COMPACT PSEUDO-RIEMANNIAN MANIFOLDS WITH PARALLEL WEYL TENSOR

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Abstract

It is shown that in every dimension $n = 3j+2$, $j = 1, 2, 3, \ldots$, there exist compact pseudo-Riemannian manifolds with parallel Weyl tensor, which are Ricci-recurrent, but neither conformally flat nor locally symmetric, and represent all indefinite metric signatures. The manifolds in question are diffeomorphic to nontrivial torus bundles over the circle. They all arise from a construction that a priori yields bundles over the circle, having as the fibre either a torus, or a 2-step nilmanifold with a complete flat torsionfree connection; our argument only realizes the torus case.

Introduction

A pseudo-Riemannian manifold (M, g) of dimension $n \geq 4$ is called *conformally symmetric* [**4**] if its Weyl conformal tensor is parallel. If, in addition, (M, g) is neither conformally flat nor locally symmetric, it is said to be *essentially conformally symmetric.*

All essentially conformally symmetric pseudo-Riemannian metrics are indefinite [**6**, Theorem 2]. Numerous examples of such metrics on open manifolds are known [**7**, **9**], which raises the question whether they exist on any compact manifolds, cf. [**10**]. This paper provides an answer:

Theorem 0.1. *In every dimension $n = 3j + 2 \geq 5$, $j = 1, 2, 3, \ldots$, there exists a compact pseudo-Riemannian manifold (M, g) of any prescribed indefinite metric signature, which is essentially conformally symmetric, Ricci-recurrent, and diffeomorphic to a torus bundle over the circle, but not homeomorphic to, or even covered by, the torus T^n.*

Here (M, g) is called *Ricci-recurrent* if, for every tangent vector field w, the Ricci tensor Ric and the covariant derivative $\nabla_w \mathrm{Ric}$ are linearly dependent at every point.

Each manifold in Theorem 0.1 arises as the quotient $M = \widehat{M}/\Gamma$ for a suitable discrete group Γ of isometries of its universal covering space $\widehat{M}$, diffeomorphic to $\mathbf{R}^n$, with a metric belonging to a family constructed by the second author in [**9**]. For every dimension $n = 3j+2$, the metrics in that family admitting such compact quotients form an infinite-dimensional space of local moduli (Remark 10.1). However, our argument provides no explicit descriptions of the metrics, or the groups Γ.

Conformal symmetry is one of the *natural linear conditions* in the sense of Besse [**1**, p. 433] that can be imposed on the covariant derivatives of the irreducible components of the curvature tensor under the action of the pseudo-orthogonal group. The analogous conditions on the other two components characterize metrics having constant scalar curvature and, respectively, parallel Ricci tensor, including the Einstein metrics. Compact Riemannian or Kähler manifolds of these two classes are the model cases of the Yamabe problem and Calabi's conjectures.

Compact conformally symmetric manifolds have generated much less interest. However, those among them having the specific form $M = \widehat{M}/\Gamma$ mentioned above are related to another familiar class of geometric structures. Namely, we show, in Remarks 4.1 and 6.2, that any such M is a bundle over the circle, and its fibre is either a torus, or a 2-step nilmanifold admitting a complete flat torsionfree connection with a nonzero parallel vector field. (Our argument only succeeds in realizing the torus case.) Complete flat torsionfree connections on compact manifolds are the subject of a vast literature, outlined in [**5**], and on nilmanifolds they exist relatively often, though not always [**2**].

One easily verifies that no essentially conformally symmetric manifold is locally reducible. The gaps in the dimension list of Theorem 0.1 cannot therefore be filled with the aid of Riemannian products. Thus, Theorem 0.1 leaves the existence question unanswered in dimensions $n \geq 4$ other than those of the form $n = 3j + 2$. While for $n \geq 5$ this may be due to the particular nature of our argument, designed to work only when $n \equiv 5 \pmod 3$, the reason why Theorem 0.1 fails to include the case $n = 4$ seems less of a coincidence. In fact, using Theorem 7.3 of the present paper, we show in [**8**] that *every four-dimensional essentially conformally symmetric Lorentzian manifold is noncompact.*

There are further instances where particular details of Theorem 0.1 reflect more general facts. Namely, two other results of [**8**] state that *the fundamental group of a compact essentially conformally symmetric manifold is always infinite*, and *for any compact essentially conformally symmetric Lorentzian manifold* (M, g)*, some two-fold covering manifold of* M *is a bundle over the circle and its fibre admits a flat torsionfree connection with a nonzero parallel vector field.*

1. Preliminaries

Let a group Γ act on a manifold $\widehat{M}$ freely by diffeomorphisms. The action of Γ on $\widehat{M}$ is called *properly discontinuous* if there exists a locally diffeomorphic surjective mapping $\pi : \widehat{M} \to M$ onto some manifold M such that the π-preimages of points of M are precisely the orbits of the Γ action. (Cf. [**5**, p. 187].) We then refer to M as the *quotient* of $\widehat{M}$ under the action of Γ and write $M = \widehat{M}/\Gamma$.

The index $j = 1, 2, 3, \ldots$ is always used to label the terms of sequences, with $x_j \to x$ meaning that $x = \lim_{j\to\infty} x_j$.

Remark 1.1. If the action of Γ on $\widehat{M}$ is free and properly discontinuous, a_j and y_j are sequences in Γ and $\widehat{M}$, while both y_j and $a_j y_j$ converge, then the sequence a_j is constant except for finitely many j.

By a *lattice* in a real vector space $\mathfrak{L}$ with $\dim \mathfrak{L} < \infty$ we mean, as usual, an additive subgroup of $\mathfrak{L}$ generated by some basis of $\mathfrak{L}$.

Remark 1.2. For $\mathfrak{L}$ as above, a countable additive subgroup $\Lambda \subset \mathfrak{L}$ is a lattice if and only if $\operatorname{span} \Lambda = \mathfrak{L}$ and Λ is closed as a subset of $\mathfrak{L}$. See [**3**, Chap. VII, Théorème 2].

Lemma 1.3. *If $k, l \in \mathbf{Z}$ and $2 \le k < l \le k^2/4$, then the polynomial $P(\lambda) = -\lambda^3 + k\lambda^2 - l\lambda + 1$ in the real variable λ has three distinct real roots λ, μ, ν such that $1/l < \lambda < 1 < \mu < k/2 < \nu < k$, and hence*

$$(1) \qquad 0 < \lambda < \mu < \nu, \quad \lambda < 1 < \nu, \quad \lambda\mu < 1 < \mu\nu, \quad \lambda\nu \ne 1.$$

Proof. Since $P(\lambda) = (k - \lambda)\lambda^2 + 1 - l\lambda$, we get $P(\lambda) \ge 1 - l\lambda > 0$ if $\lambda < 1/l$ and $P(\lambda) \le 1 - l\lambda \le 1 - kl < 0$ if $\lambda \ge k$, as well as $P(\lambda) = (k - 1/l)\lambda^2 > 0$ for $\lambda = 1/l$. Similarly, $P(1) = k - l < 0$ and $P(\lambda) = 1 + (\lambda^2 - l)\lambda \ge 1$ if $\lambda = k/2$. Therefore, P has some roots λ, μ, ν such that $1/l < \lambda < 1 < \mu < k/2 < \nu < k$. This yields (1): the inequalities $\lambda\mu < 1 \ne \lambda\nu$ follow since $\lambda\mu\nu = 1$ and $\nu > 1 \ne \mu$. q.e.d.

It is obvious that, for a nonzero polynomial P in the real variable λ,

(2) all coefficients of P are integers, its leading term is $(-\lambda)^d$, where $d = \deg P$, and its constant term equals 1 or -1

if and only if (2) holds with P replaced by the product $(1 - \lambda)P$.

Let $T : \mathfrak{L} \to \mathfrak{L}$ be an endomorphism in a real vector space $\mathfrak{L}$ with $\dim \mathfrak{L} = d < \infty$. Clearly, $T(\Lambda) = \Lambda$ for some lattice Λ in $\mathfrak{L}$ if and only if $\det T = \pm 1$ and the matrix of T in some basis of $\mathfrak{L}$ consists of integers. Condition (2) characterizes the characteristic polynomials of those endomorphisms T of $\mathfrak{L}$ for which such a lattice Λ exists.

In fact, given $a_1, \ldots, a_d \in \mathbf{R}$, let $\mathfrak{T}$ be the $d \times d$ matrix with the rows $[a_1 \ \ldots \ a_d]$, $[1 \ 0 \ \ldots \ 0]$, $[0 \ 1 \ 0 \ \ldots \ 0]$, $\ldots$, $[0 \ \ldots \ 0 \ 1 \ 0]$. Then $\mathfrak{T}$ has the characteristic polynomial $(-1)^d(\lambda^d - a_1\lambda^{d-1} - \ldots - a_{d-1}\lambda - a_d)$, which proves sufficiency of (2).

