

# The metric structure of compact rank-one ECS manifolds

Andrzej Derdzinski and Ivo Terek

ABSTRACT. Pseudo-Riemannian manifolds with nonzero parallel Weyl tensor which are not locally symmetric are known as ECS manifolds. Every ECS manifold carries a distinguished null parallel distribution  $\mathcal{D}$ , the rank  $d \in \{1, 2\}$  of which is referred to as the rank of the manifold itself. Under a natural genericity assumption on the Weyl tensor, we fully describe the universal coverings of compact rank-one ECS manifolds. We then show that any generic compact rank-one ECS manifold must be *translational*, in the sense that the holonomy group of the natural flat connection induced on  $\mathcal{D}$  is either trivial or isomorphic to  $\mathbb{Z}_2$ . We also prove that all four-dimensional rank-one ECS manifolds are noncompact, this time without assuming genericity, as it is always the case in dimension four.

## Introduction and main results

A pseudo-Riemannian manifold of dimension  $n \geq 4$  whose Weyl tensor is parallel is referred to as *conformally symmetric* [6], and it is called *essentially conformally symmetric* (briefly, *ECS*) [27] if, in addition, it is neither conformally flat nor locally symmetric. It was shown by Roter that ECS manifolds exist in every dimension [27, Corollary 3] and that they all have indefinite metric signatures [8, Theorem 2]. The local structure of ECS manifolds is fully known [10].

Conformal symmetry of  $(M, g)$  is one of the *natural linear conditions* imposed on the covariant derivatives of the  $SO(p, q)$ -irreducible components of its curvature tensor, in the sense of Besse [2, Chapter 16]. The interest in this subject is reflected in more work by other authors: Cahen and Kerbrat [3, Section 2], Hotloś [21], Mantica and Suh [25, Section 3], Schliebner [28], and Deszcz et al. in [17, Sect. 4], [16, Theorem 6.1]. The techniques used in the study of ECS manifolds are themselves also of interest, appearing in [18, Example 2.2], [29], [1, Theorem 3], [4], [5, Theorem 3.9], [22, Lemma 3], [24], [31, proofs of Theorems 1.1 and 4.5] and [30].

As shown by Olszak [26], every ECS manifold  $(M, \mathfrak{g})$  carries a distinguished null parallel distribution  $\mathcal{D}$ , whose sections are the vector fields corresponding under  $\mathfrak{g}$  to 1-forms  $\xi$  with  $\xi \wedge [W(v', v'', \cdot, \cdot)] = 0$  for all vector fields  $v', v''$ . The rank of  $\mathcal{D}$  – always equal to 1 or 2 – is referred to as the *rank of  $(M, \mathfrak{g})$*  [15]. In the rank-one case, the focus of this paper, we also call the ECS manifold in question

(\*) *translational or dilational*, depending on whether the holonomy group of the natural flat connection induced on  $\mathcal{D}$  is finite or infinite.

In [13] and [11], compact rank-one ECS manifolds of every dimension  $n \geq 5$  were constructed as suitable quotients of what we call *model ECS manifolds* (see Section 6). All such compact examples are geodesically complete and translational, but none of them is locally homogeneous. On the other hand, we show in [14] that, under a natural *genericity* assumption on the Weyl tensor, quotients of dilational model ECS manifolds cannot be compact unless they are locally homogeneous. More precisely, *genericity* refers to a certain self-adjoint endomorphism  $A$  of the vector space of parallel sections of  $\mathcal{D}^\perp/\mathcal{D}$ , and it means that only finitely many of its isometries commute with  $A$  (see also Section 4 and the end of Section 5). We have recently found [12] compact dilational examples in all odd dimensions  $n \geq 5$ , including locally homogeneous ones. They are all nongeneric and incomplete.

It is still not known whether a compact ECS manifold can be four-dimensional, or have rank two.

