Proc. Symposia in Pure Math., **54** (1993), edited by R. E. Greene and S.-T. Yau, Part 2, 157–171

GEOMETRY OF ELEMENTARY PARTICLES

Andrzej Derdzinski

Contents

1. Particles and interactions in nature.
2. Bundles over the spacetime.
3. Models of free matter particles.
4. The Yang-Mills description of interactions.
5. The standard model.
6. Grand unifications.

Introduction. This is a presentation of the *standard model of elementary particles*, i.e., their generally accepted theory, discussed here on the *classical* level (without field quantization). Our basic reference is the book [5], for which the present paper forms a self-contained summary. The author also intends to provide in [6] and [7] a more detailed treatment of particle geometry, along with its quantized version.

Although this material is covered in numerous physics textbooks (such as [2] – [4], [9], [14], [15], [18], [20] – [23]), the approach adopted here may be particularly suitable for a mathematician reader due to its consistently geometric language. Presentations of more general topics are also available in the mathematical literature; see, for instance, [1], [8], [10], [12], [13], [16], [17], [19].

The theory outlined below does *not*, by itself, provide a usable model of the real world, even though it accounts well for many qualitative effects. Its inadequacy as a source of quantitative predictions is mainly due to its classical (macroscopic) character, as opposed to the predominantly quantum (small-scale) properties displayed by elementary particles in nature. However, instead of discarding the classical model for this reason, one uses field quantization to transform it into a *quantum theory*, which supplies the required "microscopic" corrections.

Cross-references in the text are indicated by arrows ($\rightarrow$).

1. Particles and interactions in nature

1.0. Experimental data show that there are over 200 species (kinds) of subatomic particles, such as the *electron* e, *proton* p, *neutron* n, *photon* γ, *electronic*

1991 *Mathematics Subject Classification.* Primary 53C80; Secondary 81T13, 53B50, 81V05, 81V10, 81V15, 81V22.

Key words and phrases. Standard model, electroweak model, quark model, gauge theory.

Supported in part by NSF Grant DMS-8601282.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

neutrino ν_{e}, *neutral pion* π^0, etc. ($\to$ 1.5, 1.6). Particles interact in four basic ways, known as the *strong, electromagnetic, weak* and *gravitational* interactions (forces), ordered here by decreasing *strength* of the interaction, i.e., probability of its occurrence in the given circumstances. We will not discuss gravitational forces, negligibly weak on the microscopic level; see, however, 2.0.

1.1. Particle invariants assign to particle species elements of various Abelian semigroups. Here belong, for instance, the *mass* $m \in \mathbb{R}_+ = [0, \infty)$, *electric charge* $Q \in \mathbb{Z}$ ($\to$ 1.2), *average lifetime* $\tau \in (0, \infty]$, as well as *spin* $s \in \frac{1}{2}\mathbb{Z}_+ = \{0, \frac{1}{2}, 1, \dots\}$ and *parity* $\varepsilon \in \mathbb{Z}_2 = \{1, -1\}$. The spin measures the capacity of the particle to carry internal angular momentum (as if it were rotating about its axis), while the parity of the particle describes symmetry properties of its configurations with respect to space reflections ($\to$ 3.3, 3.5). Particles with $s \in \mathbb{Z}$ (resp., $s \notin \mathbb{Z}$) are known as *bosons* (resp., *fermions*).

1.2. Quantization of the electric charge. According to experimental evidence (Millikan, 1909) and theoretical arguments ($\to$ 5.1.iv), the electric charge always comes in integral multiples of the electron charge q (by convention, $q < 0$). Charges of individual particle species may be indicated by superscripts such as $^-, ^0, ^+, ^{++}$, so that $\mathrm{e} = \mathrm{e}^-$, $\mathrm{p} = \mathrm{p}^+$, $\mathrm{n} = \mathrm{n}^0$, $\gamma = \gamma^0$, $\nu_{\mathrm{e}} = \nu_{\mathrm{e}}^0$, etc. ($\to$ 1.0).

1.3. Interaction carriers are the few particle species that serve as agents mediating interactions. These are: the photon γ ($\to$ 5.1.ii) for electromagnetism, the *weak bosons* $\mathrm{W}^+, \mathrm{W}^-, \mathrm{Z}^0$ for the weak interaction ($\to$ 5.3.ii.b,c), eight kinds of *gluons* ($\to$ 5.2.ii) for the strong force and, probably, *gravitons* for gravity. Except for gravitons with $s = 2$, their spins ($\to$ 1.1) all equal 1.

1.4. Matter particles, i.e., those particle species which are not interaction carriers, may in turn be classified into *hadrons* and *leptons*, depending on whether they can or cannot participate in the strong interaction.

1.5. Leptons consist of 12 known species, all of spin $\frac{1}{2}$ ($\to$ 1.1): the electron e, *muon* μ and *tauon* τ (with positive masses and electric charge -1), the electrically neutral, massless *neutrinos* $\nu_{\mathrm{e}}, \nu_\mu, \nu_\tau$, as well as their antiparticles ($\to$ 3.2.i): $\mathrm{e}^+, \mu^+, \tau^+, \overline{\nu_{\mathrm{e}}}, \overline{\nu_\mu}, \overline{\nu_\tau}$.

1.6. Hadrons which are fermions (resp., bosons, $\to$ 1.1) are called *baryons* (resp., *mesons*), with about 100 known species of either. For instance, π^0 is a meson. Baryons can further be divided into the disjoint classes of *baryons proper* (such as p, n), and their antiparticles ($\to$ 3.2.i), the *antibaryons*.

1.7. Classification of elementary particles, as outlined above:

$$\text{all particles} \cdots \begin{cases} \text{interaction carriers} \\ \text{matter particles} \cdots \begin{cases} \text{leptons} \\ \text{hadrons} \cdots \begin{cases} \text{mesons} \\ \text{baryons} \cdots \begin{cases} \text{baryons proper} \\ \text{antibaryons} \end{cases} \end{cases} \end{cases} \end{cases}$$

2. Bundles over the spacetime

2.0. Spacetime. We work with a fixed *spacetime* $(\mathcal{M}, g)$, which is a 4-manifold $\mathcal{M}$ endowed with a pseudo-Riemannian metric g of signature $- + + +$ and with a *time orientation*, i.e., a continuous choice $x \mapsto C_x^+$ of one ("future") component of the timelike cone $C_x = \{v \in T_x\mathcal{M} : g(v,v) < 0\}$ at each $x \in \mathcal{M}$. Although one often assumes that $\mathcal{M}$ is an affine space and g is translation invariant (the *Minkowski spacetime*), nonflat spacetimes are also used, as models of gravity (in general relativity).

