheta$ on M given by $\hat\varTheta_x = \varPhi^*\varTheta_{F(x)}$ for any $x \in M$ and $\varPhi = dF_x$. Smoothness of $\hat\varTheta$ following from that of $\varTheta$ via the obvious component formula $\hat\varTheta_{j_1 \ldots j_q} = (\partial_{j_1} F^{a_1}) \ldots (\partial_{j_q} F^{a_q})(\varTheta_{a_1 \ldots a_q} \circ F)$. Nondegeneracy of a smooth mapping $F : M \to N$ into a pseudo-Riemannian manifold (N, h), meaning nondegeneracy of F^*h at every point, and thus giving rise to the pseudo-Riemannian pullback metric $g = F^*h$ on M, not be confused with the pullback fibre metric F^*h in F^*TN. The term 'submanifold metric' used for $g = F^*h$ when F (still assumed nondegenerate) is the inclusion mapping of a submanifold M of N. The fact that *all nondegenerate mappings are immersions, and the converse implication holds when h is a Riemannian metric* **(Problem 1)**. Smoothness of the orthogonal complement $E^\perp$ of a nondegenerate smooth vector subbundle E of a real vector bundle $\hat E$ endowed with a pseudo-Riemannian fibre metric, due to the existence of (smooth) orthonormal local trivializations. The normal bundle $[TM]^\perp$ of a nondegenerate immersion $F : M \to N$ into a pseudo-Riemannian manifold (N, h), that is, the orthogonal complement of TM treated as a subbundle of F^*TN, and its obvious canonical isomorphic identification $[TM]^\perp = [F^*TN]/[TM]$ with the ordinary normal bundle of the immersion F. The ensuing decomposition $E = E^+ \oplus E^-$ (February 4) of $E = F^*TN$ into the summands $E^+ = TM$ and $E^- = [TM]^\perp$. The notation $[\ ]^{\mathrm{tng}}$ and $[\ ]^{\mathrm{nrm}}$ instead of $[\ ]^+$ and $[\ ]^-$ for the summand projections $F^*TN = E \to E^\pm$. The relation $\nabla = [F^*\mathrm{D}]^{\mathrm{tng}}$ between the Levi-Civita connections ∇ of the pullback metric $g = F^*h$ on M, and D of h. Proof of this relation, in two parts. First, compatibility of $[F^*\mathrm{D}]^{\mathrm{tng}}$ with g follows: both pullbacks, and direct-summand projections of connections, preserve compatibility. Second, $[F^*\mathrm{D}]^{\mathrm{tng}}$ is torsion-free since so is D. Namely, the rank theorem [**DG**, p. 33] easily shows that *smooth local vector fields in M projectable under an* *immersion* $F : M \to N$ *realize as values all vectors tangent to* M *at all points* **(Problem 2)**. Using the fact that whenever two vector fields are tangent to a submanifold, so is their Lie bracket [**DG**, Theorem 6.1 on p. 24], along with the definitions of the torsion tensor field (February 12) and the Levi-Civita connection (February 14), we now obtain our assertion. The conclusion that $\nabla_{\dot{x}} v = [\mathrm{D}_{\dot{y}} w]^{\mathrm{tng}}$ for a smooth vector field $t \mapsto v(t)$ along a smooth curve $t \mapsto x(t) \in M$ and the vector field $w(t) = dF_{x(t)}v(t)$, the latter being tangent to N along the image curve $y(t) = F(x(t))$. Namely, $\nabla_{\dot{x}} v$ represents the pullback, under the curve mapping $t \mapsto x(t)$, of the left-hand side of the equality $\nabla = [F^*\mathrm{D}]^{\mathrm{tng}}$, while the corresponding pullback of the right-hand side must account for $[\mathrm{D}_{\dot{y}} w]^{\mathrm{tng}}$ since, applied to connections, pullbacks commute with direct-summand