This paper provides partial answers to the above questions. We start with structure theorems, calling a rank-one ECS manifold  $\mathcal{D}^\perp$ -*complete* if all the leaves of  $\mathcal{D}^\perp$  are complete relative to the induced connections (this definition makes sense for any foliation with totally geodesic leaves, such as  $\mathcal{D}^\perp$  itself), and *maximally complete* if every non-complete maximal geodesic in its universal covering intersects all leaves of  $\mathcal{D}^\perp$ . Note that maximal completeness implies  $\mathcal{D}^\perp$ -completeness, while (due to (5.2) below) it follows from completeness.

**THEOREM A.** *Any compact  $\mathcal{D}^\perp$ -complete rank-one ECS manifold is necessarily maximally complete.*

**THEOREM B.** *Every generic compact rank-one ECS manifold is both maximally complete and  $\mathcal{D}^\perp$ -complete.*

**THEOREM C.** *Any simply connected and maximally complete rank-one ECS manifold is isometric to a model ECS manifold.*

The next result is a trivial consequence of Theorems B and C.

**COROLLARY D.** *The universal covering of any generic compact rank-one ECS manifold is isometric to a model ECS manifold.*

For the Lorentzian signature, Schliebner [28] proved this last conclusion without assuming genericity. From Corollary D, we obtain the following strengthened version of [14, Theorem C], which refers to the dichotomy (\*):

**THEOREM E.** *Every generic compact rank-one ECS manifold is translational, as well as geodesically complete, and it cannot be locally homogeneous.*

Let us point out that Theorem E does not replace [14, Theorem C], but rather relies on it, since the latter is needed to prove the former. As we point out in Section 11, Corollary D combined with [11, Theorem 8.1] trivially leads to:

**COROLLARY F.** *Four-dimensional rank-one ECS manifolds are noncompact.*

In other words, if four-dimensional compact ECS manifolds do exist, they must necessarily be of rank two.

### How the paper is organized

Unless stated otherwise, all manifolds, bundles, connections, mappings, and tensor fields are assumed to be smooth. The text is divided into two parts.

**Part I.** After Sections 1–3 dealing with preliminaries, in Section 4 we elaborate on the meaning of genericity. Sections 5 and 6 lay the groundwork for proving Theorems A and C, summarizing what is already known about the structure of the universal coverings of compact rank-one ECS manifolds, and describing our model ECS manifolds. In Section 7 we prove Theorems A–C, adapting to our situation the proofs of some weaker results from [9] (namely, Lemma 7.3 and Theorem 7.1 therein). Further details are provided in Appendices A and B.

**Part II.** Section 8 introduces the *transitive-commutation property* for a group-subgroup pair, crucial for understanding the structure of the isometry group of a locally homogeneous model ECS manifold. The focus on the locally homogeneous case is justified by [14, Theorem C], where we prove that a generic dilational compact rank-one ECS manifold must be locally homogeneous. Section 9 presents what we call *standard homogeneous rank-one ECS model manifolds*, used in Section 10 to prove Theorem E.

### 1. Completeness of connections

LEMMA 1.1. *If  $\nabla$  is a connection on a manifold  $L$ , while  $X$  and  $Z$  are vector fields along a curve defined on an open interval  $J \subseteq \mathbb{R}$  of the variable  $s$  containing 0, and*

$$(1.1) \quad \nabla_s Z = \nabla_s \nabla_s X = 0, \quad X(0) = Z(0), \quad [\nabla_s X](0) = -Z(0),$$

*then  $X(s) = (1 - s)Z(s)$  for all  $s \in J$ .*

In fact,  $s \mapsto X(s) = (1 - s)Z(s)$  satisfies (1.1), and we may invoke the uniqueness of solutions of systems of second-order ordinary differential equations.