2.1. Vector bundles over the spacetime $(\mathcal{M}, g)$, used below, include the product line bundle $1 = \mathcal{M} \times \mathbb{C}$, the tangent bundle $T = T\mathcal{M}$, its dual T^*, the bundle $\Lambda^4 T^*$ of volume forms (pseudoscalars) on $\mathcal{M}$ and, for each $k \in \mathbb{Z}_+$, the bundle $S_0^k T^*$ whose fibre over $x \in \mathcal{M}$ consists of all real k-linear symmetric forms on $T_x\mathcal{M}$ with g-contraction zero (i.e., of all pseudo-spherical harmonics in $T_x\mathcal{M}$).

2.2. Weyl spinors. Whenever necessary, the spacetime $(\mathcal{M}, g)$ will be assumed to be an orientable spin manifold. Denoting L, R the orientations of $\mathcal{M}$, we may then choose fixed *Weyl spinor bundles* $\sigma_{\mathrm{L}}, \sigma_{\mathrm{R}}$ over $(\mathcal{M}, g)$, with $\sigma_{\mathrm{R}} = \overline{\sigma_{\mathrm{L}}}$ ($\rightarrow$ 3.2). They are complex vector bundles of fibre dimension 2, obtained from a common spin structure over $(\mathcal{M}, g)$ via two mutually conjugate nontrivial representations of $\mathrm{Spin}(3,1) = \mathrm{SL}(2,\mathbb{C})$ in $\mathbb{C}^2$. The (Levi-Civita) spinor connection $\mathring{\nabla}$ in σ_{L} and σ_{R} then gives rise to the Dirac operator $\mathcal{D}$ sending sections of σ_{L} to those of σ_{R} and vice versa.

2.3. Dirac spinors. If $(\mathcal{M}, g)$ is orientable and spin, we choose a fixed *Dirac spinor bundle* over $(\mathcal{M}, g)$ to be the direct sum $\sigma = \sigma_{\mathrm{L}} + \sigma_{\mathrm{R}}$ of the Weyl spinor bundles selected as in 2.2, with *unordered* summands, so that no orientation of $\mathcal{M}$ is distinguished. The operators $\mathring{\nabla}$ and $\mathcal{D}$ now are also defined in σ.

3. Models of free matter particles

3.0. Particle models. Each particle species is represented by ("lives in") a specific fibre bundle η, endowed with some additional geometric structure, over the spacetime manifold $\mathcal{M}$ ($\rightarrow$ 2.0). The sections ψ of η are to be thought of as "semiclassical states" of the particle, evolving with time in a manner described by suitable *field equations* (which form a part of the geometry of η).

3.1. Models of matter particles ($\rightarrow$ 1.4) are *vector* bundles and their field equations are *linear*. For those matter particles which are *free*, i.e., not subject to interactions, the choices of bundles and equations are quite specific ($\rightarrow$ 3.3 – 3.5). On the other hand, interaction carriers are represented by a special type of *affine* bundles ($\rightarrow$ 4.0) which, for many "practical" purposes, may also be regarded as vector bundles ($\rightarrow$ 4.4, 5.3.ii,iii).

3.2. Physical meaning of vector bundle operations. Equalities between vector or affine bundles, such as $(\eta^*)^* = \eta$, stand for natural (functorial) isomorphisms, the category in question being usually clear from the context. For a complex vector bundle η over $\mathcal{M}$, let $\overline{\eta}$ be its *conjugate bundle*, with each fibre $\overline{\eta}_x$, $x \in \mathcal{M}$, consisting of all antilinear maps $\eta_x^* \to \mathbb{C}$. Thus, we have $\overline{\eta} = \eta^*$ whenever the geometry of η involves a fixed Hermitian fibre metric (which may even be indefinite), as is the case for all models of free matter particles except neutrinos ($\to$ 3.4).

i. For a particle species represented by a complex vector bundle η ($\to$ 3.0, 3.1), the conjugate $\overline{\eta}$, along with the corresponding "conjugate geometry", may be expected to host another, related particle species, called the *antiparticle* of the original one. The resulting *antiparticle formation* (denoted $^{-}$), leaves most of the relevant particle invariants ($\to$ 1.1) either completely unchanged (e.g., ε for bosons, m, τ, s), or just changes their signs (as in the case of ε for fermions and Q; $\to$ 3.5, 5.1.iii). Antiparticles also make sense for interaction carriers ($\to$ 4.4, 5.3.ii.b,c), and, in fact, turn out to exist for all particles known in nature. However, some species (referred to as *strictly neutral*) coincide with their antiparticles and then it is natural to represent them by *real* rather than complex bundles. For instance, $\overline{\pi^0} = \pi^0$, $\overline{\gamma} = \gamma$, $\overline{\mathrm{Z}^0} = \mathrm{Z}^0$, while $(\mathrm{W}^+)^- = \mathrm{W}^-$, $\overline{\mathrm{e}} = \mathrm{e}^+ \neq \mathrm{e}$, $\overline{\nu_\mathrm{e}} \neq \nu_\mathrm{e}$, $\overline{\mathrm{p}} \neq \mathrm{p}$, $\overline{\mathrm{n}} \neq \mathrm{n}$ (notation of 1.0, 1.2, 1.3).

ii. Given k particle species living in vector bundles η_j, the direct sum $\eta = \eta_1 + \ldots + \eta_k$ stands for their *common generalization*, which is *not* a particle species in the usual sense. (For instance, the *nucleon*, generalizing protons and neutrons, does not have a well-defined electric charge.) Conversely, if a bundle η describing some particles happens to be *naturally reducible*, i.e., admit a direct sum decomposition that is natural (functorial), these particles should be regarded as forming several distinct species represented by the summands, for which η provides a common generalization.

iii. Putting together k particles of (not necessarily different) species that live in vector bundles η_j, one obtains a *composite object* whose evolving states ($\to$ 3.0) are sections of the tensor-product bundle $\eta_1 \ldots \eta_k$. Such objects include subatomic particles (hadrons, $\to$ 5.2), as well as nuclei, atoms, or even molecules. However, $\eta_1 \ldots \eta_k$ is often naturally reducible ($\to$ ii), and then it stands for *several* particle species, represented by its summands. Using the projection onto any summand η, one may thus characterize the formation of such a *composite particle* living in η as a natural (functorial) surjective bundle morphism $\eta_1 \ldots \eta_k \to \eta$.