projections (see the relation $[F^*\nabla]^{\pm} = F^*\nabla^{\pm}$ of February 27), and *the composition of pullbacks is the same as the pullback under the composite mapping, for both bundles and connections* **(Problem 3)**. Two special cases: $\nabla_{\dot{x}} \dot{x} = [\mathrm{D}_{\dot{y}} \dot{y}]^{\mathrm{tng}}$ for the velocity vector field $t \mapsto v(t) = \dot{x}(t)$ of the curve, and $\nabla_{\dot{x}} v = [\mathrm{D}_{\dot{x}} v]^{\mathrm{tng}}$ when M is a nondegenerate submanifold of (N, h). The formula $\nabla_{\dot{x}} \dot{x} = [\mathrm{D}_{\dot{x}} \dot{x}]^{\mathrm{tng}}$, arising when both special cases occur at the same time. Thus, the geodesics of a nondegenerate submanifold M of (N, h) are precisely those smooth curves $t \mapsto x(t) \in M$ for which the acceleration vector field $\mathrm{D}_{\dot{x}} \dot{x}$, in (N, h), is normal to M. The formula $d\varphi_x = 2\langle x, \cdot\rangle$ for the function $\varphi : \mathcal{V} \to \mathrm{I\!R}$ given by $\varphi(x) = \langle x, x\rangle$ on a pseudo-Euclidean vector space $\mathcal{V}$ with the inner product $\langle\,,\rangle$, obtained when one lets x depend smoothly on a parameter t and then applies d/dt to $\varphi(x)$, using Problem 3 of February 13. The conclusion that all values of $\varphi : \mathcal{V} \smallsetminus \{0\} \to \mathrm{I\!R}$ are regular, and so, due to [**DG**, Theorem 9.6 on p. 35], every (nonempty) pseudosphere $\varSigma = \{x \in \mathcal{V} : \langle x, x\rangle = c\}$, where $c \in \mathrm{I\!R} \smallsetminus \{0\}$, is a submanifold of $\mathcal{V}$ with the subset topology, having the tangent spaces $T_z\varSigma = z^\perp$ at all $z \in \varSigma$. The equality $\mathcal{V}^{\perp\perp} = \mathcal{V}$ valid whenever $\mathcal{V}$ is a vector subspace of a pseudo-Euclidean vector space $\mathcal{T}$, and immediate for dimensional reasons (February 4), due to the obvious inclusion $\mathcal{V} \subseteq \mathcal{V}^{\perp\perp} = \mathcal{V}$. Nondegeneracy of the orthogonal complement $\mathcal{V}^\perp$ under the assumption of nondegeneracy of $\mathcal{V}$, the sign pattern (metric signature) being clearly complementary to that of $\mathcal{V}$, which follows as $\mathcal{V} \oplus \mathcal{V}^\perp = \mathcal{T}$ (see February 4): a vector in $\mathcal{V}^\perp$ orthogonal to $\mathcal{V}^\perp$ must be orthogonal to $\mathcal{T}$, and hence zero. The resulting nondegeneracy of every (nonempty) pseudosphere $\varSigma = \{x \in \mathcal{V} : \langle x, x\rangle = c\}$, where $c \in \mathrm{I\!R} \smallsetminus \{0\}$, as a submanifold of the ambient pseudo-Euclidean vector space $\mathcal{V}$ with the inner product $\langle\,,\rangle$. The description of ∇-geodesics $\mathrm{I\!R} \ni t \mapsto x(t) \in \varSigma$ for the Levi-Civita connection ∇ of the submanifold metric g of $\varSigma$, realizing any initial data $x(0) = z \in \varSigma$ and $\dot{x}(0) = v \in T_z\varSigma = z^\perp$ such that $\langle v, v\rangle = \varepsilon c$ with $\varepsilon \in \{1, -1, 0\}$, which has $x(t)$ equal to $z + tv$ (if $\varepsilon = 0$), or $z \cos t + v \sin t$ (if $\varepsilon = 1$), or $z \cosh t + v \sinh t$ (if $\varepsilon = -1$): namely, $\ddot{x} = -\varepsilon x$ is normal to $\varSigma$ [**MC**, p. 2].