The next Lemma generalizes [9, Lemma 1.4] and is used in Section 7 to prove Theorem B. In its proof (and later in Appendix A) we adopt the notational convention of [9, end of Sect. 1]: given a *variation of curves* in a manifold  $M$ , that is, a  $C^\infty$  mapping  $(t, s) \mapsto x(s, t)$  from an open set in  $\mathbb{R}^2$  into  $M$ , and a connection on  $M$ , we denote by  $x_s, x_t$  (or,  $x_{ss}, x_{st}, x_{tss}$ , etc.) its partial (or, partial covariant) derivatives of orders 1, 2, 3 etc., all of which are vector fields along the variation, meaning, as usual, sections of the corresponding pullback of  $TM$ . When the connections involved are flat and torsion-free,

$$(1.2) \quad \text{all such derivatives depend symmetrically on the subscripts.}$$

LEMMA 1.2. *Let  $\mathcal{P}$  be a distribution on a manifold  $L$ ,  $\nabla$  be a connection on  $L$ , and assume that:*

- (a)  $\nabla$  is flat and torsionfree,
- (b)  $\mathcal{P}$  is trivialized by a vector space  $\mathcal{X}$  of complete parallel vector fields,
- (c) there is a vector space  $\mathcal{Y}$  of complete vector fields on  $L$  which is isomorphically mapped via the quotient projection onto a vector space of sections trivializing the quotient bundle  $TL/\mathcal{P}$  over  $L$ , and parallel relative to the connection induced on  $TL/\mathcal{P}$ .

*Then  $\nabla$  is complete.*

PROOF. The evaluation  $\mathcal{X} \rightarrow \mathcal{P}_z$  at each  $z \in L$  is an isomorphism, and so every  $\nabla$ -geodesic of  $L$  starting tangent to  $\mathcal{P}$  is complete, such a geodesic being an integral curve of a complete parallel vector field on  $L$ .

Given  $z \in L$  and  $\mathbf{v} \in \mathcal{Y}$ , let  $y: \mathbb{R} \rightarrow L$  be the integral curve of  $\mathbf{v}$  with  $y(0) = z$ . As  $\nabla_{\mathbf{v}} \mathbf{v}$  is always tangent to  $\mathcal{P}$ , we may choose  $\zeta, \eta: \mathbb{R} \rightarrow \mathcal{X}$  with

$$(1.3) \quad [\zeta(t)]_{y(t)} = [\nabla_{\mathbf{v}} \mathbf{v}]_{y(t)} = \dot{y}(t), \quad \dot{\eta} = -\zeta.$$

Each  $\eta(t)$  is complete, and the variation  $\mathbb{R}^2 \ni (t, s) \mapsto x(t, s) = e^{s\eta(t)} y(t)$  has

$$(1.4) \quad x(t, 0) = y(t), \quad x_s(t, 0) = [\eta(t)]_{y(t)}, \quad x_{ss}(t, s) = 0.$$

Hence  $x_{tt}(t, s) = (1 - s)[\zeta(t)]_{x(t, s)}$  for all  $(t, s) \in \mathbb{R}^2$ , as one sees applying Lemma 1.1 to  $\mathbf{Z}(s) = [\zeta(t)]_{x(t, s)}$  and  $\mathbf{X}(s) = x_{tt}(t, s)$ , with fixed  $t$ , the equalities (1.1) being immediate from (1.2)–(1.4). (Note that  $\nabla[\zeta(t)] = 0$  and  $x_{ttss} = x_{sstt} = 0$ .) In particular,  $t \mapsto x(t, 1) = e^{\eta(t)}y(t)$  is a complete geodesic whose initial velocity is, when  $\eta(0) = 0$ , equal to  $\mathbf{v}_z + \dot{\eta}(0)$ . However, every vector in  $T_zM$  is of this form for suitable  $\mathbf{v}$  and  $\eta$ , as the values  $\mathbf{v}_z$  realize all values at  $z$  in the complementary subbundle to  $\mathcal{P}$  spanned by  $\mathcal{Y}$ , while the values  $\dot{\eta}(0)$  realize all elements of  $\mathcal{P}_z$ .  $\square$

LEMMA 1.3. *If  $\nabla$  is a complete connection on a manifold  $L$  and a non-constant function  $f: L \rightarrow \mathbb{R}$  has  $\nabla df = 0$ , then  $f$  is surjective.*

Namely, along a maximal geodesic,  $f$  is an affine function of its parameter.

## 2. Properly discontinuous $\mathbb{R}^k$ -subactions

Three well-known facts are phrased here as a remark for easy reference.