3.3. Matter bosons of spin $k \in \mathbb{Z}_+$ and parity $(-1)^k$ (resp., $(-1)^{k-1}$, $\to$ 1.1) live in the bundle $\eta = S_0^k T^*$ (resp., in the tensor product $\eta = (S_0^k T^*)\Lambda^4 T^*$), if they are strictly neutral, and in its complexification $\eta^{\mathbb{C}}$ otherwise ($\to$ 2.1, 3.2.i). Their field equations ($\to$ 3.0) consist of the *Klein-Gordon equation*

$$\Box \psi = (mc/\hbar)^2 \psi \tag{1}$$

and, if $k \geq 1$, also of the *divergence condition*

$$\operatorname{div} \psi = 0 \tag{2}$$

imposed on sections ψ of η or $\eta^{\mathbb{C}}$. Here $\Box = \mathrm{Trace}_g \circ \mathring{\nabla}^2$ is the d'Alembertian (wave operator) of the Levi-Civita connection $\mathring{\nabla}$ in η and $\operatorname{div} \psi$ stands for the obvious

g-contraction of $\mathring{\nabla}\psi$, while $\hbar$, m and c are, respectively, Planck's constant divided by 2π, the mass of the particle in question, and the speed of light. In particular, states of a particle with mass m, spin 0 and parity $+1$ are real or complex valued functions ψ on the spacetime $(\mathcal{M}, g)$ satisfying (1) with the pseudo-Riemannian Laplacian $\Box$ of $(\mathcal{M}, g)$. As another example, sections ψ of $T^* = T^*\mathcal{M}$, with (1) and (2), describe the states of a strictly neutral particle with mass m, spin 1 and parity -1.

3.4. Neutrinos of all species $\nu_{\mathrm{e}}, \nu_\mu, \nu_\tau$ ($\to$ 1.5) are represented by a fixed Weyl spinor bundle σ_{L} ($\to$ 2.2), while the corresponding antineutrinos $\overline{\nu_{\mathrm{e}}}, \overline{\nu_\mu}, \overline{\nu_\tau}$ live in $\sigma_{\mathrm{R}} = \overline{\sigma_{\mathrm{L}}}$. In all cases, the field equations consist of *Weyl's equation*

$$\mathcal{D}\psi = 0 \tag{3}$$

for sections ψ of σ_{L} or σ_{R}, $\mathcal{D}$ being the Dirac operator. As the choice of bundles indicates, each (anti)neutrino species *distinguishes an orientation of space*. This is manifested through the phenomenon known as *parity violation*, predicted by Lee and Yang and discovered by Wu in 1956.

3.5. Fermions other than neutrinos, with spin $k + \frac{1}{2}$, $k \in \mathbb{Z}_+$ and parity $(-1)^k$ (resp., $(-1)^{k-1}$, $\to$ 1.1) live in the subbundle η of the tensor product $(S_0^k T^*)\sigma$ obtained by requiring that the Clifford product involving T^* and σ be zero (resp., in its conjugate $\overline{\eta}$), where σ is a fixed Dirac spinor bundle over $(\mathcal{M}, g)$ ($\to$ 2.3, 2.1). The field equations consist of *Dirac's equation*

$$(\mathcal{D} + mc/\hbar)\psi = 0 \tag{4}$$

and, if $k \geq 1$, also of the divergence condition (2), imposed on sections ψ of η or $\overline{\eta}$, with $m, c, \hbar$ as in 3.3. In particular, particles of spin $\frac{1}{2}$ such as the electron e, proton p, neutron n live in σ, while their antiparticles (the *positron* e^+, *antiproton* $\overline{\mathrm{p}}$, *antineutron* $\overline{\mathrm{n}}$) are represented by $\overline{\sigma}$, and each is governed by the Dirac equation (4) with the appropriate mass m. (It is convenient here to distinguish σ from $\overline{\sigma}$, even though they are naturally isomorphic. Also, in contrast with hadrons, lepton parities are not well-defined, i.e., cannot be determined by experiment, and so the choice of σ rather than $\overline{\sigma}$ for e is just a matter of convention.)

4. The Yang-Mills description of interactions

4.0. Interaction bundles. In the formalism of Yang and Mills (1954), a given interaction is described by an *interaction bundle* δ, which is a real or complex vector bundle, of some fibre dimension N, over the spacetime manifold $\mathcal{M}$ ($\to$ 2.0). Moreover, δ is endowed with a fixed geometry, consisting mainly of a G-structure, i.e., a reduction P of the full principal frame bundle of δ to a (usually compact) subgroup G of $\mathrm{GL}(N, \mathbb{F})$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. (In most cases we will replace P by an equivalent tensorial object in δ, such as a Hermitian fibre metric when $G = \mathrm{U}(N) \subset \mathrm{GL}(N, \mathbb{C})$.) Suppressing P from the notation, we denote $\mathcal{C}(\delta)$ the affine bundle over $\mathcal{M}$, the C^∞ sections of which coincide with the connections ∇

in δ, compatible with P. The translation-space bundle (i.e., the associated vector bundle) of $\mathcal{C}(\delta)$ then is the tensor product $\mathfrak{g}(\delta)T^*$, where $T^* = T^*\mathcal{M}$ and $\mathfrak{g}(\delta)$ stands for the G-adjoint bundle of Lie algebras, corresponding to P. The interaction-carrier particles now live in $\mathcal{C}(\delta)$, that is, their states are just G-connections ∇ in δ, evolving so as to obey the *Yang-Mills equation*

$$\operatorname{div} R^{\nabla} = 0, \tag{5}$$

where the curvature R^{∇} of ∇ is regarded as a $\mathfrak{g}(\delta)$-valued 2-form on $\mathcal{M}$.

A natural question to ask now is what interaction bundles and in particular which structure groups G correspond in this way to the known physical interactions ($\to$ 1.0). The answer ($\to$ 5.1 – 5.3) is currently believed to be $G = \mathrm{U}(1)$ for electromagnetism, $G = \mathrm{U}(2)$ for the unified *electroweak* (electromagnetic plus weak) interaction, and $G = \mathrm{SU}(3)$ for the strong force. On the other hand, the weak interaction, on its own, is not of the Yang-Mills type ($\to$ 5.3.iii).

4.1. Interactions involving matter. Let η be the *free-particle bundle* (or, *generic-particle bundle*, $\to$ 4.4) of the given matter-particle species, i.e., the vector bundle over $\mathcal{M}$ where the particle lives when it is considered free ($\to$ 3.1). Subject to an interaction described by the interaction bundle δ, this particle is represented by the *interacting-particle bundle* α obtained from δ and η via a specific natural (functorial) construction, a basic example of which is the tensor product $\alpha = \delta\eta$. (As $\overline{\delta\eta} = \overline{\delta}\overline{\eta}$, the role of δ for the corresponding antiparticle species must be played by $\overline{\delta}$, $\to$ 3.2.i.) To account for the interaction, the field equations now are to be imposed on pairs (ψ, ∇) consisting of sections ψ of α, and ∇ of $\mathcal{C}(\delta)$, so that they govern simultaneous evolution of matter particles and interaction carriers, and have the "coupled" form

$$\mathcal{F}(\psi, \mathring{\nabla} \otimes \nabla) = 0, \qquad \operatorname{div} R^{\nabla} = \mathcal{J}(\psi),$$

where

- **i**. $\mathring{\nabla}$ stands for the canonical (Levi-Civita) connection in η, and $\mathring{\nabla} \otimes \nabla$ for the connection in α naturally induced by $\mathring{\nabla}$ and ∇,
- **ii**. $\mathcal{F}$ is such that the free field equations ((1) and (2), or (3), or (4) and (2)) can be written as $\mathcal{F}(\psi, \mathring{\nabla}) = 0$ for sections ψ of η, and so $\mathcal{F}$ must be sufficiently general to also make sense in α,
- **iii**. $\mathcal{J}$ is a differential operator (to be determined in each case from natural considerations), sending sections of α onto $\mathfrak{g}(\delta)$-valued 1-forms on $\mathcal{M}$.