Homework: Problems 1, 2 and 3, italicized above.

April 5: The observation that *nonconstant geodesics of* $(\varSigma, g)$ *are suitably parametrized circles, hyperbolas or lines in planes through* 0 *within* $\mathcal{V}$, *depending on whether* $\langle\,,\rangle$ *restricted to the plane is definite, or nondegenerate and indefinite or, respectively, degenerate and semi-definite* **(Problem 1)**. The fact that, *given a linear connection* ∇ *in a real/complex vector bundle* E *over a manifold* M, *and a nonempty connected open set* U, *with* $\mathcal{V}$ *denoting the space of* ∇*-parallel local sections of* E *defined on* U, *at every* $x \in U$ *the evaluation operator* $\mathcal{V} \ni \psi \mapsto \psi_x \in E_x$ *is injective* **(Problem 2)**, and it becomes *a linear isomorphism for flat connections* ∇ *and all sufficiently small connected neighborhoods* U *of any given point* $z \in M$ which, conversely, implies flatness of ∇ **(Problem 3)**. The "Hessian-metric

equation" $\nabla df = -Kfg$ with the unknown function f on a pseudo-Riemannian manifold (M, g), for a constant K, and the idea of "encoding" its solutions f as $\overline{\nabla}$-parallel sections $(f, w) = (f, \nabla f)$ of the vector bundle $E = [M \times \mathrm{I\!R}] \oplus TM$ over M which is the direct sum of the product line bundle $M \times \mathrm{I\!R}$ and T^*M, the connection $\overline{\nabla}$ in E being given by $\overline{\nabla}_v (f, w) = (d_v f - g(v, w), \nabla_v w + Kfv)$. The curvature tensor $\bar{R}$ of $\overline{\nabla}$, having the form $\bar{R}(u, v)(f, w) = (0, R(u, v)w - K[g(u, w)v - g(v, w)u])$, as one easily sees using the general curvature formula (February 7) combined with the first and second "universal shortcuts" of February 7 and February 16 (so that $\nabla u, \nabla v, \nabla w, [u, v]$ and df all vanish at the point in question). The obvious consequence: the metric g has constant sectional curvature K if and only if $\overline{\nabla}$ is flat, that is, if and only if $\overline{\nabla}$ satisfies the condition described in Problem 3 [**MC**, p. 1]. Another fact, easily verified in local coordinates: for any connection ∇ on a manifold M, any smooth curve $t \mapsto x(t) \in M$, and any smooth function $f : M \to \mathrm{I\!R}$, with $(\ )^{\dot{}} = d/dt$, at every t and $x = x(t)$ one has $[f(x)]^{\ddot{}} = [\nabla df](\dot{x}, \dot{x}) + d_w f$, where $w = \nabla_{\dot{x}} \dot{x}$ [**MC**, p. 1]. The simplified form $[f(x)]^{\ddot{}} = [\nabla df](\dot{x}, \dot{x})$ of the last formula in the case of ∇-geodesics $t \mapsto x(t)$. Corollary: symmetry of the Hessian (Problem 1 of April 1) and the April 3 – April 5 description of pseudosphere geodesics imply that, for any linear homogeneous function on a pseudo-Euclidean vector space, its restriction f to a pseudo-sphere of "radius squared" $c \neq 0$ satisfies the Hessian-metric equation $\nabla df = -Kfg$ with $K = 1/c$. An immediate consequence: due to the unlimited solvability of this last equation, $\overline{\nabla}$ is flat (see Problem 2), and so – as shown above – the pseudo-sphere in question, with its submanifold metric, is a pseudo-Riemannian manifold of constant curvature $K = 1/c$. The conclusion, immediate from the chain of inequalities of April 3, that nK is the lowest positive eigenvalue of $-\Delta$ for an n-dimensional Euclidean sphere with its submanifold metric of constant sectional curvature $K > 0$, and the corresponding eigenspace consists precisely of all linear functionals restricted to the sphere. A brief mention of the Poincaré inequality $\|\nabla f\|^2 \geq \lambda_1 \|f\|^2$ for smooth functions f on a compact Riemannian manifold without (or, with) boundary that have integral zero (or, respectively, vanish on the boundary), λ_1 being the lowest positive eigenvalue of $-\Delta$ in this function space, and the inequality is strict except when $\Delta f = \lambda_1 f$.
Homework: Problems 1, 2 and 3, italicized above.