2. Conformally symmetric Ricci-recurrent metrics

In this section, $f, p, n, V, \langle\,,\rangle$ and A stand for the following objects:

$$(3) \quad \begin{array}{l} \text{a nonconstant periodic function } f : \mathbf{R} \xrightarrow{C^\infty} \mathbf{R} \text{ with a period } p > 0, \\ \text{an integer } n \geq 4 \text{ and a real vector space } V \text{ of dimension } n-2, \\ \text{a pseudo-Euclidean inner product } \langle\,,\rangle \text{ on } V, \\ \text{a nonzero, traceless, } \langle\,,\rangle\text{-self-adjoint linear operator } A : V \to V. \end{array}$$

As in [**9**], the data (3) lead to a pseudo-Riemannian metric

$$(4) \qquad \widehat{g} \,=\, \kappa\, dt^2 \,+\, dt\, ds \,+\, h \quad \text{ on the manifold } \widehat{M} = \mathbf{R}^2 \times V \,\approx\, \mathbf{R}^n.$$

The products of differentials stand here for symmetric products, t, s are the Cartesian coordinates on $\mathbf{R}^2$ treated, with the aid of the projection $\widehat{M} \to \mathbf{R}^2$, as functions $\widehat{M} \to \mathbf{R}$, and h is the pullback to $\widehat{M}$ of the flat (constant) pseudo-Riemannian metric on V formed by the inner product $\langle\,,\rangle$, while $\kappa : \widehat{M} \to \mathbf{R}$, with $\kappa(t,s,v) = f(t)\langle v,v\rangle + \langle Av,v\rangle$.

Lemma 2.1. *For any choice of the data* (3)*, the metric* $\widehat{g}$ *given by* (4) *is essentially conformally symmetric and Ricci-recurrent.*

Proof. See [**9**, Theorem 3], where weaker assumptions are used: rather than being defined on $\mathbf{R}$ and periodic, f is just a nonconstant real-valued C^∞ function on an open interval $I \subset \mathbf{R}$, and $\widehat{M}$ is replaced with $I \times \mathbf{R} \times V$. In the notation of [**9**], our t, s and h appear as $x^1, 2x^n$ and $k_{\lambda\mu} dx^\lambda dx^\mu$, while our $f(t)$ is $[2(n-2)]^{-1} C \exp\left(\int Q\, dx^1\right)$. q.e.d.

Next, we set $\mathrm{G} = \mathbf{Z} \times \mathbf{R} \times \mathcal{E}$, where $\mathcal{E}$ is the vector space of all C^∞ solutions $u : \mathbf{R} \to V$ to the differential equation $\ddot{u}(t) = f(t)u(t) + Au(t)$. Clearly, $\Omega(u,w) = \langle \dot{u}, w\rangle - \langle u, \dot{w}\rangle$ is, for any $u, w \in \mathcal{E}$, a constant function $\mathbf{R} \to \mathbf{R}$, which defines a nondegenerate skew-symmetric bilinear form $\Omega : \mathcal{E} \times \mathcal{E} \to \mathbf{R}$. On the other hand, setting $(Tu)(t) = u(t-p)$, we obtain a linear isomorphism $T : \mathcal{E} \to \mathcal{E}$ with

$$(5) \quad T^*\Omega = \Omega, \quad \text{ that is, } \quad \Omega(Tu, Tw) = \Omega(u,w) \text{ whenever } u, w \in \mathcal{E}.$$

For $(k,q,u), (l,r,w) \in \mathrm{G}$ and $(t,s,v) \in \widehat{M} = \mathbf{R}^2 \times V$, we set

$$(6) \quad \begin{array}{ll} \text{a)} & (k,q,u) \cdot (l,r,w) = (k+l, q+r-\Omega(u, T^l w), T^{-l}u + w), \\ \text{b)} & (k,q,u) \cdot (t,s,v) = (t+kp, s+q-\langle \dot{u}(t), 2v+u(t)\rangle, v+u(t)), \end{array}$$

which, by (5), defines a Lie-group structure in G and an action of the Lie group G on the manifold $\widehat{M}$. With all triples assumed to be elements of G, (6.a) gives

$$
(7)\qquad
\begin{array}{rl}
\mathrm{i)} & (k,q,u)^{-1} = (-k,-q,-T^k u),\\
\mathrm{ii)} & (k,q,u)\cdot(0,r,0) = (0,r,0)\cdot(k,q,u) \;=\; (k,\, q+r,\, u),\\
\mathrm{iii)} & (0,r,0)^l\cdot(k,q,u) = (k,q+lr,u),\\
\mathrm{iv)} & (k,q,u)\cdot(0,r,w)\cdot(k,q,u)^{-1} = (0,r-2\varOmega(u,w),T^k w),\\
\mathrm{v)} & (0,q,u)\cdot(0,r,w)\cdot(0,q,u)^{-1}\cdot(0,r,w)^{-1} = (0,2\varOmega(w,u),0).
\end{array}
$$

Our G also acts on the manifold $\mathbf{R}^2\times\mathcal{E}$, diffeomorphic to $\mathbf{R}^{2n-2}$, by

$$(8)\qquad (k,q,u)\cdot(t,z,w) \;=\; (t+kp,\; z+q-\varOmega(u,w),\; T^k(w+u)).$$

The following mapping is easily verified to be equivariant relative to the actions of G given by (8) and (6.b):

$$(9)\qquad \mathbf{R}^2\times\mathcal{E}\ni(t,z,w)\mapsto(t,s,v)=(t,\; z-\langle\dot w(t),w(t)\rangle,\; w(t))\in\widehat{M}.$$

Lemma 2.2. *The group* G *acts on* $(\widehat{M},\widehat{g})$ *by isometries.*

Proof. Using any fixed basis e_λ of V, $\lambda=3,\dots,n$, we obtain $h=h_{\lambda\mu}\,dv^\lambda dv^\mu$ and $\kappa=(fh_{\lambda\mu}+a_{\lambda\mu})v^\lambda v^\mu$ in the product coordinates t,s,v^λ for $\widehat{M}$, where the coordinate functions v^λ on V send each $v\in V$ to its components in the expansion $v=v^\lambda e_\lambda$, while $h_{\lambda\mu}=\langle e_\lambda,e_\mu\rangle$ and $a_{\lambda\mu}=\langle Ae_\lambda,e_\mu\rangle$. For any given $(k,q,u)\in\mathrm{G}$, the mapping $F:\widehat{M}\to\widehat{M}$ with $F(t,s,v)=(k,q,u)\cdot(t,s,v)$ has the components F^*t,F^*s,F^*v^λ (that is, $t\circ F$, etc.) equal to $t+kp$, $s+q-h_{\lambda\mu}\dot u^\lambda(t)[2v^\mu+u^\mu(t)]$ and $F^\lambda=v^\lambda+u^\lambda(t)$. Evaluating their differentials and noting that $\ddot u(t)=f(t)u(t)+Au(t)$, we get $F^*\widehat{g}=(F^*\kappa)(dF^*t)^2+(dF^*t)\,dF^*s+h_{\lambda\mu}\,dF^\lambda\,dF^\mu=\widehat{g}$, as required. q.e.d.

REMARK 2.3. We will use the symbol T for a more general translation operator, acting on functions $\mathbf{R}\ni t\mapsto\eta(t)$ valued in scalars, vectors or operators by $(T\eta)(t)=\eta(t-p)$.

3. First-order and Lagrangian subspaces

We again assume $f,p,n,V,\langle\,,\rangle$ and A to be as in (3), while $\widehat{M},\Gamma$, $\mathcal{E},\varOmega$ and T stand for the corresponding objects defined in Section 2.

By a *first-order subspace* of the solution space $\mathcal{E}$ we mean any $(n-2)$-dimensional vector subspace $\mathcal{L}\subset\mathcal{E}$ having the property that $u(t)\neq 0$ whenever $u\in\mathcal{L}\smallsetminus\{0\}$ and $t\in\mathbf{R}$. For any first-order subspace $\mathcal{L}$,

(10) the evaluation operators $u\mapsto u(t)$ form a C^∞ curve, parametrized by $t\in\mathbf{R}$, of linear isomorphisms $\mathcal{L}\to V$.

Since Ω is nondegenerate, $\dim \mathcal{L}' = \dim \mathcal{E} - \dim \mathcal{L}$ for any vector subspace $\mathcal{L} \subset \mathcal{E}$ and $\mathcal{L}' = \{u \in \mathcal{E} : \Omega(u,w) = 0 \text{ for all } w \in \mathcal{L}\}$. Thus

(11) $2 \dim \mathcal{L} \le \dim \mathcal{E}$ whenever $\mathcal{L}$ is a Lagrangian subspace of $\mathcal{E}$,

where $\mathcal{L}$ is called *Lagrangian* if $\Omega(u,w) = 0$ for all $u, w \in \mathcal{L}$.

REMARK 3.1. If $\mathcal{L} \subset \mathcal{E}$ is a first-order subspace, the restriction of the mapping (9) to $\mathbf{R}^2 \times \mathcal{L}$ is a diffeomorphism $\mathbf{R}^2 \times \mathcal{L} \to \widehat{M}$, equivariant relative to the actions (8) and (6.b) of the subgroup H of G whose underlying set is $\{0\} \times \mathbf{R} \times \mathcal{L}$. In fact, by (10), the restriction is a diffeomorphism, and it is equivariant since so is (9).