4.2. Lack of naturality. Genuine (observable) physical objects can move in space and "age" with time. The bundles η representing particles should therefore be *natural* in the sense that isometries between open subsets of $(\mathcal{M}, g)$ have (single or multiple-valued) functorial lifts to bundle maps in η. It is so, in fact, for the models of free matter particles ($\to$ 3.3 – 3.5), but not, in general, for interaction bundles δ, or the bundles $\mathcal{C}(\delta)$ and α as in 4.0, 4.1, where interaction carriers and interacting matter particles live. To achieve some sort of naturality in the latter cases, one uses additional procedures (formation of bound states or breaking of symmetry), as described below.

4.3. Bound states. The interacting-particle bundles $\alpha_1, \ldots, \alpha_k$ obtained as in 4.1 using some $\eta_1, \ldots, \eta_k$ and a fixed δ, may sometimes admit natural surjective morphisms of their tensor product $\alpha_1 \ldots \alpha_k$ onto a *natural* bundle η which is the free-particle bundle of a matter particle. The resulting composite particle ($\rightarrow$ 3.2.iii), living in η, then may be called a *bound state* of the original k particles, as it is held together by the given interaction (force), yet, being free, does not exert comparably strong forces of this type on its environment. To obtain such bound states, one only needs to eliminate δ-related factors present in the α_j using natural multilinear bundle maps (examples: $\rightarrow$ 5.1.i, 5.2.i).

4.4. Symmetry breaking means enriching the original geometry of a given interaction bundle δ ($\rightarrow$ 4.0), mainly by choosing a reduction of its G-structure to some proper subgroup H of G. Very often this procedure is just formal and lacks direct physical meaning. When that is the case, H is usually assumed trivial, and the resulting choice of a trivialization for δ and the G-structure leads to the identification $\mathcal{C}(\delta) = T^* + \ldots + T^*$. Thus, the interaction carriers, living in $\mathcal{C}(\delta)$ ($\rightarrow$ 4.0) may to some extent be regarded as forming $\dim G$ separate species of "matter-like", strictly neutral particles with spin 1 and parity -1 ($\rightarrow$ 3.2.ii, 3.3). Similarly, under such a trivialization, interacting-particle bundles α ($\rightarrow$ 4.1) become direct sums of natural bundles (e.g., if $\alpha = \delta\eta$, one obtains $\alpha = \eta + \ldots + \eta$ with N summands, N being the fibre dimension of δ). Consequently, for $N > 1$, a single free-particle bundle η may lead to *several* "observed" matter-particle species, which justifies referring to η as the *generic-particle bundle.* However, the decompositions in question depend on the trivialization used and so have no "absolute" physical significance. For instance, a state of a *single* species for one trivialization will usually correspond to a *mixture* of species for another.

4.5. Spontaneous symmetry breaking takes place when the reduction from G to H in 4.4 is actually present in nature. This may only happen if the interaction is sufficiently *weak.* (Solidifying of fluids at *low temperatures* is a useful analogy.) The reason why it is then worthwhile to keep G (instead of just H) in the picture is the purely accidental manner in which the specific reduction is selected, so that the possible ways the symmetry *could* have become broken still enjoy full G-symmetry (example: $\rightarrow$ 5.3.i).

5. The standard model

5.0. The standard model of elementary particles is a system of theories that, in view of experimental evidence, as well as its internal coherence, is generally accepted as a correct description of the microworld. Besides the field quantization and renormalization procedures, not discussed here, its principal ingredients are the *electroweak* and *quark-gluon models* (both based on the Yang-Mills formalism, $\rightarrow$ 4.0, 4.1), the main ideas of which are outlined below.

5.1. Electromagnetism (Weyl, 1929). The *electromagnetism bundle* (i.e., the electromagnetic interaction bundle, $\rightarrow$ 4.0), corresponding to the electron charge q, is a complex line bundle λ with a Hermitian fibre metric $\langle\, , \rangle$ (a U(1)-structure) over the spacetime $(\mathcal{M}, g)$. For a matter-particle species of charge kq, $k \in \mathbb{Z}$ ($\rightarrow$ 1.2), represented by the free-particle bundle η, the interacting-particle bundle ($\rightarrow$ 4.1) is the tensor product $\alpha = \lambda^k \eta$ (where $\lambda^{-1} = \overline{\lambda}$, $\rightarrow$ 3.2).

- **i**. Since $\lambda^{k_1} \ldots \lambda^{k_n} = \lambda^{k_1 + \cdots + k_n}$ due to the natural isomorphism induced by $\langle\,,\rangle : \lambda\overline{\lambda} \rightarrow \lambda^0 = 1 = \mathcal{M} \times \mathbb{C}$, electromagnetic bound states, arising from mutual "cancellation" of the λ factors ($\rightarrow$ 4.3), can only be obtained when the charges of the consitituent particles add up to zero (so that the system they form is electrically neutral). This is, e.g., the case for non-ionized atoms, but not for nuclei (which would fly apart due to electric repulsion, were it not for the strong force).
- **ii**. Electromagnetic forces are strong enough for their U(1) symmetry not to be broken spontaneously ($\rightarrow$ 4.5). From formal symmetry breaking ($\rightarrow$ 4.4) we obtain $\mathcal{C}(\lambda) = T^*$ and $\alpha = \lambda^k \eta = \eta$ (with η, α as above), so that there is a single interaction-carrier particle species (the photon γ, living in $\mathcal{C}(\lambda)$), which is strictly neutral, with spin 1 and parity -1 ($\rightarrow$ 3.3), while matter particles subject to the electromagnetic interaction lead to the same picture as the free ones.
- **iii**. As $\overline{\lambda^k \eta} = \lambda^{-k} \overline{\eta}$, the electric charge becomes reversed under antiparticle formation ($\rightarrow$ 3.2.i).
- **iv**. It is the electric charge quantization ($\rightarrow$ 1.2) that enables a *single* bundle λ to account for electromagnetic properties of *all* matter particles (which in turn is the first step toward a unified description of particle interactions).

5.2. The quark model (Gell-Mann, Zweig, 1964) presumes that all hadrons are bound states (composites) of peculiar particles called *quarks*, which come in 6 *flavors* (species) u, d, s, c, b, t, and of their antiparticles, the *antiquarks* $\overline{\mathrm{u}}, \ldots, \overline{\mathrm{t}}$. The complicated strong forces involving hadrons then may be viewed as residual effects of the much stronger (and simpler) interactions of (anti)quarks, just as some interatomic forces (of electromagnetic origin) are caused by uneven distribution of electric charge in each (neutral) atom.