April 8: Proof of these claims in the case of the closed interval $[0, \pi]$ carrying the obvious metric $dt \otimes dt$, so that $\Delta = (\ )^{\ddot{}}$ with $(\ )^{\dot{}} = d/dt$ for the standard variable t, where *one has $\lambda_1 = 1$, and the λ_1-eigenspace is spanned by the function $t \mapsto \sin t$* **(Problem 1)**. The proof is based on observing that, whenever $\varphi : [0, \pi] \to \mathrm{I\!R}$ and $H : (0, \pi) \to \mathrm{I\!R}$ are functions with continuous derivatives $\dot{\varphi} = d\varphi/dt$ and $\dot{H} = dH/dt$ such that $\varphi(0) = \varphi(\pi) = 0$ and there exist finite limits of $tH(t)$ and $t^2\dot{H}(t)$ as $t \to 0^+$, and of $(t - \pi)H(t)$ and $(t - \pi)^2\dot{H}(t)$ as $t \to \pi^-$, one necessarily has $\int_0^\pi (\dot{H} - H^2)\varphi^2\, dt \leq \int_0^\pi \dot{\varphi}^2\, dt$, the inequality being strict unless $\varphi = qe^{-F}$ on $(0, \pi)$ for some constant q and some antiderivative F of H. Namely, nonnegativity of $\int_0^\pi (\dot{\varphi}^2 - \dot{H}\varphi^2 + H^2\varphi^2)\, dt$ follows since it equals $\int_0^\pi (\dot{\varphi} + H\varphi)^2\, dt$, as one sees noting that $(\dot{\varphi}^2 + H\varphi)^2 = \dot{\varphi}^2 + 2H\varphi\dot{\varphi} + H^2\varphi^2$, while the integrals of $2H\varphi\dot{\varphi}$ and $-\dot{H}\varphi^2$ coincide: $2H\varphi\dot{\varphi} + \dot{H}\varphi^2 = (H\varphi^2)^{\dot{}}$. The Poincaré inequality for $[0, \pi]$ and the claim about the equality case *are now immediate if one sets $H(t) = -\cot t$* **(Problem 2)**. The general identity $d[g(\nabla f, \nabla f)] = 2[\nabla df](\nabla f, \cdot\,)$ for functions f on pseudo-Riemannian manifolds [**MC**, p. 1], immediate either from the component calculation $[f^{,k} f_{,k}]_{,j} = 2f^{,k} f_{,kj}$, of from the symmetry of the Hessian $\nabla df = g(A\cdot\,, \cdot\,)$ for $A = \nabla w$ and $w = \nabla f$ (Problem 1 of April 1) combined with the Leib-