Lemma 3.2. *First-order subspaces $\mathcal{L}$ of the solution space $\mathcal{E}$ are in a bijective correspondence with C^∞ functions $B : \mathbf{R} \to \mathrm{End}(V)$ such that $\dot{B} + B^2 = f + A$, where f stands for the function $t \mapsto f(t)\,\mathrm{Id}$. The correspondence assigns to B the space $\mathcal{L}$ of all solutions $u : \mathbf{R} \to V$ to the differential equation $\dot{u}(t) = B(t)u(t)$, and, for this $\mathcal{L}$,*

i) *$\mathcal{L}$ is a Lagrangian subspace of $\mathcal{E}$ if and only if $B(t)$ is self-adjoint relative to $\langle\,,\rangle$ for every $t \in \mathbf{R}$,*
ii) *$\mathcal{L}$ is T-invariant if and only if B is periodic with period p,*
iii) *if $T(\mathcal{L}) = \mathcal{L}$, the determinant of $T : \mathcal{L} \to \mathcal{L}$ is $\exp\left(-\int_0^p \mathrm{tr}\, B(t)\, dt\right)$.*

Proof. The assignment $B \mapsto \mathcal{L}$ described in the lemma sends B with $\dot{B} + B^2 = f + A$ to a first-order subspace of $\mathcal{E}$ in view of uniqueness of solutions for ordinary differential equations. The surjectivity and injectivity of $B \mapsto \mathcal{L}$ are both obvious from (10): given $\mathcal{L}$, we choose $B(t)$ to be the inverse of the evaluation isomorphism $u \mapsto u(t)$ followed by the operator $u \mapsto \dot{u}(t)$, which is clearly the unique choice of B producing the given solution space $\mathcal{L}$. Now (i) is immediate from (10).

The equation $\dot{u} = Bu$ for $u : \mathbf{R} \to V$ implies $(Tu)^{\cdot} = (TB)(Tu)$ (cf. Remark 2.3). This second equation gives $T(\mathcal{L}) \subset \mathcal{L}$ whenever B is periodic with period p (that is, $TB = B$). Conversely, if $T(\mathcal{L}) = \mathcal{L}$, the two equations combined with (10) yield $TB = B$, proving (ii).

For a basis u_λ of $\mathcal{L}$, $\lambda = 3, \ldots, n$, and a fixed volume form $[\ldots]$ in V, defining $\eta : \mathbf{R} \to \mathbf{R}$ by $\eta = [u_3, \ldots, u_n]$ we get $\dot{\eta} = \eta \,\mathrm{tr}\, B$, so that, if $\mathcal{L}$ is T-invariant, integration shows that $\log|\eta| - \log|T\eta| = \tau$, where $\tau = \int_0^p \mathrm{tr}\, B(t)\, dt$ (cf. (ii) and Remark 2.3), and, therefore, $\eta = e^\tau T\eta$. On the other hand, $\eta \det T$, for $T : \mathcal{L} \to \mathcal{L}$, equals $[Tu_3, \ldots, Tu_n] = T\eta$. Hence $\det T = \eta^{-1} T\eta = e^{-\tau}$, which gives (iii). q.e.d.

REMARK 3.3. If $\mathcal{L}$ and B are related as in Lemma 3.2, while $T(\mathcal{L}) = \mathcal{L}$ and $B(t)$ commutes with $B(t')$ for all $t, t' \in \mathbf{R}$, then $T : \mathcal{L} \to \mathcal{L}$ is given by e^{-S}, where $S = \int_0^p B(t)\, dt \in \mathrm{End}(V)$ acts as an operator $\mathcal{L} \to \mathcal{L}$ with $(Su)(t) = Su(t)$.

In fact, let $J(t,s) = \int_s^t B(t')\,dt' \in \mathrm{End}(V)$ for $t,s \in \mathbf{R}$. Since $de^{J(t,s)}/dt = B(t)e^{J(t,s)}$, the unique solution $w \in \mathcal{L}$ to the initial value problem $\dot{w} = Bw$, $w(s) = v$ is, for any $s \in \mathbf{R}$ and $v \in V$, given by $w(t) = e^{J(t,s)}v$. Applying this to $w = u$ or $w = Tu$, where $u \in \mathcal{L}$ is fixed, and $s = 0$, we see that $u(t) = e^{J(t,0)}u(0)$ and $(Tu)(t) = e^{J(t,p)}u(0)$, for each $t \in \mathbf{R}$ (since $(Tu)(p) = u(0)$). Thus, $(Tu)(t) = e^{J(t,p)-J(t,0)}u(t) = e^{-J(0,p)}u(t)$.

4. Discrete subgroups of G

For $f, p, n, V, \langle\,,\rangle, A$ and $\mathcal{E}, \varOmega, T, \mathrm{G}$ as in Section 2, let

(12) $\varPi : \mathrm{G} \to \mathbf{Z}, \quad \Delta : \mathrm{Ker}\,\varPi \to \mathcal{E}, \quad \text{and} \quad \Delta_t : \mathrm{Ker}\,\varPi \to V, \ \text{for } t \in \mathbf{R},$

be the following homomorphisms (with $\mathcal{E}, V$ treated as additive groups):

(13) $\varPi(k,q,u) = k, \qquad \Delta(0,q,u) = u, \qquad \Delta_t(0,q,u) = u(t).$

Throughout this section, given a subgroup Γ of G, we use the notation

(14) $\Sigma = \Gamma \cap \mathrm{Ker}\,\varPi, \quad \Xi = \Sigma \cap \mathrm{Ker}\,\Delta, \quad \Lambda = \Delta(\Sigma), \quad \mathcal{L} = \mathrm{span}\,\Lambda.$

Thus, Ξ and Λ are the kernel and image of $\Delta : \Sigma \to \mathcal{E}$, while $\mathcal{L}$ is the vector subspace of $\mathcal{E}$ spanned by the additive subgroup Λ. Since $\Sigma \subset \{0\} \times \mathbf{R} \times \Lambda \subset \{0\} \times \mathbf{R} \times \mathcal{L}$, identifying $\{0\} \times \mathbf{R} \times \mathcal{L}$ with $\mathbf{R} \times \mathcal{L}$

(15) we treat Σ as a subset of $\mathbf{R} \times \mathcal{L}$ such that $\Sigma \subset \mathbf{R} \times \Lambda$.

REMARK 4.1. If a subgroup $\Gamma \subset \mathrm{G}$ acts on $\widehat{M} = \mathbf{R}^2 \times V$ freely and properly discontinuously with a compact quotient $M = \widehat{M}/\Gamma$, then

i) the image $\varPi(\Gamma)$ equals $m\mathbf{Z}$ for some integer $m > 0$,
ii) M is the total space of a C^∞ bundle over the circle $\mathbf{R}/\varPi(\Gamma)$, and the mapping $\widehat{M} \ni (t,s,v) \mapsto p^{-1}t \in \mathbf{R}$ descends to the bundle projection $\mathrm{pr} : \widehat{M} \to \mathbf{R}/\varPi(\Gamma)$,
iii) for every $t \in \mathbf{R}$, the submanifold $\widehat{M}_t = \{t\} \times \mathbf{R} \times V$ of $\widehat{M}$ is invariant under the subgroup Σ of Γ, the action of Σ on $\widehat{M}_t$ is properly discontinuous, and the inclusion $\widehat{M}_t \to \widehat{M}$ descends to an embedding $\widehat{M}_t/\Sigma \to M$, the image of which is the fibre $M_{\mathrm{pr}(t)}$ of the bundle M over the point $\mathrm{pr}(t)$ in the base circle,
iv) Σ acts on each $\widehat{M}_t$ by affine transformations whose linear parts preserve the vector $(0,1,0) \in \{0\} \times \mathbf{R} \times V \subset \widehat{M}$ (cf. (6.b)); this gives rise to a flat torsionfree connection with a nonzero parallel vector field on the compact fibre $M_{\mathrm{pr}(t)}$.

In fact, (i) and (ii) are obvious as the assignment $(t,s,v) \mapsto t/p$ descends to a surjective submersion $M \to \mathbf{R}/\varPi(\Gamma)$, which would be an unbounded C^∞ function $M \to \mathbf{R}$ if $\varPi(\Gamma)$ were the trivial group. As (6.b) clearly implies Σ-invariance of $\widehat{M}_t$, assertion (iii) is immediate from the definition of proper discontinuity in Section 1.

Theorem 4.2. *Suppose that* Γ *is a subgroup of* G*, the action of* Γ *on* $\widehat{M}$ *is free and properly discontinuous, and the quotient manifold* $M = \widehat{M}/\Gamma$ *is compact. Then*

a) *for every* $t \in \mathbf{R}$*, the image* $\Delta_t(\Sigma)$ *spans* V *as a vector space,*
b) $\Sigma \cap \operatorname{Ker} \Delta_t = \Xi$ *whenever* $t \in \mathbf{R}$*,*
c) $\Xi = \{0\} \times \mathbf{Z}\theta \times \{0\} = \{(0, l\theta, 0) : l \in \mathbf{Z}\}$ *for some* $\theta \in [0, \infty)$*,*
d) $2\varOmega(u, w) \in \mathbf{Z}\theta$ *for* θ *defined in* (c) *and all* $u, w \in \Lambda$*,*
e) *whenever* (k, q, u) *is an element of* Γ*, we have* $T^k(\Lambda) = \Lambda$ *and* $T^k(\mathcal{L}) = \mathcal{L}$*, while* $\varPsi(\Sigma) = \Sigma$*, for* $\Sigma \subset \mathbf{R} \times \mathcal{L}$ *as in* (15) *and* $\varPsi : \mathbf{R} \times \mathcal{L} \to \mathbf{R} \times \mathcal{L}$ *given by* $\varPsi(r, w) = (r - 2\varOmega(u, w), T^k w)$*,*
f) Γ *has no Abelian subgroup of finite index, unless* $T^k : \mathcal{L} \to \mathcal{L}$ *equals the identity for some* $(k, q, u) \in \Gamma$ *with* $k \geq 1$*,*
g) $\mathcal{L}$ *is a first-order subspace of* $\mathcal{E}$*, so that* $\dim \mathcal{L} = n - 2$*,*
h) *one of the following two cases occurs*:
   I) Ξ *is the trivial group,* $\mathcal{L} \subset \mathcal{E}$ *is a Lagrangian subspace,* Σ *with* (15) *is a lattice in* $\mathbf{R} \times \mathcal{L}$*, and* Λ *is the isomorphic image of* Σ *under the projection* $\mathbf{R} \times \mathcal{L} \to \mathcal{L}$*,*
   II) Ξ *is isomorphic to* $\mathbf{Z}$ *and* Λ *is a lattice in* $\mathcal{L}$*.*