All quarks (resp., antiquarks) have spin $\frac{1}{2}$ and parity $+1$ (resp., -1), so that, by 3.5, the free-particle bundle ($\rightarrow$ 4.1) for each flavor is a fixed Dirac spinor bundle σ over the spacetime $(\mathcal{M}, g)$ (resp., its conjugate $\overline{\sigma}$). The *strong-interaction bundle* ($\rightarrow$ 4.0) is a complex vector bundle ρ of fibre dimension 3 with a Hermitian fibre metric $\langle\,,\rangle$ and a fixed section Ω of $\Lambda^3 \rho^*$ which are *compatible* in the sense that $|\Omega| = 1$, i.e., $\Omega(\xi_1, \xi_2, \xi_3) = 1$ for some orthonormal basis ξ_1, ξ_2, ξ_3 of each fibre ρ_x, $x \in \mathcal{M}$. Obviously, the pair consisting of $\langle\,,\rangle$ and Ω is nothing else than an SU(3)-structure in ρ ($\rightarrow$ 4.0). The interacting-particle bundle ($\rightarrow$ 4.1) of each quark (resp., antiquark) flavor is the tensor product $\rho\sigma$ (resp., $\overline{\rho}\overline{\sigma}$).

- **i**. Bound states of quarks are obtained as in 4.3 by "naturally cancelling" ρ and $\overline{\rho}$ in the $\rho\sigma, \overline{\rho}\overline{\sigma}$ factors, which, essentially, can only be done using one of the bundle morphisms $\langle\,,\rangle, \Omega, \overline{\Omega}$ of the tensor products $\rho\overline{\rho}, \rho^3, \overline{\rho}^3$ onto $1 = \mathcal{M} \times \mathbb{C}$. The resulting composite particles are quark-antiquark pairs, three-quark systems, or three-antiquark systems, and may be easily identified with mesons, baryons, and, respectively, antibaryons ($\rightarrow$ 1.6).

ii. The interquark forces are far too strong to allow spontaneous breaking of symmetry ($\to$ 4.5). Under formal symmetry breaking ($\to$ 4.4), one may regard *gluons*, i.e., the strong-interaction carriers, living in $\mathcal{C}(\rho)$, as forming $\dim \mathrm{SU}(3) = 8$ species, while each (anti)quark flavor, when subject to the strong interaction, appears to come in 3 versions (*colors*), with all reservations stated in 4.4.

iii. Quarks do not seem to exist freely (outside of hadrons) in nature, which is probably due to the *extreme strength* of their mutual interaction.

iv. Another strange property of quarks is that their electric charges are *fractional* multiples of the electron charge q ($-2q/3$ for u, c, t and $q/3$ for d, s, b). This does not contradict the charge quantization ($\to$ 1.2), both in view of iii and since q may be replaced by $q/3$. In fact, values of the electric charge (and, similarly, of all other particle invariants additive under the composite-particle formation), thus assigned to (anti)quarks, lead to remarkable agreement with the observed spectrum of hadrons.

5.3. The electroweak model (Glashow, Salam, Weinberg, 1961-1967) treats electromagnetism and the weak force as manifestations of a single interaction, ascribing the observed differences between them to spontaneous symmetry breaking.

The *electroweak-interaction bundle* ($\to$ 4.0) is a complex vector bundle ι of fibre dimension 2 with a Hermitian fibre metric (a U(2)-structure; see, however, iv), inducing a fibre norm denoted $|\ |$, over the spacetime $(\mathcal{M}, g)$. The geometry of ι also involves a *natural fibre metric* $(\,,)$ in the affine bundle $\mathcal{C}(\iota)$, i.e., a fibre metric in its translation-space bundle $\mathfrak{g}(\iota)T^*$ ($\to$ 4.0), obtained by combining g in $T^* = T^*\mathcal{M}$ with a fibre metric in $\mathfrak{g}(\iota)$ (also denoted $(\,,)$) that comes from an Ad-invariant inner product in the Lie algebra $\mathfrak{g} = \mathfrak{u}(2)$. Since $(X, Y) = (a - b)(\operatorname{Trace} X)(\operatorname{Trace} Y) - 2a \operatorname{Trace}(XY)$ for $X, Y \in \mathfrak{g}(\iota_x) \subset \operatorname{End} \iota_x$, $x \in \mathcal{M}$, and some constants $a, b > 0$, $(\,,)$ is determined up to a factor by its *Weinberg angle* $\theta \in (0, \pi/2)$ with $\tan^2\theta = a/b$.

The generic (free) particle bundle ($\to$ 4.1) is a fixed Dirac spinor bundle $\sigma = \sigma_{\mathrm{L}} + \sigma_{\mathrm{R}}$ ($\to$ 2.3), and it represents the *electron generation* $(\mathrm{e}, \nu_{\mathrm{e}})$, i.e., the electron and its neutrino (with their antiparticles $\mathrm{e}^+, \overline{\nu_{\mathrm{e}}}$ accounted for implicitly). The same approach applies to either of the remaining *lepton generations* $(\mu, \nu_\mu), (\tau, \nu_\tau)$ ($\to$ 1.5). For the interacting-particle bundle ($\to$ 4.1), we choose $\alpha = \iota\sigma_{\mathrm{L}} + (\Lambda^2\iota)\sigma_{\mathrm{R}}$.

i. Spontaneous symmetry breaking ($\to$ 4.5) in the electroweak model consists in selecting a section ϕ of ι with a constant length $|\phi| > 0$, which amounts to a structure-group reduction from U(2) to U(1). The line subbundle $\lambda = \phi^\perp$ of ι, with the corresponding fibre metric, then is interpreted as the electromagnetism bundle ($\to$ 5.1). Thus, $\iota = 1 + \lambda$ with $1 = \mathcal{M} \times \mathbb{C} = \operatorname{Span} \phi$, while $\Lambda^2\iota = \lambda$ under the isometric bundle isomorphism $\lambda_x \ni \xi \mapsto \xi \wedge \phi/|\phi| \in \Lambda^2\iota_x$. Hence $\alpha = (1 + \lambda)\sigma_{\mathrm{L}} + \lambda\sigma_{\mathrm{R}}$, i.e., the interacting-particle bundle $\alpha = \sigma_{\mathrm{L}} + \lambda\sigma$ now stands for two particle species represented by σ_{L} and $\lambda\sigma$, which are obviously identified with ν_{e} and e, carrying their correct electric charges ($\to$ 3.2.ii, 5.1, 3.4, 3.5, 1.5).