niz rule: $d_v[g(w,w)] = 2g(\nabla_v w, w) = 2g(Av, w) = 2g(Aw, v) = 2[\nabla df](w,v)$. Isometries between pseudo-Riemannian manifolds. The classification theorem: as a special case of a result due to Élie Cartan [*Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **54**, 1926, 214–264], any pseudo-Riemannian manifold (M,g) of constant curvature $K \neq 0$ is locally isometric to a pseudo-sphere $\Sigma = \{x \in \mathcal{V} : \langle x, x\rangle = 1/K\}$ of "radius squared" $c = 1/K$ in a pseudo-Euclidean vector space $\mathcal{V}$ ('locally isometric' being shorthand for: every point of the former manifold has a neighborhood isometric to an open submanifold of the latter). Proof [**MC**, p. 2]: $\mathcal{V}$ consists here of functions f with $\nabla df = -Kfg$ defined on a fixed sufficiently small connected neighborhood U of any given point in M, with the inner product $\langle\,,\rangle$ in $\mathcal{V}$ characterized by $\langle f, f\rangle = g(\nabla f, \nabla f) + Kf^2$, which is easily seen to be constant due to the condition $\nabla df = -Kfg$ and the general identity $d[g(\nabla f, \nabla f)] = 2[\nabla df](\nabla f, \cdot\,)$ (see above). One also *easily verifies nondegeneracy of* $\langle\,,\rangle$ **(Problem 3)**. As a next step, we define $\Phi : U \to \mathcal{V}$ by $\varphi(x) = f$, for the unique $f \in \mathcal{V}$ with $f(x) = 1/K$ and $df_x = 0$. Naturality of the (seemingly strange) definition of Φ, obvious when one uses $\langle\,,\rangle$ to identify $\mathcal{V}$ with $\mathcal{V}^*$, since Φ now becomes the mapping $U \to \mathcal{V}^*$ assigning to each x the evaluation (Dirac delta) functional $\mathcal{V} \ni f \mapsto f(x) \in \mathrm{I\!R}$. The observations that *the values of* Φ *all lie in the pseudosphere* $\Sigma = \{y \in \mathcal{V} : \langle y, y\rangle = 1/K\}$ **(Problem 4)**, while, *for an open interval* I *and a curve* $I \ni t \mapsto f(t, \cdot\,)$ *valued in a finite-dimensional vector space* $\mathcal{V}$ *of smooth functions on a manifold* M, *smoothness of the curve is equivalent to smoothness of the mapping* $I \times M \ni (t,x) \mapsto f(t,x) \in \mathrm{I\!R}$ **(Problem 5)**, as one notes choosing a basis of $\mathcal{V}$. The differential of Φ at any point $z = z(t)$ of a smooth curve $t \mapsto z(t) \in M$, sends $v = \dot z(t)$ to $d\Phi_z v = \dot f$, the unique function $\dot f \in \mathcal{V}$ with $\dot f(z) = 0$ and $d\dot f_z = g(v, \cdot\,)$, as one sees using the partial derivatives $\dot f = \partial f/\partial t$ and $\partial_j f$ (for fixed local coordinates in U), all of which are functions of (t,x), expressing our definition of Φ as $f = 1/K$ and $\partial_j f = 0$ at $(t, z(t))$ (these and subsequent relations being valid at $(t, z(t))$), and then applying $(\;)^{\dot{}} = d/dt$ to the last two equalities, obtaining (from the chain rule) $\dot f + \dot z^k \partial_k f = 0$ and $\partial_j \dot f + \dot z^k \partial_k \partial_j f = 0$, at $(t, z(t))$. Now, since $f_{,jk} = \partial_k \partial_j f - \Gamma_{kj}^l \partial_l f$, the Hessian-metric equation satisfied by f, combined with the relations $f = 1/K$ and $\partial_j f = 0$ at $(t, z(t))$, allows us to rewrite these last two equalities as $\dot f = 0$ and $0 = \partial_j \dot f + \dot z^l f_{,jl} = \partial_j \dot f - Kf\dot z^l g_{jl} = \partial_j \dot f - \dot z^l g_{jl}$. With $z(t) = z$ and $\dot z(t) = v$ at fixed t, our claim about $d\Phi_z v$ follows. Finally, our description of $\dot f = d\Phi_z v$ clearly gives $\langle \dot f, \dot f\rangle = g(v,v)$ or, in other words, g equals the Φ-pullback of the submanifold metric of the pseudosphere $\Sigma = \{y \in \mathcal{V} : \langle y, y\rangle = 1/K\}$, which makes $\Phi : U \to \Sigma$ an immersion, and hence locally diffeomorphic due to equality of the dimensions and the inverse mapping theorem. Thus, Φ is, locally, is an isometry between open sets in U and open subsets of Σ, which completes the proof.
Homework: Problems 1 – 5, italicized above.