REMARK 2.1. First, the composition of two fibrations (including covering projections) is clearly a fibration. Secondly, if a Lie group  $G$  acts on a manifold  $\widehat{M}$  with a subgroup  $\Gamma$  acting on  $\widehat{M}$  freely and properly discontinuously, then  $\Gamma$  is a discrete subset of  $G$ . Finally, whenever a compact topological manifold is contractible, it consists of a single point. (Otherwise, it would have a nontrivial top  $\mathbb{Z}_2$  cohomology group.)

LEMMA 2.2. *If  $\mathbb{R}^k$  acts freely on a contractible manifold  $\widehat{M}$  and a subgroup  $\Gamma$  of  $\mathbb{R}^k$  acts on  $\widehat{M}$  properly discontinuously with a compact quotient  $\widehat{M}/\Gamma$ , then  $k = \dim \widehat{M}$ , the action of  $\mathbb{R}^k$  on  $\widehat{M}$  is simply transitive, and  $\Gamma$  is a lattice in  $\mathbb{R}^k$ . Consequently,  $\widehat{M}$  and  $\widehat{M}/\Gamma$  are, respectively, an affine  $k$ -space and a  $k$ -dimensional torus.*

PROOF. As  $\Gamma$  is a discrete subset of  $\mathbb{R}^k$  (Remark 2.1), it forms a lattice in the subspace  $Y \subseteq \mathbb{R}^k$  which it spans and, due to commutativity, the action of  $\mathbb{R}^k$  on  $\widehat{M}$  descends to a free action of the torus  $Y/\Gamma$  on  $\widehat{M}/\Gamma$  which, according to [19, Corollary 4.2.11, p. 213], turns  $\widehat{M}/\Gamma$  into the total space of a principal torus bundle over some compact base  $B$ . By Remark 2.1, the composition  $\widehat{M} \rightarrow \widehat{M}/\Gamma \rightarrow B$  is a bundle projection with the fibre  $Y$ , and its homotopy long exact sequence [20, Theorem 4.49, p. 376] implies that  $B$  has trivial homotopy groups, being therefore contractible [23, Lemma 2.1]. Due to Remark 2.1,  $B$  consists of a single point and the resulting relations  $\dim \widehat{M} = \dim Y \leq k \leq \dim \widehat{M}$ , the last one immediate since the action of  $\mathbb{R}^k$  is free, yield our assertion.  $\square$

### 3. Spectra of endomorphisms

Given an endomorphism  $B$  of a  $k$ -dimensional vector space  $\mathcal{X}$ , by the *spectrum* of  $B$  we mean the unordered system  $\beta(1), \dots, \beta(k)$  formed by the complex characteristic roots of  $B$  listed with their multiplicities. If  $B = [d\sigma_q/dq]_{q=1}$  is the infinitesimal generator of a Lie-group homomorphism  $(0, \infty) \ni q \mapsto \sigma_q \in \text{GL}(\mathcal{X})$  and the spectrum of each  $\sigma_q$  is  $q^{\alpha(1)}, \dots, q^{\alpha(k)}$ , with  $q^{\alpha(j)} = q^{\text{Re } \alpha(j)} e^{i(\log q)\text{Im } \alpha(j)}$ , where  $\alpha(j) \in \mathbb{C}$  do not depend on  $q$ , for  $j = 1, \dots, k$ , then

$$(3.1) \quad B \text{ has the spectrum } \alpha(1), \dots, \alpha(k).$$

In fact, the complex-linear extension of  $B$  to  $\mathcal{X}^{\mathbb{C}}$  has, in some basis, an upper triangular matrix with the diagonal entries  $\beta(1), \dots, \beta(k)$  forming the spectrum of  $B$ . Thus  $\sigma_q = \exp[(\log q)B]$  has the spectrum  $q^{\beta(j)}$ ,  $j = 1, \dots, k$ , and, up to a rearrangement,  $q^{\beta(j)} = q^{\alpha(j)}$ . Hence  $[\beta(j) - \alpha(j)] \log q \in 2\pi i\mathbb{Z}$  and so  $\beta(j) = \alpha(j)$ .