Proof. If $\Delta_t(\Sigma)$ spanned a proper subspace V' of V, a nonzero linear functional $V \to \mathbf{R}$ vanishing on V' would descend to an unbounded C^∞ function on the compact manifold M_t (cf. Remark 4.1(iii)). This yields (a). Next, if we had $u(t) = 0$ for some $(0, q, u) \in \Sigma$ with $u \in \mathcal{L} \setminus \{0\}$ and some $t \in \mathbf{R}$, choosing $v \in V$ such that $q = \langle \dot{u}(t), 2v \rangle$ (which exists as $\dot{u}(t) \neq 0$), we would get $(0, q, u) \cdot (t, s, v) = (t, s, v)$ with any $s \in \mathbf{R}$ (cf. (6.b)), and so the action of Γ would not be free. Hence, if $(0, q, u) \in \Sigma \cap \operatorname{Ker} \Delta_t$, then $u = 0$, which implies (b).

Obviously, $\Xi = \{0\} \times \mathrm{K} \times \{0\}$ for some additive subgroup K of $\mathbf{R}$. If K were not a closed subset of $\mathbf{R}$, any fixed sequence θ_j of mutually distinct nonzero elements of K with $\theta_j \to 0$, and any $(t, s, v) \in \widehat{M}$, would yield the sequences $(0, \theta_j, 0)$ and $(0, \theta_j, 0) \cdot (t, s, v)$ contradicting, by (7.ii), the conclusion of Remark 1.1. This proves (c) (cf. Remark 1.2).

By (7.v), the commutator of $(0, q, u), (0, r, w) \in \Sigma$ is $(0, 2\varOmega(w, u), 0)$, which, in view of (14), is an element of Ξ. Thus, (c) gives (d).

Next, let $(k, q, u) \in \Gamma$. By (7.iv) and (7.i), the inner automorphisms corresponding to (k, q, u) and $(k, q, u)^{-1}$ send any $(0, r, w) \in \Sigma$ to $(0, r - 2\varOmega(u, w), T^k w)$ and $(0, r + 2\varOmega(T^k u, w), T^{-k} w)$, which must again be elements of the normal subgroup $\Sigma \subset \Gamma$. Thus, both T^k and T^{-k}

leave Λ invariant, which yields (e), as $\mathcal{L} = \operatorname{span}\Lambda$ and $\Omega(T^k u, w) = \Omega(u, T^{-k} w)$ by (14) and (5).

If Γ has an Abelian subgroup Γ' of finite index, replacing Γ by Γ', we may assume that Γ is Abelian. From (7.iv) for any $(0, r, w) \in \Sigma$ and any $(k, q, u) \in \Gamma$ with $k \geq 1$ (which exists, cf. Remark 4.1(i)) we now get, by (14), $T^k w = w$ for all $w \in \mathcal{L} = \operatorname{span}\Lambda$, and (f) follows.

The evaluation operator $\mathcal{L} \to V$ is surjective for each t, since it maps $\Lambda = \Delta(\Sigma)$ onto $\Lambda_t = \Delta_t(\Sigma)$, while $\mathcal{L} = \operatorname{span}\Lambda$ and $V = \operatorname{span}\Lambda_t$ by (14) and (a). Consequently, $\dim\mathcal{L} \geq \dim V = n - 2$. Proving (g) is now reduced to showing that $\dim\mathcal{L} \leq n - 2$.

We first assume that Ξ is trivial, and so $\Delta : \Sigma \to \Lambda$ is an isomorphism. Thus, Σ is Abelian, and (6.a) with $k = l = 0$ implies that $\mathcal{L} = \operatorname{span}\Lambda$ is a Lagrangian subspace, which has two consequences. First, (g) holds in this case, as $\dim\mathcal{L} \leq n-2$ by (11). Secondly, again by (6.a), the subgroup H of G with the underlying set $\{0\} \times \mathbf{R} \times \mathcal{L}$ is Abelian, and, under the obvious identification $\{0\} \times \mathbf{R} \times \mathcal{L} \approx \mathbf{R} \times \mathcal{L}$, it coincides with the additive group of $\mathbf{R} \times \mathcal{L}$. Hence Σ with (15) is an additive subgroup of $\mathbf{R} \times \mathcal{L}$. The properly discontinuous action of Σ on $\widehat{M}_t$, having the compact quotient M_t (see Remark 4.1(iii)) corresponds, under the equivariant diffeomorphism defined in Remark 3.1 (for our $\mathcal{L}$) combined with the obvious identification $\{t\} \times \mathbf{R} \times \mathcal{L} \approx \mathbf{R} \times \mathcal{L}$, to the action of Σ by vector-space translations on the ambient space $\mathbf{R} \times \mathcal{L}$. Therefore, Σ is a lattice in $\mathbf{R} \times \mathcal{L}$, and assertion (h-I) follows.

Finally, let Ξ be nontrivial. Thus, θ in (c) is positive. By (7.iii),

$$(16) \qquad (k, q + l\theta, u) \in \Gamma \ \text{ whenever } \ (k, q, u) \in \Gamma \ \text{ and } \ l \in \mathbf{Z}.$$

Consequently, Λ is closed as a subset of $\mathcal{L}$. In fact, otherwise there would exist a sequence u_j of mutually distinct nonzero elements of Λ, such that $u_j \to 0$. Choosing $(0, q_j, u_j) \in \Sigma$ with $q_j \in [0, \theta]$ (cf. (16)), and then replacing the q_j by a convergent subsequence, we could use any $(t, s, v) \in \widehat{M}$ to obtain a sequence $(0, q_j, u_j) \cdot (t, s, v)$ that converges (see (6.b)), contrary to Remark 1.1. Similarly, $\Lambda_t = \Delta_t(\Sigma)$ is, for each t, a closed subset of V. Namely, if for some t and some sequence $(0, q_j, u_j) \in \Sigma$ the vectors $u_j(t) \in V$ were mutually distinct and nonzero, while $u_j(t) \to 0$, (16) would allow us to modify q_j in such a way that $q_j - \langle \dot{u}_j(t), u_j(t) \rangle$ is a bounded sequence in $\mathbf{R}$. Any convergent subsequence of $(0, q_j, u_j) \cdot (t, s, 0)$, with any $s \in \mathbf{R}$, would now, by (6.b), again contradict Remark 1.1.

Being closed, both Λ and $\Lambda_t = \Delta_t(\Sigma)$ are lattices in the respective spaces $\mathcal{L}$ and V (see Remark 1.2 and (a)). Thus, they are free Abelian groups of ranks $\dim\mathcal{L} \geq n - 2$ and, respectively, $\dim V = n - 2$.

The restriction to Λ of the evaluation operator $\mathcal{L} \to V$ is, by (13), a surjective homomorphism onto Λ_t, and, by (b), it is injective. Hence $\dim \mathcal{L} = \dim V$, so that we have (g), while (c) gives (h-II). q.e.d.

5. Simplifying assumptions

For any given objects (3), with $\widehat{M}$ and G as in Section 2, we ask whether some subgroup $\Gamma \subset \mathrm{G}$ acts on $\widehat{M}$ properly discontinuously, producing a *compact* n-dimensional quotient manifold $M = \widehat{M}/\Gamma$. For the purpose of answering this question, we may always require Γ to satisfy the following additional conditions:

(17)
a) the integer m defined in Remark 4.1(i) is equal to 1,
b) $\Omega(u,w) \in \mathbf{Z}\theta$ for all $u, w \in \Lambda$ (cf. (14) and Theorem 4.2(c)),
c) Σ is a lattice in $\mathbf{R} \times \mathcal{L}$, where $\Sigma \subset \mathbf{R} \times \mathcal{L}$ as in (15).

In fact, assuming that $m = 1$ leads to no loss of generality, as the use of mp instead of p in (3) causes m to be replaced by 1.

Next, by (7.ii), the formula $\phi(r) = (0, r, 0)$ defines a homomorphism ϕ from the additive group $\mathbf{R}$ into the center of G. The image $\phi(\mathbf{R})$ thus consists of isometries of $(\widehat{M}, \widehat{g})$ (cf. Lemma 2.2) descending to the quotient manifold $M = \widehat{M}/\Gamma$, so as to form a group K of isometries of M. The action of K on M is free, and the kernel of the projection homomorphism $\phi(\mathbf{R}) \to \mathrm{K}$ is the group Ξ in (14). In fact, the kernel obviously contains Ξ. Now let $(t, s, v) \in \widehat{M}$ and $r \in \mathbf{R}$. If $(0, r, 0) \cdot (t, s, v) = (t, s + r, v)$ lies in the Γ-orbit of (t, s, v), and hence equals $(k, q, u) \cdot (t, s, v)$ for some $(k, q, u) \in \Gamma$, then, by (6.b) and Theorem 4.2(b), $k = u = 0$ and $q = r$, and so $(0, r, 0) \in \Xi$.