ii. Extending connections from $\lambda = \phi^\perp$ to ι so as to make ϕ parallel, we obtain an injective morphism $\mathcal{C}(\lambda) \to \mathcal{C}(\iota)$ of affine bundles. Also, isometric embeddings of the *real* vector bundles λ, $\mathcal{M}\times\mathbb{R}$ into $\mathfrak{g}(\iota)$ with the fibre metric $(\,,\,)$ can be defined by $\xi \mapsto \frac{1}{2}a^{-1/2}|\phi|^{-1}X$ and $(x,r) \mapsto \frac{1}{2}ira^{-1/2}(\sec\theta)Y$, where, at each $x \in \mathcal{M}$, $X, Y \in \operatorname{End}\iota_x$ satisfy $X\phi = |\phi|^2\xi$, $X(z\xi) = -z|\xi|^2\phi$, $z \in \mathbb{C}$ (thus, X depends on $\xi \in \lambda_x$), and $Y = \mathrm{Id}$ on $\operatorname{Span}\phi$, $Y = -(\cos 2\theta)\,\mathrm{Id}$ on $\phi^\perp$, with a and the Weinberg angle θ selected before. Orthogonality of the images of these morphisms now establishes a $(\,,\,)$-orthogonal direct-sum decomposition $\mathcal{C}(\iota) = \mathcal{C}(\lambda) + \lambda T^* + T^*$, which is unique and natural in a suitable category. Thus, by 3.2.ii, the electroweak-interaction carriers, living in $\mathcal{C}(\iota)$, form the following particle species, corresponding to the summands $\mathcal{C}(\lambda), \lambda T^*, T^*$:

a. The photon γ, represented by $\mathcal{C}(\lambda)$ ($\to$ 5.1.ii).

b. The charged weak boson W^-, living in λT^* (and hence carrying the electron charge, $\to$ 5.1). Its antiparticle W^+ ($\to$ 3.2.i) is another weak-interaction carrier, implicit in our discussion. Since $\lambda T^* = \lambda \otimes_{\mathbb{R}} T^* = \lambda \otimes_{\mathbb{C}} (T^*)^{\mathbb{C}}$, the free-particle bundle of $\mathrm{W}^\pm$ (cf. 5.1) is the complexification $(T^*)^{\mathbb{C}}$ of $T^* = T^*\mathcal{M}$, which makes either of $\mathrm{W}^\pm$ a matter particle of spin 1 ($\to$ 3.3).

c. The neutral weak boson Z^0, living in T^* and hence strictly neutral, of spin 1 ($\to$ 3.3).

iii. The masses m of most particle species are positive. Exceptions, with $m = 0$, are only possible when the field equations cannot, for formal reasons, contain a nonzero *mass term* (as in (1), (4)), which in fact is the case for neutrinos, satisfying (3), and interaction carriers (with unbroken symmetry), governed by (5). Thus, whether the given particle is massive or massless depends only on its free-particle bundle. (Specifically, $m = 0$ for affine bundles and $\sigma_{\mathrm{L}}, \sigma_{\mathrm{R}}$, while $m > 0$ for the remaining vector bundles in 3.3, 3.5.) Since spontaneous symmetry breaking establishes the "massive" bundles $(T^*)^{\mathbb{C}}, T^*$ as models of the W and Z bosons, their masses must be positive, in contrast with the photon living in $\mathcal{C}(\lambda)$. This is consistent with experimental evidence such as the short range for the weak force, as opposed to the electromagnetic and (interquark) strong interactions, which are long-range. Without the electroweak unification, a description of the weak interaction based on the Yang-Mills formalism ($\to$ 4.0) would not be possible precisely because of its short-range character, i.e., massiveness of its carriers.

iv. Physicists usually choose the structure group of the electroweak theory to be the (twofold) covering group $\mathrm{U}(1) \times \mathrm{SU}(2)$ of $\mathrm{U}(2)$ rather than $\mathrm{U}(2)$ itself. However, taken at the face value, this would amount to imposing unnecessary additional conditions on the geometry and physics of the model.

v. A dynamical approach to the electroweak model (not presented here) also involves a device known as the *Higgs boson*, which may be just a formal mechanism, but could as well turn out to be a (still undiscovered) matter particle with some unusual properties.

6. Grand unifications

6.0. Grand unified theories are attempts to go beyond the standard model by describing both strong and electroweak forces in terms of a single interaction subject to spontaneous symmetry breaking, just as the electroweak model does for the weak force and electromagnetism ($\to$ 5.3). Examples: $\to$ 6.3, 6.5.

6.1. The physical status of grand unifications is unclear, since they all use large structure groups G ($\to$ 6.2), and hence predict the existence of additional interaction carriers (note that $\dim G$ is, basically, the number of carrier species, $\to$ 4.4). The new kind of extremely weak interactions mediated by such particles would, in particular, cause the protons to decay spontaneously (at a very slow rate). However, a decade of intensive searches brought no evidence of proton-decay processes in nature.

6.2. The most obvious direct-sum unification of all microworld forces, using the interaction bundle $\rho+\iota$ with ρ,ι as in 5.2, 5.3, and with the corresponding $\mathrm{SU}(3)\times\mathrm{U}(2)$-structure, is not acceptable precisely because of its reducibility, which violates physical and geometric simplicity requirements.

6.3. The SU(5) **grand unified theory** (Georgi and Glashow, 1974) uses an interaction bundle ($\to$ 4.0) which is a complex vector bundle β of fibre dimension 5 over the spacetime manifold $\mathcal{M}$, while its geometry is an SU(5)-structure, i.e., consists of a Hermitian fibre metric $\langle\,,\rangle$ in β and a section Ω of $\Lambda^5\beta^*$ compatible with $\langle\,,\rangle$ as in 5.2.

We restrict our consideration to just one, e.g., the first, of the *basic-fermion generations* $(\mathrm{e},\nu_{\mathrm{e}},\mathrm{u},\mathrm{d}),(\mu,\nu_\mu,\mathrm{c},\mathrm{s}),(\tau,\nu_\tau,\mathrm{t},\mathrm{b})$, which the model describes separately. (These generations, ordered by increasing masses, are naturally distinguished by their common pattern $(-1,0,2/3,-1/3)$ of electric charges, $\to$ 1.5, 5.2.iv.)

The generic (free) particle bundle ($\to$ 4.1) of the whole generation is a fixed Dirac spinor bundle $\sigma=\sigma_{\mathrm{L}}+\sigma_{\mathrm{R}}$ ($\to$ 2.3). As in 5.3, we let $\alpha=\beta\sigma_{\mathrm{L}}+(\Lambda^2\beta)\sigma_{\mathrm{R}}$ be the interacting-particle bundle. The spontaneous symmetry breaking ($\to$ 4.5) in the SU(5) model involves 3 steps:

- **a**. Choosing a subbundle $\iota\subset\beta$ of complex fibre dimension 2 which, with the Hermitian fibre metric inherited from $\langle\,,\rangle$, is interpreted as the electroweak-interaction bundle, while the natural fibre metric $(\,,)$ in $\mathcal{C}(\iota)$ ($\to$ 5.3) is induced by one in $\mathcal{C}(\beta)$. Since the latter is unique, up to a factor, in view of simplicity of SU(5), the SU(5) theory predicts a specific value of the Weinberg angle θ, such that $\tan^2\theta=3/5$, in some contrast with the experimental bounds $0.29\le\tan^2\theta\le 0.32$. (There are plausible explanations of this discrepancy.)
- **b**. Selecting a complex line bundle χ with a Hermitian fibre metric and a fixed isometric isomorphism $\chi^3=\Lambda^2\iota$ (which amounts to a structure-group reduction in a suitable larger bundle). As Ω provides the identification $1=\Lambda^5\beta=(\Lambda^2\iota)\Lambda^3\iota^\perp=\chi^3\Lambda^3\iota^\perp$, the complex vector bundle $\rho=\overline{\chi\iota^\perp}$ of fibre dimension 3 over $\mathcal{M}$ then satisfies $\Lambda^3\rho=1$, i.e., carries a natural

SU(3)-structure. Thus, ρ may be regarded as the strong-interaction bundle ($\to$ 5.2).

c. Breaking the symmetry in ι exactly as in 5.3.i, i.e., by choosing a section ϕ of constant positive length and thinking of $\lambda = \phi^\perp \subset \iota$ as the electromagnetism bundle.

Since $\lambda = \Lambda^2\iota$ ($\to$ 5.3.i), we may write $\chi = \lambda^{1/3}$ and $\beta = \lambda^{-1/3}\overline{\rho} + \iota$. This leads to the decomposition $\alpha = \sigma_{\mathrm{L}} + \lambda\sigma + \lambda^{-1/3}\overline{\rho}\sigma + \left[\overline{\lambda^{-2/3}\rho\sigma_{\mathrm{L}}} + \lambda^{-2/3}\rho\sigma_{\mathrm{R}}\right]$, the four summands of which represent, as in 3.2.ii, the first-generation fermions along with their correct electric charges and strong-interaction properties ($\to$ 5.1, 5.2). In fact, $\sigma_{\mathrm{L}}, \lambda\sigma$ and $\lambda^{-1/3}\overline{\rho}\sigma$ clearly correspond to ν_{e}, e and the $\overline{\mathrm{d}}$ antiquark ($\to$ 3.4, 3.5, 5.2), while the fourth summand only differs from the correct model $\lambda^{-2/3}\rho\sigma = \lambda^{-2/3}\rho\sigma_{\mathrm{L}} + \lambda^{-2/3}\rho\sigma_{\mathrm{R}}$ of the interacting u quark ($\to$ 5.2.iv) by having one of its own summands replaced by the conjugate (such complications cannot be avoided, for dimensional reasons).

One also has (see [5], formula (7.18)) the natural orthogonal decomposition $\mathcal{C}(\beta) = \mathcal{C}(\rho) + \mathcal{C}(\lambda) + \lambda T^* + T^* + \lambda^{4/3}\rho T^* + \lambda^{1/3}\rho T^*$, which leads, besides the known interaction carriers (gluons in $\mathcal{C}(\rho)$, the photon in $\mathcal{C}(\lambda)$, W^+, W^- in λT^*, Z^0 in T^*, $\to$ 5.2.ii, 5.3.ii), also to new ones, represented by the last two summands. The latter, hypothetical particles, denoted $\mathrm{X}^{-4/3}$, $\mathrm{Y}^{-1/3}$ (and called, along with their antiparticles $\mathrm{X}^{4/3}$, $\mathrm{Y}^{1/3}$, the X and Y *bosons*), carry fractional electric charges ($\to$ 1.2, 5.1) and, if they really exist, should cause proton decay ($\to$ 6.1).

6.4. Some spinor bundles. For each positive integer k, a $\mathrm{Spin}(4k+2)$-structure in a complex vector bundle ζ of fibre dimension 4^k over $\mathcal{M}$ can be described as a Hermitian fibre metric $\langle\,,\rangle$ in ζ along with a real vector subbundle $\kappa \subset \mathrm{End}_{\mathbb{R}}\zeta$ of fibre dimension $4k+2$ such that, for each $x \in \mathcal{M}$, $X \in \kappa_x$ and $\xi, \xi' \in \zeta_x$, $X : \zeta_x \to \zeta_x$ is antilinear, while X^2 is a multiple of Id and $\langle \xi, X\xi' \rangle = \langle \xi', X\xi \rangle$. A natural construction of an example is based on using a complex vector bundle β of fibre dimension $2k+1$ with an $\mathrm{SU}(2k+1)$-structure, i.e., a Hermitian fibre metric (also denoted $\langle\,,\rangle$) and a compatible $\Omega \in \Lambda^{2k+1}\beta^*$ ($\to$ 5.2). The induced Hermitian fibre metric $\langle\,,\rangle$ in the exterior-algebra bundle $\Lambda\beta^* = \Lambda^{\mathrm{even}}\beta^* + \Lambda^{\mathrm{odd}}\beta^*$ then is also sesquilinear in the bundle $\Lambda^{\mathrm{even}}\beta^* + \overline{\Lambda^{\mathrm{odd}}\beta^*} = \Lambda^{\mathrm{even}}\overline{\beta} + \Lambda^{\mathrm{odd}}\beta$, i.e., in $\Lambda\beta^*$ endowed with the new complex structure for which the multiplication by i equals the old $(-1)^j i \cdot \mathrm{Id}$ on $\Lambda^j\beta^*$. The Hodge star $*$ determined by $\langle\,,\rangle$ and Ω, although antilinear in $\Lambda\beta^*$, is a linear endomorphism of $\Lambda^{\mathrm{even}}\overline{\beta} + \Lambda^{\mathrm{odd}}\beta$. We now define a complex vector bundle $\zeta = \zeta[\beta]$ with a $\mathrm{Spin}(4k+2)$-structure by $\zeta[\beta] = \mathrm{Ker}\,(* - \mathrm{Id}) \subset \Lambda^{\mathrm{even}}\overline{\beta} + \Lambda^{\mathrm{odd}}\beta$, with the Hermitian fibre metric obtained by restricting $\langle\,,\rangle$ and with $\kappa \subset \mathrm{End}_{\mathbb{R}}(\zeta[\beta])$ given as the image of the real bundle morphism $\beta^* \to \mathrm{End}_{\mathbb{R}}(\Lambda\beta^*)$ sending ψ to $\mathfrak{e}_\psi + \mathfrak{i}_\psi$, where $\mathfrak{e}_\psi(\omega) = \psi \wedge \omega$ and $\mathfrak{i}_\psi(\psi_1 \wedge \ldots \wedge \psi_k) = \sum_{j=1}^k (-1)^{j-1}\langle \psi_j, \psi \rangle \psi_1 \wedge \ldots \wedge \widehat{\psi_j} \wedge \ldots \wedge \psi_k$. Under the projection $\frac{1}{2}(* + \mathrm{Id}) : \Lambda^{\mathrm{even}}\overline{\beta} + \Lambda^{\mathrm{odd}}\beta \to \zeta$, ζ becomes isomorphic to $1 + \beta + \Lambda^2\overline{\beta} + \Lambda^3\beta + \ldots$ (summation up to $\Lambda^k\beta$ or $\Lambda^k\overline{\beta}$). In particular, β is isometrically embedded in $\zeta[\beta]$.