It is a trivial fact from linear algebra that, whether  $\mathcal{X}$  is finite-dimensional or not, every family  $\mathcal{F}$  of eigenvectors of an endomorphism  $\Psi \in \text{End}(\mathcal{X})$  corresponding to mutually distinct eigenvalues is linearly independent. As a consequence:

**LEMMA 3.1.** *Given  $\Psi$  and  $\mathcal{F}$  as above, let  $(x_\alpha)_{\alpha \in A}$  be an indexed family of vectors such that  $x_\alpha \in \mathcal{F}$  whenever  $\alpha \in A$ . If  $A_0 \subseteq A$  is a nonempty finite set with the property that  $\sum_{\alpha \in A_0} x_\alpha \in \ker \Psi$ , then  $x_\alpha \in \ker \Psi$  for every  $\alpha \in A_0$ .*

**PROOF.** There are positive integers  $n$  and  $k_1, \dots, k_n$ , as well as  $\alpha_1, \dots, \alpha_n \in A$ , such that  $\sum_{\alpha \in A_0} x_\alpha = \sum_{i=1}^n k_i x_{\alpha_i}$ , where  $x_{\alpha_i} \neq x_{\alpha_j}$  whenever  $i \neq j$ . If  $\lambda_i$  is the eigenvalue of  $\Psi$  associated with  $x_{\alpha_i}$ , it follows that  $\sum_{i=1}^n k_i \lambda_i x_{\alpha_i} = 0$ , whence  $\lambda_i = 0$ . Therefore  $n = 1$  and  $\lambda_1 = 0$ .  $\square$

The assumption and conclusion of Lemma 3.1 apply to  $\Psi = q d/dq$  in the space of all complex-valued  $C^\infty$  functions of the variable  $q \in (0, \infty)$  and the family  $\mathcal{F}$  formed by all the power functions  $(0, \infty) \ni q \mapsto q^{a+bi} = q^a e^{ib \log q}$  with  $a, b \in \mathbb{R}$ . The proof of Theorem E, in Section 10, uses the following consequence:

$$(3.2) \quad \begin{array}{l} \text{the sum of several terms of the form } q^{a+bi} \text{ can be constant} \\ \text{as a function of } q \text{ only if each term in the sum is } q^0 = 1. \end{array}$$

### 4. Generic endomorphisms

Throughout this section, we let  $(V, \langle \cdot, \cdot \rangle)$  be a pseudo-Euclidean vector space of dimension  $m$ , denote by  $\mathcal{A}$  the space of all traceless self-adjoint endomorphisms of  $V$ , and say that  $A \in \mathcal{A}$  is *generic* if only finitely many linear isometries of

$(V, \langle \cdot, \cdot \rangle)$  commute with  $A$ . For  $m = 2$ , unless  $A = 0$ , there are at most four linear isometries commuting with  $A$ , cf. [9, Remark 6.2], and so

$$(4.1) \quad \text{every } A \in \mathcal{A} \setminus \{0\} \text{ is generic when } m = 2.$$

In all dimensions  $m$ , generic endomorphisms always exist. In fact, any  $A \in \mathcal{A}$  with  $m$  distinct eigenvalues is generic, since its  $m$  eigenlines are mutually orthogonal and hence nondegenerate. Furthermore,

$$(4.2) \quad \text{the set of generic endomorphisms is an open and dense subset of } \mathcal{A}.$$