In case I of Theorem 4.2(h), (17.b) and (17.c) always hold, in view of Theorem 4.2(d) with $\theta = 0$. To obtain (17.b) in case II of Theorem 4.2(h), we replace the quotient manifold $M = \widehat{M}/\Gamma$ by its own quotient M' under the free action of the $\mathbf{Z}_2$ subgroup of the circle $\mathrm{K} = \phi(\mathbf{R})/\Xi$. This means replacing Γ by the subgroup $\Gamma' = \Gamma \cup (\zeta \cdot \Gamma)$ of G generated by Γ and the central element $\zeta = (0, \theta/2, 0)$, for θ defined in Theorem 4.2(c). Since $\theta' = \theta/2$ corresponds to Γ' just as θ did to Γ, Theorem 4.2(d) now yields (17.b). Finally, (17.c) follows in case II from (17.b). In fact, as $\Lambda = \Delta(\Sigma)$ by (14) and Λ is a lattice in $\mathcal{L}$ (see Theorem 4.2(h-II)), we may fix $(q_\lambda, u_\lambda) \in \Sigma \subset \mathbf{R} \times \mathcal{L}$, $\lambda = 3, \ldots, n$, such that u_λ form a basis of $\mathcal{L}$ generating the additive subgroup Λ. We now show that Σ coincides with the lattice Σ' in $\mathbf{R} \times \mathcal{L}$ generated by the basis consisting of $(\theta, 0)$ and our (q_λ, u_λ). Namely, the projection $(r, w) \mapsto w$ forms surjective group homomorphisms $\Delta : \Sigma \to \Lambda$ and

$\Sigma' \to \Lambda$. Hence $\sum_\lambda k(\lambda)u_\lambda$ is the Δ-image of the product (in Σ)

$$(18)\quad (\theta,0)^l(q_3,u_3)^{k(3)}\dots(q_n,u_n)^{k(n)} \in \Sigma,\ \text{with integers } l \text{ and } k(\lambda).$$

Thus, every $(r,w) \in \Sigma$ is of the form (18): (r,w) has the same Δ-image w as $(\tilde r,w) = (q_3,u_3)^{k(3)}\dots(q_n,u_n)^{k(n)}$, with $k(\lambda) \in \mathbf{Z}$ such that $w = \sum_\lambda k(\lambda)u_\lambda$, and so $(r,w) = (\theta,0)^l(\tilde r,w)$ for some $l \in \mathbf{Z}$. (By Theorem 4.2(c), $(\theta,0)$ generates Ξ, the kernel of $\Delta : \Sigma \to \Lambda$.)

On $\mathbf{R}\times\Lambda$ there are two group structures: one additive, with Σ' as a subgroup, the other, given by $(q,u)(r,w) = (q+r-\Omega(u,w), u+w)$ (cf. (6.a)), having Σ as a subgroup. The mapping $\chi : \mathbf{R}\times\Lambda \to S^1\times\Lambda$ with $\chi(r,w) = (r+\mathbf{Z}\theta, w)$ is, by (17.b), a homomorphism from *both* groups into the direct product of $S^1 = \mathbf{R}/\mathbf{Z}\theta$ and Λ.

Any $(r,w) \in \Sigma$ of the form (18) is related to the linear combination $(r',w) = l(\theta,0) + \sum_\lambda k(\lambda)(q_\lambda,u_\lambda) \in \Sigma'$ with the same coefficients $l, k(\lambda) \in \mathbf{Z}$ by $(r',w) = (r,w) + l'(\theta,0) = (\theta,0)^{l'}(r,w)$ for some $l' \in \mathbf{Z}$. In fact, as $\chi(r,w) = \chi(r',w)$ (both being $\sum_\lambda k(\lambda)q_\lambda + \mathbf{Z}\theta$), (r,w) and (r',w) differ, relative to either group structure in $\mathbf{R}\times\Lambda$, by an element of $\operatorname{Ker}\chi = \Xi = \mathbf{Z}\theta\times\{0\}$. Therefore, $\Sigma = \Sigma'$.

6. A criterion for the existence of compact quotients

Given $f,p,n,V,\langle\,,\rangle,A$ with (3), let $B : \mathbf{R} \to \operatorname{End}(V)$ be a C^∞ function, periodic with period p, such that $\dot B + B^2 = f + A$. These data lead to further objects: $\widehat M = \mathbf{R}^2\times V$ and the group G acting on $\widehat M$ defined in Section 2, the vector space $\mathcal{L}$ of dimension $n-2$ formed by all solutions $u : \mathbf{R}\to V$ to the differential equation $\dot u(t) = B(t)u(t)$, the translation operator $T : \mathcal{L}\to\mathcal{L}$ with $(Tw)(t) = w(t-p)$, the $(n-1)$-dimensional vector space $\mathcal{W} = \mathbf{R}\times\mathcal{L}$, and its one-dimensional subspace $\mathcal{I} = \mathbf{R}\times\{0\}$.

Theorem 6.1. *For any objects $f,p,n,V,\langle\,,\rangle$ and A as in* (3)*, the following two conditions are equivalent*:

i) *some subgroup $\Gamma\subset$ G acts on $\widehat M$ freely and properly discontinuously with a compact quotient manifold $M = \widehat M/\Gamma$, and satisfies condition* (17.a);
ii) *there exist a C^∞ function $B : \mathbf{R}\to\operatorname{End}(V)$, periodic of period p, with $\dot B + B^2 = f + A$, a lattice Σ in $\mathcal{W}$, a linear functional $\varphi\in\mathcal{L}^*$, and $\theta\in[0,\infty)$ such that*
   a) $\Sigma\cap\mathcal{I} = \mathbf{Z}\theta\times\{0\}$,
   b) $\Psi(\Sigma) = \Sigma$ *for $\Psi : \mathcal{W}\to\mathcal{W}$ given by $\Psi(r,w) = (r+\varphi(w), Tw)$,*
   c) $\Omega(u,w)\in\mathbf{Z}\theta$ *whenever $u,w\in\Lambda$,*

where $\mathcal{L}, T, \mathcal{W}, \mathcal{I}$ correspond to B as above, Λ is the image of Σ under the projection $\mathcal{W} \to \mathcal{L}$, and $\Omega(u,w) \in \mathbf{R}$ is the constant function $\langle \dot{u}, w\rangle - \langle u, \dot{w}\rangle$.

We now show that (i) implies (ii) in Theorem 6.1, postponing the proof of the converse statement until Section 8.

Let Γ be as in (i). According to Section 5, we may also assume (17). By Theorem 4.2(e),(g) with $k=1$, the space $\mathcal{L}$ in (14) is a T-invariant first-order subspace of the solution space $\mathcal{E}$ defined in Section 2, and so, by Lemma 3.2, $\mathcal{L}$ arises from a C^∞ function $B : \mathbf{R} \to \mathrm{End}(V)$ with $\dot{B} + B^2 = f + A$. Lemma 3.2(ii) shows that B is periodic with period p. By (17.c), Σ defined in (14) is a lattice in $\mathcal{W} = \mathbf{R} \times \mathcal{L}$. Choosing θ as in Theorem 4.2(c), we obtain assertions (ii-a) and (ii-c), cf. (17.b). Finally, let us set $\varphi(w) = -2\Omega(u,w)$ for $w \in \mathcal{L}$, with Ω defined in the lines preceding (6) and $u \in \mathcal{E}$ chosen so that $(1,q,u) \in \Gamma$ for some $q \in \mathbf{R}$. (Note that we assume (17.a).) Now (ii-b) is immediate from Theorem 4.2(e) with $k=1$.

REMARK 6.2. As we will show in Section 8, if condition (ii) in Theorem 6.1 is satisfied, then (i) holds for a subgroup $\Gamma \subset \mathrm{G}$. In addition, this Γ may be chosen so that the compact quotient manifold $M = \widehat{M}/\Gamma$ is a bundle over the circle with some fibre N which is either a torus (when $\mathcal{L}$ is a Lagrangian subspace of the solution space $\mathcal{E}$, cf. Section 3), or a 2-step nilmanifold (when $\mathcal{L}$ is not Lagrangian).

REMARK 6.3. Since assertion (ii) in Theorem 6.1 includes the condition $\Psi(\Sigma) = \Sigma$, it implies (2) for P denoting the characteristic polynomials of both Ψ and $T : \mathcal{L} \to \mathcal{L}$. (Cf. the end of Section 1.) Thus, if (ii) holds, $T : \mathcal{L} \to \mathcal{L}$ must have determinant ± 1.

7. Nonexistence in dimension four

As mentioned in the Introduction, results of this section are used in the forthcoming paper [**8**].

If $F : V \to V$ is an endomorphism with the traceless part E in a two-dimensional real vector space V, then the traceless part of F^2 is $(\mathrm{tr}\, F)E$. In fact, the matrix of E in some basis is one of

$$\begin{bmatrix} \mu & 0 \\ 0 & -\mu \end{bmatrix}, \quad \begin{bmatrix} 0 & -\mu \\ \mu & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

with $\mu \in \mathbf{R}$. Hence E^2 is a multiple of 1 (the identity). The traceless and scalar parts of $F^2 = [E + (\mathrm{tr}\, F)/2]^2 = E^2 + (\mathrm{tr}\, F)E + (\mathrm{tr}\, F)^2/4$ thus are $(\mathrm{tr}\, F)E$ and $E^2 + (\mathrm{tr}\, F)^2/4$.