April 10: The effect of operations applied to connections on the curvature tensor, expressed via the curvature operators $B = R(v,w)$, where the vector fields v, w are fixed. First, $B = B_1 \oplus \ldots \oplus B_q$ for the direct-sum connection ∇ in $E = E_1 \oplus \ldots \oplus E_q$, where B_j are associated with the given connections in E_j, $j = 1, \ldots, q$, *as one sees using the component description of* R (February 7) *in a direct-sum type local trivialization for* E **(Problem 1)**. Second, $R'(v,w) = R(v,w)$ (restricted to E') in the case of a ∇-parallel subbundle E' of E, a connection ∇ in E and the resulting connection ∇' in E' and the resulting Third, $\overline{R}(v,w)\Phi = [R'(v,w)] \circ \Phi - \Phi \circ [R(v,w)]$, for sections v, w of TM and Φ of $\overline{E}$ (or, elements

v, w of T_xM and Φ of $\hat{E}_x$, where $x \in M$), vector bundles E, E' with linear connections ∇, ∇' over a base manifold M, and $\overline{\nabla}$ in $\overline{E} = \mathrm{Hom}(E, E')$ obtained by applying the Hom functor, $R, R', \hat{R}$ being the curvature tensors of $\nabla, \nabla', \hat{\nabla}$. *This easily follows* **(Problem 2)** from the Leibniz-rule characterization of $\overline{\nabla}$ (February 7). One has two special cases, and an "alternative interpretation" of the formula for $\overline{R}(v, w)\Phi$. In one case, $E = E'$ and $\nabla = \nabla'$, resulting in the commutator relation $\overline{R}(v, w)\Phi = [R(v, w), \Phi]$, in the other ∇' is the standard flat connection D in the product line bundle $M \times \mathrm{I\!K}$, so that $\overline{\nabla}$ equals the dual of ∇ in the dual $E^* = \mathrm{Hom}(E, M \times \mathrm{I\!K})$ of E, and $\overline{R}(v, w)\xi = -\xi[R(v, w)]$, with ξ now denoting a section of E^*. Affine diffeomorphisms between manifolds with fixed connections *which, in the case of vector spaces with standard flat connections, are just the ordinary affine isomorphisms* **(Problem 3)**. The local classification of flat torsion-free connections ∇ in tangent bundles: such ∇, in TM, is locally diffeomorphically equivalent to the standard flat connection D in TN, for a vector space N. Proof: one easily verifies that the curvature tensor $\bar{R}$ of the connection $\overline{\nabla}$ in the vector bundle $E = [M \times \mathrm{I\!R}] \oplus T^*M$ over M, given by $\overline{\nabla}_v(f, \xi) = (d_v f - \xi v, \nabla_v \xi)$, with ∇ also denoting the dual of ∇ in T^*M, has the form $\bar{R}(v, w)(f, \xi) = (0, -\xi[R(v, w)])$. Flatness of ∇ now implies that of $\overline{\nabla}$. For any fixed $z \in M$, let $\mathcal{V}$ be the vector space of $\overline{\nabla}$-parallel local sections (f, ξ) of E having $f(z) = 0$ and defined on a connected neighborhood U of z, small enough so that $\mathcal{V}$ has the same dimension m as M, and we can choose a basis (x^j, dx^j) of $\mathcal{V}$, $j = 1, \ldots, n$. Having linearly independent differentials at z, the functions x^j restricted to a suitable smaller neighborhood U' of z form – due to the inverse mapping theorem – a local coordinate system, that is, a diffeomorphism F between U' an open set in $\mathrm{I\!R}^n$. Identifying U' with the image $F(U')$ and ∇ with the connection on the resulting open set $U' = F(U') \subseteq \mathrm{I\!R}^m$ arising as the F-push-forward of ∇, we see that the differentials dx^j of the coordinate functions x^j are ∇-parallel. Thus, the component functions Γ_{jk}^l of ∇ relative to the standard coordinate system of $\mathrm{I\!R}^m$ must vanish identically, being characterized by $\nabla_{\partial_j}\partial_k = \Gamma_{jk}^l \partial_l$ (February 5), so that, with ∇ also denoting the dual of ∇, the formula $\nabla_{\partial_j} e^a = -\Gamma_{jb}^a e^b$ of February 7, or [**DG**, pp. 87-88], now yields $0 = \hat{\nabla}_{\partial_j} dx^l = -\Gamma_{jk}^l dx^k$, the local trivializations ∂_j and dx^j being dual to each other [**DG**, formula (5.26) on p. 21], which completes the proof. An equivalent assertion: $\exp_z : U \to U'$ is then an affine diffeomorphism between suitably chosen connected neighborhoods of 0 in T_zM and z in M (which is immediate as it holds for the standard flat connection, having $\exp_z v = z + v$). The conclusion about flat pseudo-Riemannian manifolds (M, g) being locally isometric to pseudo-Euclidean vector spaces with constant metrics: $\exp_z : U \to U'$ as above, being affine, pushes the pseudo-Euclidean of T_zM, restricted to U, onto a *parallel* metric on U', which must equal g since it equals g at the point z **(Problem 4)**. The central argument needed to prove the Myers theorem – much more general than the theorem itself, as stated in [**DG**, p. 119]: *if a length* $\boldsymbol{L}$ *geodesic* $[0, \pi] \ni t \mapsto x(t)$ *in a Riemannian manifold of dimension* $n \geq 2$ *is locally minimizing – meaning*: *not longer than all nearby smooth curves joining its endpoints – and* $r(\dot{x}, \dot{x}) \geq (n-1)Kg(\dot{x}, \dot{x})$ *at all* $x = x(t)$*, where* $K \in (0, \infty)$*, then* $\boldsymbol{L} \leq \pi/\sqrt{K}$. Proof of the above statement [**DG**, p. 119].
Homework: Problems 1 – 5, italicized above.