Indeed, note that an endomorphism  $A \in \mathcal{A}$  is generic if and only if its isotropy group  $G_A$  under the action of  $O(V, \langle \cdot, \cdot \rangle)$  on  $\mathcal{A}$  by conjugation is finite. However, finiteness of  $G_A$  amounts to its being countable, since  $G_A$ , given by the polynomial equation  $CAC^* = A$ , is an algebraic variety and so it has finitely many connected components [32, Theorem 3]. Consequently, genericity of  $A$  is equivalent to triviality of its isotropy algebra, which in turn means that the rank of  $F(A)$  equals  $r$ , for  $r = \dim O(V, \langle \cdot, \cdot \rangle)$  and  $F: \mathcal{A} \rightarrow \text{Hom}(\mathfrak{so}(V, \langle \cdot, \cdot \rangle), \mathcal{A})$  given by  $F(A)(B) = [A, B]$ . We now fix some generic endomorphism  $A_0 \in \mathcal{A}$ , some bases of  $\mathfrak{so}(V, \langle \cdot, \cdot \rangle)$  and  $\mathcal{A}$ , and a nonzero  $r \times r$  subdeterminant of the matrix representing  $F(A_0)$  in these bases. By analyticity, such a subdeterminant is nonzero on an open and dense subset of  $\mathcal{A}$ , thus proving (4.2).

In [14, Section 5] we show that a nilpotent endomorphism  $A \in \mathcal{A}$  is generic if and only if  $A^{m-1} \neq 0$ , in which case there is a basis  $(v_1, \dots, v_m)$  of  $V$  such that

$$(4.3) \quad \begin{aligned} &Av_j = v_{j-1} \text{ and } \langle v_i, v_k \rangle = \varepsilon \delta_{ij} \text{ for all } i, j \in \{1, \dots, m\}, \text{ where } \varepsilon = \pm 1 \\ &\text{is the semi-definiteness sign of } \langle A^{m-1} \cdot, \cdot \rangle, k = m+1-j, \text{ and } v_0 = 0. \end{aligned}$$

In addition, such a basis is also unique up to an overall sign change.

It follows [14, Corollary 5.3] that for every  $q \in (0, \infty)$ ,

$$(4.4) \quad \begin{aligned} &\text{there are only two linear isometries } C, -C \text{ of } (V, \langle \cdot, \cdot \rangle) \\ &\text{with } CAC^{-1} = q^2A \text{ and, in a basis satisfying (4.3), they are} \\ &\text{given by } Cv_j = \delta q^{m+1-2j}v_j, \text{ for some sign factor } \delta = \pm 1. \end{aligned}$$

## 5. The universal coverings

In this section we fix a rank-one ECS manifold  $(M, \mathfrak{g})$  of dimension  $n \geq 4$  with arbitrary indefinite metric signature, and let  $\Gamma$  be the fundamental group of  $M$ . Consider the universal covering projection  $\pi: \tilde{M} \rightarrow M$ , and set  $\tilde{\mathfrak{g}} = \pi^*\mathfrak{g}$ , so that  $(\tilde{M}, \tilde{\mathfrak{g}})$  is a simply connected rank-one ECS manifold on which  $\Gamma$  acts freely and properly discontinuously by isometries, with quotient  $M = \tilde{M}/\Gamma$ . We will also write  $\tilde{\mathcal{D}}$  for the Olszak distribution of  $(\tilde{M}, \tilde{\mathfrak{g}})$ , defined in the Introduction.

As the Levi-Civita connection of  $(\tilde{M}, \tilde{g})$  induces a connection on  $\tilde{\mathcal{D}}$ , and the latter is flat [10, Lemma 2.2(f)], simple connectivity of  $\tilde{M}$  allows us to fix

$$(5.1) \quad \begin{array}{l} \text{a null parallel vector field } \mathbf{w} \text{ spanning } \tilde{\mathcal{D}}, \text{ leading to a surjective} \\ \text{function } t: \tilde{M} \rightarrow I \text{ onto an open interval } I \subseteq \mathbb{R}, \text{ with } dt = \tilde{g}(\mathbf{w}, \cdot). \end{array}$$

In addition, as shown in [15, end of Section 12],

$$(5.2) \quad \text{the leaves of } \tilde{\mathcal{D}}^\perp \text{ coincide with the level sets of } t: \tilde{M} \rightarrow I.$$