REMARK 7.1. If $\dot{\rho} + \psi\rho = \delta$ for C^1 functions $\rho, \psi, \delta : \mathbf{R} \to \mathbf{R}$, periodic with period $p > 0$, and $\delta \neq 0$ everywhere in $\mathbf{R}$, then $\rho \neq 0$ everywhere in $\mathbf{R}$. In fact, the derivative $\dot{\rho}$ has the same nonzero signum at each zero of ρ, and so ρ can have at most one zero in $\mathbf{R}$, while ρ with just one zero could not be periodic.

Lemma 7.2. *For $f, p, n, V, \langle\,,\rangle, A$ as in* (3) *with $n = 4$, and a C^∞ function $B : \mathbf{R} \to \mathrm{End}(V)$, periodic of period p, with $\dot{B} + B^2 = f + A$, the determinant of the translation operator $T : \mathcal{L} \to \mathcal{L}$, defined at the beginning of Section* 6, *is not equal to* ± 1.

Proof. The traceless part of the equality $\dot{B} + B^2 = f + A$ is $\dot{E} + (\mathrm{tr}\, B)E = A$, where $E(t)$ denotes the traceless part of $B(t)$. (See the beginning of this section.) Choosing $v, v' \in V$ with $\langle Av, v'\rangle \neq 0$ and setting $\rho(t) = \langle E(t)v, v'\rangle$, $\psi(t) = \mathrm{tr}\, B(t)$, we now obtain $\dot{\rho} + \psi\rho = \delta$, where $\delta = \langle Av, v'\rangle \neq 0$. Thus, $\rho \neq 0$ everywhere by Remark 7.1. Also, $\psi = (\delta - \dot{\rho})/\rho$, so that $\int_0^p \mathrm{tr}\, B(t)\, dt = \int_0^p \psi\, dt \neq 0$ (as $\int_0^p (\dot{\rho}/\rho)\, dt = 0$). Our claim now follows from Lemma 3.2(iii). q.e.d.

Theorem 7.3. *If $f, p, V, \langle\,,\rangle, A$ are as in* (3) *for $n = 4$, then, for $\widehat{M}, \mathrm{G}$ defined in Section* 2, *no subgroup Γ of G acts on $\widehat{M}$ properly discontinuously so as to produce a compact quotient manifold $M = \widehat{M}/\Gamma$.*

In fact, for such Γ we might assume (17.a). Since (i) implies (ii) in Theorem 6.1, it would follow from Remark 6.3 that the translation operator $T : \mathcal{L} \to \mathcal{L}$ has determinant ± 1, contrary to Lemma 7.2.

8. Proof that (ii) implies (i) in Theorem 6.1

Given B, Σ, φ and θ with the properties listed in (ii), along with Ψ, Ω as in (ii-b) and (ii-c), $\mathbf{R} \times \mathcal{L}$ with the operation $(q, u) \cdot (r, w) = (q + r - \Omega(u, w), u + w)$ forms a Lie group, which we denote by H. In fact, the embedding $(q, u) \mapsto (0, q, u)$ identifies H with a Lie subgroup of G, namely, $\Delta^{-1}(\mathcal{L})$ for Δ defined in (12).

In view of (7.v) and (7.ii), H is 2-step nilpotent or Abelian, while $\Psi : \mathrm{H} \to \mathrm{H}$ is a group automorphism (cf. (5)). By (ii-c) and (ii-a), Σ is a subgroup of H, which gives rise to the quotient manifold $N = \mathrm{H}/\Sigma$, with Σ acting on H by left translations.

Next, N is compact (a nilmanifold). In fact, suppose first that $\theta = 0$. By (ii-c), $\Omega(u, w) = 0$ for all $u, w \in \mathcal{L}$. (Namely, Σ spans $\mathbf{R} \times \mathcal{L}$, and so Λ spans $\mathcal{L}$.) Thus, H is the additive group of $\mathbf{R} \times \mathcal{L}$, and N is a torus. Now let $\theta > 0$. Since Σ is a lattice in $\mathbf{R} \times \mathcal{L}$, some compact set $Q' \subset \mathbf{R} \times \mathcal{L}$ intersects every orbit of Σ (where Σ acts by translations in the additive group $\mathbf{R} \times \mathcal{L}$), and so the image of Q' under the projection

$\mathbf{R}\times\mathcal{L}\to\mathcal{L}$ is a compact set $\widehat{Q}\subset\mathcal{L}$ intersecting every orbit of the translation group $\Lambda\subset\mathcal{L}$. Compactness of N follows: the compact set $Q=[0,\theta]\times\widehat{Q}\subset\mathrm{H}=\mathbf{R}\times\mathcal{L}$ intersects every orbit of Σ (now acting by left translations in H). To see this, consider the orbit of $(r,w)\in\mathrm{H}$. The property of $\widehat{Q}$, mentioned above, allows us to find $(q,u)\in\Sigma$ with $u+w\in\widehat{Q}$, and, for $l\in\mathbf{Z}$ such that $l\theta+q+r-\Omega(u,w)\in[0,\theta]$, we get $(\theta,0)^l\cdot(q,u)\cdot(r,w)\in Q$ (cf. (7.iii)), while $(\theta,0)\in\Sigma$ by (ii-a).

Since $\Omega:\mathcal{E}\times\mathcal{E}\to\mathbf{R}$ is nondegenerate (see the lines preceding (5)), for some $u\in\mathcal{E}$ the functional $-2\Omega(u,\,\cdot\,)$ restricted to $\mathcal{L}$ coincides with φ. Let us now fix any such u and any $\widetilde{q}\in\mathbf{R}$. For the unique $\widetilde{u}\in\mathcal{L}$ with $\widetilde{u}(0)=u(0)$, (5) gives $\Omega(T\widetilde{u},Tw)=\Omega(\widetilde{u},w)$, and so

$$\mathrm{H}\ni(r,w)\;\mapsto\;(r+\widetilde{q}+\Omega(\widetilde{u}-2u,w),T(w+\widetilde{u}))\in\mathrm{H}\tag{19}$$

describes the composite mapping in which the Lie-group automorphism $\Psi:\mathrm{H}\to\mathrm{H}$ is followed by the right translation by the element $(\widetilde{q},T\widetilde{u})$ of $\mathrm{H}=\mathbf{R}\times\mathcal{L}$. As $\Psi(\Sigma)=\Sigma$ (see (ii-b)) and $N=\mathrm{H}/\Sigma$, where Σ acts on H by *left* translations, (19) descends to the quotient nilmanifold $N=\mathrm{H}/\Sigma$, producing a diffeomorphism $\Phi:N\to N$.

We now define M to be the total space of a C^∞ bundle over the circle $\mathbf{R}/\mathbf{Z}$, choosing our $(n-1)$-dimensional nilmanifold N to be the fibre, and using the diffeomorphism $\Phi:N\to N$ to glue together the boundary components $\{0\}\times N$ and $\{p\}\times N$ of $[0,p]\times N$, which we treat as copies of N, while $\mathbf{R}/\mathbf{Z}$ is viewed as the result of identifying the two endpoints in $[0,p]$. Thus, $M=(\mathbf{R}\times N)/\mathbf{Z}$, where the action of $\mathbf{Z}$ on $\mathbf{R}\times N$ is given by $k(t,y)=(t+kp,\Phi^k(y))$ for $k\in\mathbf{Z}$ and $(t,y)\in\mathbf{R}\times N$, or, equivalently, generated by the diffeomorphism $(t,y)\mapsto(t+p,\Phi(y))$.

From now on we use the embedding $(q,u)\mapsto(0,q,u)$ to treat H and Σ as subgroups of G. Let Γ be the subgroup of G generated by Σ and the element $(1,q,u)$, with $q=\widetilde{q}+\langle u(0),\dot{u}(0)-B(0)u(0)\rangle$ and u chosen above. Also, let $\pi:\widehat{M}\to M$ be the locally diffeomorphic surjective mapping arising as the composite $\widehat{M}\to\mathbf{R}^2\times\mathcal{L}=\mathbf{R}\times\mathrm{H}\to\mathbf{R}\times N\to M$, in which the first arrow is the inverse of the equivariant diffeomorphism described in Remark 3.1, the second sends $(t,(r,w))$ to (t,ξ), where ξ stands for the coset $\Sigma\cdot(r,w)$ in $N=\mathrm{H}/\Sigma$, and the third one is the quotient projection $\mathbf{R}\times N\to M=(\mathbf{R}\times N)/\mathbf{Z}$.

It suffices to show that Γ acts on $\widehat{M}$ freely and the π-preimages of points of M coincide with the orbits of Γ. (See Section 1.) To this end, first note that, by (7.iv) and (ii-b), Σ is invariant under the inner automorphism corresponding to $(1,q,u)$, and so any $(l,r,w)\in\Gamma$, being a finite product of factors from the set $\Sigma\cup\{(1,q,u),(1,q,u)^{-1}\}$, equals $(1,q,u)^k\cdot(0,r',w')$ for some $k\in\mathbf{Z}$ and $(0,r',w')\in\Sigma$. Secondly,

($*$) under the equivariant diffeomorphism $\widehat{M} \to \mathbf{R} \times \mathrm{H}$ forming the first arrow in the composite $\pi : \widehat{M} \to M$, the actions (6.b) of H and Σ on $\widehat{M}$ correspond to their actions on $\mathbf{R} \times \mathrm{H}$ via left translations of the H factor. (Cf. Remark 3.1 and (8) with $k = 0$.)