April 12: Proof of Myers's theorem [**DG**, pp. 119-120]. Further conclusion: M then also has a finite fundamental group, due to the resulting compactness of its universal covering. The scalar curvature function $s : M \to \mathrm{I\!R}$ of a pseudo-Riemannian manifold (M, g), withy

$s = \mathrm{tr}_g\, r$, so that $s = g^{jk} R_{jk}$. The exterior derivative of a smooth local $(0,2)$ tensor field b on a manifold with a connection, which is the $(0,3)$ tensor field $Z = db$ given by $Z_{jkl} = b_{kl,j} - b_{jl,k}$. The identities $R_{jkl}{}^{p}{}_{,p} = R_{jl,k} - R_{kl,j}$ for torsion-free connections, and $2g^{kl} R_{jk,l} = s_{,j}$ (known as the Bianchi identity for the Ricci tensor), valid in all pseudo-Riemannian manifolds, and obtained by successive contractions of the second Bianchi identity (February 16). The coordinate-free versions of these identities: $\delta R = -dr$ and $2\delta r = ds$. Schur's theorem: in dimensions $n \geq 3$, if a connected pseudo-Riemannian manifold (M, g) has $r = sg/n$ (that is, its Ricci tensor equals a function times g), then the scalar curvature s is constant. Its one-line proof [**DG**, p. 131], based on applying div to the equality $r = sg/n$. Einstein metrics/manifolds defined by requiring that $r = \lambda g$ with a constant λ, which by Schur's theorem need not be assumed constant in dimensions $n \neq 2$. The relation $R = Kg \wedge g$ for a constant K, characterizing metrics/manifolds of constant (sectional) curvature, showing, via the equality $\mathrm{Ric}(g \wedge g) = (n-1)g$, that constancy of K comes for free if $n \neq 2$, and constant sectional curvature implies the Einstein property. Equivalence of flatness, and constancy of the Gaussian curvature, to both of the above conditions when $n = 1$ or, respectively, $n = 2$. The second fundamental form B of a smooth vector subbundle E of a given real/complex vector bundle $\hat{E}$ over a manifold M, relative to a linear connection ∇ in $\hat{E}$, which is a section of $\mathrm{Hom}(TM, \mathrm{Hom}(E, \hat{E}/E))$ such that $B_w\psi$, also written as $B(w, \psi)$, equals $\pi \nabla_w \psi$ for any smooth local sections w of TM and ψ of E, with $\pi : \hat{E} \to \hat{E}/E$ denoting the quotient projection morphism. Proof of well-definedness of B, that is, of the pointwise dependence of $B_w\psi$ on w and ψ, based on the component expression $B_w\psi = B_{ja}^{\lambda} \pi e_\lambda$ with $B_{ja}^{\lambda} = \Gamma_{ja}^{\lambda}$, obviously valid – due to the formula $\nabla_w \psi = w^j(\partial_j \psi^a + \Gamma_{jb}^{a} \psi^b) e_a$ of February 5 – whenever e_a, e_λ, or e_a, or πe_λ are local trivializing sections for $\hat{E}$, or E or, respectively, $\hat{E}/E$, the last property being a consequence of the first two [**DG**, p. 72]. The observations that $B = 0$ everywhere if and only if E is ∇-parallel (February 9), and $\mathrm{Hom}(TM, \mathrm{Hom}(E, \hat{E}/E)) = L(TM, E; \hat{E}/E)$ for real vector bundles. The special case provided by the second fundamental form B of an immersion $F : M \to N$ into a manifold N, relative to a fixed connection D in TN, which is here a section of $L(TM, TM; [F^*TN]/[TM])$, since the injective vector-bundle morphism $TM \to F^*TN$ sending (x, v), with $v \in T_xM$, to $(x, dF_x v)$ allows us, from now on, to treat TM in this situation as a subbundle of F^*TN, while referring to $[F^*TN]/[TM]$ as the normal bundle of the immersion F. The above formula $B_{ja}^{\lambda} = \Gamma_{ja}^{\lambda}$ now rewritten as $B_{jk}^{\lambda} = \Gamma_{jk}^{\lambda}$ for local coordinates x^j in M and y^j, y^λ in which F has the form $(x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, 0, \dots, 0)$, and e_a, e_λ are replaced by the corresponding coordinate vector fields. The conclusion that B *must be symmetric at every point whenever* D *is torsion-free* (**Problem 1**). A submanifold M of a manifold N with a fixed connection D in TN being called totally geodesic if, given any D-geodesic $I \ni t \mapsto x(t) \in N$ and any parameter $t' \in I$ such that the point $x = x(t')$ lies in M and $\dot{x}(t') \in T_xM$, one has $x(t) \in M$ for all $t \in I$ sufficiently close to t'. Examples: open submanifolds of affine subspaces in an affine space N carrying the standard flat connection D. Conclusion: since $B_{jk}^{\lambda} = \Gamma_{jk}^{\lambda}$, *in the torsion-free case the totally geodesic property of* M *is equivalent to vanishing of* B (**Problem 2**).