6.5. The Spin(10) **grand unification** (Fritzsch and Minkowski, 1975), usually called the SO(10) theory, uses an interaction bundle ($\to$ 4.0) which is a complex vector bundle ζ of fibre dimension 16 with a Spin(10)-structure, as in 6.4 with $k = 2$. The generic-particle bundle for a whole basic-fermion generation $(\mathrm{e}, \nu_{\mathrm{e}}, \mathrm{u}, \mathrm{d})$ ($\rightarrow$ 6.3) is, this time, a fixed Weyl spinor bundle σ_{L} ($\rightarrow$ 2.2), and one chooses the interacting-particle bundle ($\rightarrow$ 4.1) to be $\alpha = \zeta\sigma_{\mathrm{L}}$.

The first step of spontaneous symmetry breaking consists in selecting a bundle β with an SU(5)-structure along with a structure-preserving isomorphism $\zeta = \zeta[\beta]$ ($\rightarrow$ 6.4, with $k = 2$), so that $\beta \subset \zeta$ and $\zeta = 1+\beta+\Lambda^2\overline{\beta}$, where $1 = \mathcal{M}\times\mathbb{C}$. Regarding β as the interaction bundle of the SU(5) theory, one then proceeds with the further steps of symmetry breaking ($\rightarrow$ 6.3.a,b,c). Since now $\alpha = (1 + \beta + \Lambda^2\overline{\beta})\sigma_{\mathrm{L}} = \sigma_{\mathrm{L}} + \beta\sigma_{\mathrm{L}} + \overline{(\Lambda^2\beta)\sigma_{\mathrm{R}}}$, with the last two summands resembling the choice of α in the SU(5) model ($\rightarrow$ 6.3), we obtain, after 6.3.a,b,c, $\alpha = (\sigma_{\mathrm{L}} + \sigma_{\mathrm{L}}) + \lambda\sigma_{\mathrm{L}} + \overline{\lambda}\sigma_{\mathrm{L}} + \lambda^{-1/3}\overline{\rho}\sigma_{\mathrm{L}} + \lambda^{1/3}\rho\sigma_{\mathrm{L}} + \lambda^{-2/3}\rho\sigma_{\mathrm{L}} + \lambda^{2/3}\overline{\rho}\sigma_{\mathrm{L}}$. Because $\sigma = \sigma_{\mathrm{L}} + \sigma_{\mathrm{R}}$ with $\sigma_{\mathrm{R}} = \overline{\sigma_{\mathrm{L}}}$, this decomposition differs from the expression $\sigma + \lambda\sigma + \lambda^{1/3}\rho\sigma + \lambda^{-2/3}\rho\sigma$ by containing, instead of some summands, their conjugates. Ignoring such discrepancies (as in 6.3), we interpret the latter four summands ($\rightarrow$ 3.2.ii) as models of a "modified" neutrino ν_{e} ($\rightarrow$ 6.6), the electron e, and the u, d quarks, along with their correct electromagnetic and strong-interaction properties ($\rightarrow$ 3.5, 5.1, 5.2).

As dim Spin(10) = 45, the Spin(10) theory predicts even more interaction-carrier species than the SU(5) model (cf. 6.1). See [11] for details.

6.6. Massive neutrinos. Neutrinos might conceivably have very small, positive masses (the experimental evidence makes this appear rather unlikely, but is still inconclusive). Then they would be described by Dirac's equation (4) in the "chiral" Dirac spinor bundle $\sigma = \sigma_{\mathrm{L}} + \sigma_{\mathrm{R}}$ the summands of which are *ordered*, in contrast with 2.3, to account for parity violation ($\rightarrow$ 3.4).

References

1. E.Binz, J. Śniatycki and H. Fischer, *Geometry of classical fields*, North-Holland Mathematics Studies, vol. 154, North-Holland, Amsterdam, 1988.
2. N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*, Cambridge Univ. Press, Cambridge, 1983.
3. N. N. Bogolyubov [N. N. Bogolubov], A. A. Logunov, A. I. Oksak and I. T. Todorov, *General principles of quantum field theory*, Mathematical Physics and Applied Mathematics, vol. 10, Kluwer, Dordrecht, 1990.
4. N. N. Bogolyubov [N. N. Bogoliubov] and D. V. Shirkov, *Quantum fields*, Benjamin/Cummings, Reading, Mass., 1983.
5. A. Derdzinski, *Geometry of the standard model of elementary particles*, Texts and Monographs in Physics, Springer-Verlag, Berlin-Heidelberg-New York, 1992.
6. ———, *Quantum electrodynamics: A mathematical introduction*, in preparation.
7. ———, *Notes on particle physics*, in preparation.
8. G. G. Emch, *Mathematical and conceptual foundations of 20th-century physics*, North-Holland Mathematics Studies, vol. 100, North-Holland, Amsterdam, 1984.
9. B. Felsager, *Geometry, particles and fields*, Odense Univ. Press, Odense, 1983.
10. F. G. Friedlander, *The wave equation on a curved space-time*, Cambridge Univ. Press, Cambridge, 1975.
11. H. Fritzsch and P. Minkowski, *Unified interactions of leptons and hadrons*, Ann. Phys. **93** (1975), 193-266.
12. R. Hermann, *Vector bundles in mathematical physics I, II*, Benjamin, New York, 1970.

13. ———, *Interdisciplinary mathematics I – XXIV*, Math Sci Press, Brookline, Mass., 1973-1988.
14. K. Huang, *Quarks, leptons and gauge fields*, World Scientific, Singapore, 1982.
15. J. Leite Lopes, *Gauge field theories*, Pergamon Press, Oxford, 1981.
16. Y. I. Manin, *Mathematics and physics*, Progress in Physics, vol. 3, Birkhäuser, Boston, 1981.
17. ———, *Gauge field theory and complex geometry*, Springer-Verlag, Berlin, 1988.
18. R. N. Mohapatra, *Unification and supersymmetry*, Contemporary Physics, Springer-Verlag, New York, 1986.
19. R. Penrose and W. Rindler, *Spinors and space-time I, II*, Cambridge Univ. Press, Cambridge, 1984-1986.
20. P. Ramond, *Field theory*, Benjamin/Cummings, Reading, Mass., 1981.
21. L. H. Ryder, *Elementary particles and symmetries*, Documents on Modern Physics, Gordon and Breach, New York, 1986.
22. ———, *Quantum field theory*, Cambridge Univ. Press, Cambridge, 1985.
23. A. Sudbery, *Quantum mechanics and the particles of nature: An outline for mathematicians*, Cambridge University Press, Cambridge, 1986.

Department of Mathematics, The Ohio State University, Columbus, OH 43210
E-mail address: andrzej@mps.ohio-state.edu