The action of Γ on $\widehat{M}$ is free: if $(l, r, w) \in \Gamma$ has a fixed point, writing $(l, r, w) = (1, q, u)^k \cdot (0, r', w')$ as above, we obtain $\Pi(l, r, w) = k$, in view of (13); thus, $k = 0$ by (6.b), and $(l, r, w) \in \Sigma$, which shows that (l, r, w) must be the identity, since Σ acts on $\widehat{M}$ freely (see ($*$)).

Also, two elements of the same orbit of Γ in $\widehat{M}$ have the same image under π. In fact, diffeomorphisms $F : \widehat{M} \to \widehat{M}$ with $\pi \circ F = \pi$ form a group, which must contain Γ, as it contains both Σ and $(1, q, u)$. (The former, by ($*$); the latter, since the action (6.b) of $(1, q, u)$ sending $\widehat{M}_0 = \{0\} \times \mathbf{R} \times V$ onto $\widehat{M}_p = \{p\} \times \mathbf{R} \times V$ is easily verified to coincide with (19) if one identifies $\widehat{M}_0$ with $\{0\} \times \mathrm{H} \approx \mathrm{H}$ and $\widehat{M}_p$ with $\{p\} \times \mathrm{H} \approx \mathrm{H}$ via the diffeomorphism in ($*$), and uses the relation between $\mathcal{L}$ and B along with the definitions of T and Ω in Section 2.)

Finally, let $(t, s, v), (t', s', v') \in \widehat{M}$ and $\pi(t, s, v) = \pi(t', s', v')$. Thus, $t' = t - kp$ for some $k \in \mathbf{Z}$, and, replacing (t', s', v') by the product $(1, q, u)^k \cdot (t', s', v')$, we may assume that $t' = t$ (cf. (6.b)). Then, by ($*$), (t, s, v) and (t', s', v') lie in the same orbit of Σ.

9. The main lemma

Given $p \in (0, \infty)$, let $\mathcal{F}_p$ be the set of all septuples $(\alpha, \beta, \gamma, f, a, b, c)$ consisting of C^∞ functions $\alpha, \beta, \gamma, f : \mathbf{R} \to \mathbf{R}$ of the variable t, periodic of period p, and constants $a, b, c \in \mathbf{R}$ with $a + b + c = 0$ such that either $b < a < c$ or $b < c < a$, which satisfy, everywhere in $\mathbf{R}$, the inequalities $\alpha > \beta > \gamma$ and the ordinary differential equations

$$\dot{\alpha} + \alpha^2 = f + a, \qquad \dot{\beta} + \beta^2 = f + b, \qquad \dot{\gamma} + \gamma^2 = f + c. \tag{20}$$

We denote by $\mathcal{C}$ the subset of $\mathcal{F}_p$ formed by those $(\alpha, \beta, \gamma, f, a, b, c)$ in which the functions α, β, γ and f are all constant.

Remark 9.1. If $(\alpha, \beta, \gamma, f, a, b, c) \in \mathcal{F}_p$ and one of α, β, γ, f is constant, so are the other three. (In fact, a function α such that $\dot{\alpha} + \alpha^2$ is constant cannot be periodic unless it is constant.)

We now define a mapping $\operatorname{spec} : \mathcal{F}_p \to \mathbf{R}^3$ by $\operatorname{spec}(\alpha, \beta, \gamma, f, a, b, c) = (\lambda, \mu, \nu)$ for the unique $\lambda, \mu, \nu > 0$ such that

$$\log \lambda = -\int_0^p \alpha(t)\, dt, \quad \log \mu = -\int_0^p \beta(t)\, dt, \quad \log \nu = -\int_0^p \gamma(t)\, dt. \tag{21}$$

Lemma 9.2. *The image* $\operatorname{spec}(\mathcal{F}_p \smallsetminus \mathcal{C}) \subset \mathbf{R}^3$ *is the set of all* (λ, μ, ν) *satisfying conditions* (1).

We precede the proof of Lemma 9.2 with three other lemmas.

Remark 9.3. The most important part of Lemma 9.2 is the surjectivity claim, derived from a much stronger assertion. Namely, to conclude that the preimage of every triple (λ, μ, ν) with (1) under the mapping $\mathrm{spec} : \mathcal{F}_p \smallsetminus \mathcal{C} \to \mathbf{R}^3$ is nonempty, we show that it is, in fact, infinite-dimensional. See the final paragraph of this section.

Any two out of the three equations (20) can be solved as follows.

Lemma 9.4. *Let* $p \in (0, \infty)$, $a, b \in \mathbf{R}$ *and* $a \neq b$. *Triples* (α, β, f) *formed by* C^∞ *functions* $\mathbf{R} \to \mathbf{R}$, *periodic of period* p, *with* $\dot\alpha + \alpha^2 = f + a$ *and* $\dot\beta + \beta^2 = f + b$, *are in a natural bijective correspondence with* C^∞ *functions* $\rho : \mathbf{R} \to \mathbf{R}$ *which are periodic with period* p *and nonzero everywhere in* $\mathbf{R}$. *The correspondence is given by* $\rho = \alpha - \beta$, *and* ρ *determines* (α, β, f) *via the relations* $2\alpha = \rho + (a - b - \dot\rho)/\rho$, $2\beta = -\rho + (a - b - \dot\rho)/\rho$ *and* $f = \dot\alpha + \alpha^2 - a$.

(This is a trivial exercise: for $\rho = \alpha - \beta$ and $\psi = \alpha + \beta$, the equations $\dot\alpha + \alpha^2 = f + a$, $\dot\beta + \beta^2 = f + b$ give $\dot\rho + \psi\rho = a - b$, so that $\rho \neq 0$ everywhere by Remark 7.1.) We will use the notation

$$\mathcal{D} \;=\; \{(r, s) \in \mathbf{R}^2 : r > 0,\;\; s > 0,\;\; r \neq s\}. \tag{22}$$

Lemma 9.5. *The mapping* $(\alpha, \beta, \gamma, f, a, b, c) \mapsto (\rho, \sigma, r, s)$, *with* $\rho = \alpha - \beta$, $\sigma = \beta - \gamma$, $r = a - b$, *and* $s = c - b$, *sends the set* $\mathcal{F}_p$ *defined in Section* 9 *for any fixed* $p \in (0, \infty)$ *bijectively onto the set* $\mathcal{S}_p$ *of all quadruples* (ρ, σ, r, s) *in which* $(r, s) \in \mathcal{D}$ *and* ρ, σ *are positive* C^∞ *functions of* $t \in \mathbf{R}$, *periodic with period* p, *such that* $d\,[\log(\sigma/\rho)]/dt = \rho + \sigma - r\rho^{-1} - s\sigma^{-1}$. *The inverse mapping is given by* $a = (2r - s)/3$, $b = -(r+s)/3$, $c = (2s-r)/3$, $2\alpha = \rho + (r - \dot\rho)/\rho$, $2\beta = -\rho + (r - \dot\rho)/\rho$, $2\gamma = -\sigma - (s + \dot\sigma)/\sigma$ *and* $f = \dot\alpha + \alpha^2 - a$.

In fact, the condition $d\,[\log(\sigma/\rho)]/dt = \rho + \sigma - r\rho^{-1} - s\sigma^{-1}$ amounts to the equality between two expressions for 2β, obtained by applying Lemma 9.4 first to (α, β, f) and a, b, then to (β, γ, f) and b, c.

Lemma 9.6. *For any* $p \in (0, \infty)$, *the formula* $x = \log(\sigma/\rho)$ *defines a bijective correspondence* $(\rho, \sigma, r, s) \mapsto (x, r, s)$ *between the set* $\mathcal{S}_p$ *defined in Lemma* 9.5 *and* $\mathcal{X}_p \times \mathcal{D}$, *where* $\mathcal{X}_p$ *is the space of all* C^∞ *function* x *of the variable* $t \in \mathbf{R}$, *periodic with period* p, *and* $\mathcal{D}$ *is the set* (22). *In terms of* (x, r, s), *the quadruple* $(\rho, \sigma, r, s) \in \mathcal{S}_p$ *is given by* $2\rho = (1 + e^x)^{-1}[\dot x + \sqrt{\dot x^2 + 4(1 + e^x)(r + se^{-x})}\,]$ *and* $\sigma = e^x\rho$.

Proof. If $d\,[\log(\sigma/\rho)]/dt = \rho + \sigma - r\rho^{-1} - s\sigma^{-1}$ and $x = \log(\sigma/\rho)$, replacing σ in the equality $\dot x = \rho + \sigma - r\rho^{-1} - s\sigma^{-1}$ by $e^x\rho$, we get the required formula for ρ by solving a quadratic equation. q.e.d.