Homework: Problems 1 and 2, italicized above.

April 15:

The lowest dimensions $n \in \{0, 1, 2, 3, 4\}$.

$\dim \Lambda^p = 3$.

Homework: Problems 1 and 2, italicized above.

April 17: $\sqrt{\alpha}^{\pm} = ab \pm cd$,
left (or, right) multiplications by pure quaternions. The geometric interpretation of a Euclidean, or pseudo-Euclidean, inner product defined only up to a factor, phrased as the angle-geometry lemma [**IP**] or, respectively, the null-cone lemma [**IP**]. Conformal relatedness of two pseudo-Riemannian metrics on a given manifold, meaning that one equals the other times a function without zeros (and usually assumed positive when both metrics are Riemannian), which one also expresses by calling one metric the result of a conformal change of the other. Pseudo-Riemannian (or, Riemannian) conformal structures on a manifold: equivalence classes of metrics modulo conformal relatedness (with a positive-function factor in the Riemannian case).

April 19: Conformal flatness of a metric – its being locally conformally related to a flat metric. The result of Jan A. Schouten [*Über die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Maßbestimmung auf eine Mannigfaltigkeit mit euklidischer Maßbestimmung*, Math. Zeitschr. **11**, 1921, 58-88]: a pseudo-Riemannian manifold of any dimension $m \geq 1$ is conformally flat if and only if W and dh are both identically zero, W and dh being its Weyl tensor and the exterior derivative of its Schouten tensor (February 26). Here W (or, by definition, h) equals zero if $m \leq 3$ (or, if $m = 1$) and, as we will see later, vanishing of W implies that of dh when $m \geq 4$.
Homework: Problems 1, 2 and 3, italicized above.