We now prove Lemma 9.2. First, Lemmas 9.5 and 9.6 yield a bijection $\mathcal{F}_p \to \mathcal{X}_p\times\mathcal{D}$, which we use to identify $\mathcal{F}_p$ and $\mathcal{X}_p\times\mathcal{D}$. This identification turns spec into a mapping $\mathcal{X}_p \times \mathcal{D} \to \mathbf{R}^3$ given by $\mathrm{spec}(x,r,s) = (\lambda,\mu,\nu)$ with $\lambda,\mu,\nu > 0$ characterized by $(4\log\lambda,\, 4\log\mu,\, 4\log\nu) = (-\delta-\varepsilon,\, \delta-\varepsilon,\, \delta-\varepsilon+2\zeta)$, where

$$\begin{aligned} \delta &= \textstyle\int_0^p(1+e^x)^{-1}[\dot x^2+4(1+e^x)(r+se^{-x})]^{1/2}\,dt,\\ \varepsilon &= \textstyle\int_0^p(1+se^{-x}/r)^{-1}[\dot x^2+4(1+e^x)(r+se^{-x})]^{1/2}\,dt,\\ \zeta &= \textstyle\int_0^p(1+e^{-x})^{-1}[\dot x^2+4(1+e^x)(r+se^{-x})]^{1/2}\,dt. \end{aligned} \tag{23}$$

Thus, δ,ε,ζ are all positive, $\varepsilon\ne\zeta$ (as $s\ne r$), and $\varepsilon<\delta+\zeta$ (since $(1+se^{-x}/r)^{-1}<1=(1+e^x)^{-1}+(1+e^{-x})^{-1}$). Consequently, λ,μ,ν satisfy (1). In other words, the values of $\mathrm{spec}:\mathcal{F}_p\to\mathbf{R}^3$ all lie in the open set $\mathcal{U}$ of all $(\lambda,\mu,\nu)\in\mathbf{R}^3$ with (1).

Furthermore, spec restricted to $\mathcal{C}\subset\mathcal{F}_p$ is a diffeomorphism $\mathcal{C}\to\mathcal{U}$. (This makes sense as $\mathcal{C}$ is a submanifold of $\mathbf{R}^7$, namely, the graph of a C^∞ mapping $\mathcal{C}'\ni(\alpha,\beta,\gamma)\mapsto(f,a,b,c)\in\mathbf{R}^4$, where $\mathcal{C}'$ is the open set in $\mathbf{R}^3$ formed by all (α,β,γ) such that $|\beta|<\alpha$ and $|\beta|<-\gamma\ne\alpha$.) In fact, by (21), $\mathrm{spec}:\mathcal{C}\to\mathcal{U}$ is the composite $\mathcal{C}\to\mathcal{C}'\to\mathcal{U}$ given by

$$(\alpha,\beta,\gamma,f,a,b,c)\mapsto(\alpha,\beta,\gamma)\mapsto(\lambda,\mu,\nu)=(e^{-p\alpha},e^{-p\beta},e^{-p\gamma}), \tag{24}$$

and $(\alpha,\beta,\gamma)\mapsto(\lambda,\mu,\nu)$ in (24) is a diffeomorphism $\mathcal{C}'\to\mathcal{U}$.

Finally, let $\mathcal{Y}$ be any vector subspace of $\mathcal{X}_p$ such that $2\le\dim\mathcal{Y}<\infty$ and $\mathcal{Y}$ contains the space of all constant functions (which we denote by $\mathbf{R}$). The subset $\mathcal{Y}\times\mathcal{D}$ of $\mathcal{X}_p\times\mathcal{D}$ is at the same time an open set in $\mathcal{Y}\times\mathbf{R}^2$, containing $\mathbf{R}\times\mathcal{D}$. At every $(x,r,s)\in\mathbf{R}\times\mathcal{D}$, the restriction of $\mathrm{spec}:\mathcal{X}_p\times\mathcal{D}\to\mathcal{U}$ to $\mathcal{Y}\times\mathcal{D}$ is a submersion, since, as we just verified, its own restriction to $\mathbf{R}\times\mathcal{D}$ is a diffeomorphism. (Under our identification $\mathcal{F}_p\approx\mathcal{X}_p\times\mathcal{D}$, the subset $\mathcal{C}$ of $\mathcal{F}_p$ corresponds to $\mathbf{R}\times\mathcal{D}$, since, by Lemma 9.6, x is constant if and only if both ρ and σ are, which, in view of Lemma 9.5, amounts to constancy of α,β,γ and f.)

The preimage of any given point $(\lambda,\mu,\nu)\in\mathcal{U}$ under the mapping $\mathrm{spec}:\mathcal{X}_p\times\mathcal{D}\to\mathcal{U}$ thus contains a submanifold of dimension $\dim\mathcal{Y}-1$ intersecting $\mathbf{R}\times\mathcal{D}$ at just one point, which completes the proof of Lemma 9.2. The assertion about infinite dimensionality in Remark 9.3 now follows, since $\dim\mathcal{Y}$ can be arbitrarily large.

10. Proof of Theorem 0.1

We fix $p\in(0,\infty)$ as well as $k,l\in\mathbf{Z}$ with $2\le k<l\le k^2/4$, and choose the corresponding λ,μ,ν with (1) as in Lemma 1.3. According to Lemma 9.2, there exist C^∞ functions $\alpha,\beta,\gamma,f:\mathbf{R}\to\mathbf{R}$, periodic of period p, and constants $a,b,c\in\mathbf{R}$ with $a+b+c=0$, which

satisfy equations (20) and (21), while f is nonconstant (Remark 9.1), $\alpha > \beta > \gamma$, and $(a,b,c) \neq (0,0,0)$ (since $a \neq b \neq c \neq a$).

Let $A_0 \in \mathrm{End}(V_0)$ and a C^∞ function $B_0 : \mathbf{R} \to \mathrm{End}(V_0)$, for a fixed 3-dimensional real vector space V_0, be defined by requiring A_0 and all $B_0(t)$ to be simultaneously diagonalized by some fixed basis of V_0, with the eigenvalues a,b,c and, respectively, $\alpha(t),\beta(t),\gamma(t)$. By declaring the above basis orthonormal, we now introduce in V_0 a pseudo-Euclidean inner product of *arbitrary* signature.

Our f,p,V_0,A_0 and the inner product thus are objects of type (3) with $n=5$, while $\dot{B}_0 + B_0^2 = f + A_0$. By Lemma 3.2, the space $\mathcal{L}_0$ of all solutions $u : \mathbf{R} \to V_0$ to the equation $\dot{u} = B_0 u$ is a Lagrangian subspace of the solution space $\mathcal{E}_0$ for the equation $\ddot{u} = fu + A_0 u$, and the the translation operator $T_0 : \mathcal{E}_0 \to \mathcal{E}_0$, given by $(T_0 u)(t) = u(t-p)$, leaves $\mathcal{L}_0$ invariant. According to Remark 3.3, $T_0 : \mathcal{L}_0 \to \mathcal{L}_0$ is diagonalizable with the eigenvalues λ,μ,ν characterized by (21), so that P appearing in Lemma 1.3 is its characteristic polynomial. As P satisfies (2), there exists a lattice Λ_0 in $\mathcal{L}_0$ with $T_0(\Lambda_0) = \Lambda_0$. (See the end of Section 1.)

Next, we generalize this construction from $n=5$ to $n=3j+2$, for any integer $j \geq 1$, using the original f and p, but replacing each of $V_0, A_0, B_0, \mathcal{E}_0, \mathcal{L}_0, T_0$ and Λ_0 by its jth Cartesian power $V, A, B, \mathcal{E}, \mathcal{L}, T$ and Λ. Now V, A and each $B(t)$ is the direct sum of j copies of V_0, A_0, or $B_0(t)$, so that $\mathcal{E}, \mathcal{L}$ and T arise in the same way from $\mathcal{E}_0, \mathcal{L}_0$ and T_0. (We represent direct sums by Cartesian products.) Thus, $T(\Lambda) = \Lambda$ for the lattice $\Lambda = \Lambda_0 \times \ldots \times \Lambda_0$ in $\mathcal{L} = \mathcal{L}_0 \times \ldots \times \mathcal{L}_0$. In V we choose a pseudo-Euclidean inner product $\langle\,,\rangle$ which is the orthogonal direct sum of inner products selected as above in each V_0 summand; the signature may vary from one summand to another, and so the resulting signature of $\langle\,,\rangle$ is completely arbitrary.

The objects f,p,V,A and $\langle\,,\rangle$ are again of type (3), this time with $n = 3j+2$, and $\Sigma = \mathbf{Z}\theta \times \Lambda$, for any fixed $\theta \in (0,\infty)$, is a lattice in the vector space $\mathcal{W} = \mathbf{R} \times \mathcal{L}$. Our B, Σ and θ, along with the zero functional φ, obviously satisfy condition (ii) in Theorem 6.1. (Each $B(t)$ is diagonalized by an orthonormal basis of V, so that $\mathcal{L}$ is Lagrangian, cf. Lemma 3.2(i), and $\Omega(u,w) = 0$ in (ii-c).)

Theorem 0.1 is now immediate. Specifically, by Theorem 6.1, there exists a subgroup $\Gamma \subset \mathrm{G}$ acting on $\widehat{M}$ freely and properly discontinuously, such that the quotient manifold $M = \widehat{M}/\Gamma$ is compact. Lemmas 2.1 and 2.2 imply that M carries a metric g with the properties required in Theorem 0.1. (The signature of the metric $\widehat{g}$ in (4) is the result of augmenting, by one plus and one minus, the sign pattern of h, that is, of $\langle\,,\rangle$.) The final clause of Theorem 0.1 is in turn a consequence of Remark 6.2 and Theorem 4.2(f), as $\Gamma = \pi_1 M$.

REMARK 10.1. The freedom of choosing $(\alpha, \beta, \gamma, f, a, b, c)$ is infinite-dimensional (Remark 9.3), which gives rise to an infinite-dimensional space of local-isometry types of the resulting metrics g. In fact, the Ricci tensor of the metric $\widehat{g}$ in (4) is a constant multiple of $f(t)\,dt \otimes dt$, and the 1-form dt is $\widehat{g}$-parallel [**9**, p. 93]. Therefore, the function f constitutes a local geometric invariant of $\widehat{g}$, defined up to affine changes of the variable t and multiplications of f by nonzero constants.

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