By Lemma 1.3,  $(M, g)$  is incomplete when  $I \neq \mathbb{R}$ . Moreover, as the Olszak distribution is a local geometric invariant of the given ECS metric,  $t$  in (5.1) is unique up to affine substitutions, and so for every  $\gamma \in \text{Iso}(\tilde{M}, \tilde{g})$  there is  $(q, p) \in \text{Aff}(\mathbb{R})$  such that  $t \circ \gamma = qt + p$ , giving rise to two homomorphisms:

$$(5.3) \quad \text{a) } \text{Iso}(\tilde{M}, \tilde{g}) \ni \gamma \mapsto (q, p) \in \text{Aff}(\mathbb{R}), \quad \text{b) } \text{Iso}(\tilde{M}, \tilde{g}) \ni \gamma \mapsto q \in \mathbb{R} \setminus \{0\}.$$

The following principle will be repeatedly used:

$$(5.4) \quad \begin{array}{l} \text{replacing } \Gamma \text{ with a finite index subgroup } \Gamma_0 \text{ amounts to replacing } M \\ \text{with the quotient } \tilde{M}/\Gamma_0, \text{ which is also compact (as the total space of} \\ \text{a finite-sheeted covering of } M) \text{ and has } \tilde{M} \text{ as its universal covering.} \end{array}$$

Using (5.4), we from now assume that

$$(5.5) \quad \text{the image of } \Gamma \text{ under (5.3-b) is contained in } (0, \infty).$$

In [15, Section 12], we show that, if (5.5) holds and  $M$  is compact,

$$(5.6) \quad \begin{array}{l} \text{a) there exists a smooth positive function } \psi \text{ on } \tilde{M} \text{ such} \\ \text{that the 1-form } \psi dt \text{ is closed and } \Gamma\text{-invariant, and} \\ \text{b) the vector field } \mathbf{w} \text{ in (5.1) is complete.} \end{array}$$

With  $q$  related to  $\gamma$  as in (5.3-b), these  $\psi$  and  $\mathbf{w}$  satisfy the conditions

$$(5.7) \quad \psi \circ \gamma = q^{-1}\psi \text{ and } \gamma_*\mathbf{w} = q^{-1}\mathbf{w}, \text{ for every } \gamma \in \Gamma,$$

due to the relation  $\gamma^*(dt) = q dt$  and  $\Gamma$ -invariance of  $\psi dt$ .

The Levi-Civita connection of  $(\tilde{M}, \tilde{g})$  induces one on the quotient bundle  $\tilde{\mathcal{D}}^\perp/\tilde{\mathcal{D}}$ , which is flat by [10, Lemma 2.2(f)]. Thus,  $\tilde{M}$  being simply connected, the real vector space  $V$  of parallel sections of  $\tilde{\mathcal{D}}^\perp/\tilde{\mathcal{D}}$  has the full dimension  $m = n - 2$ . The space  $V$  also inherits a natural pseudo-Euclidean inner product  $\langle \cdot, \cdot \rangle$  from  $\tilde{g}$ , and the Weyl tensor  $W$  of  $(\tilde{M}, \tilde{g})$  induces, cf. [9, Section 4],

$$(5.8) \quad \begin{array}{l} \text{a traceless self-adjoint operator } A: V \rightarrow V, \\ \text{given by } A(\mathbf{v} + \tilde{\mathcal{D}}) = W(\mathbf{u}, \mathbf{v})\mathbf{u} + \tilde{\mathcal{D}}, \text{ where } \mathbf{u} \\ \text{is any vector field on } \tilde{M} \text{ such that } \tilde{g}(\mathbf{u}, \mathbf{w}) = 1. \end{array}$$



Clearly,  $\mathbf{u}$  in (5.8) is unique modulo  $\tilde{\mathcal{D}}^\perp$ , so that  $A(X + \tilde{\mathcal{D}})$  is well-defined and, by (5.7),  $\gamma_*\mathbf{u} + \tilde{\mathcal{D}} = q\mathbf{u} + \tilde{\mathcal{D}}$  for every  $\gamma \in \Gamma$ . Every  $\gamma \in \Gamma$  induces a linear isometry  $C: V \rightarrow V$ , acting via  $C(\mathbf{v} + \tilde{\mathcal{D}}) = \gamma_*\mathbf{v} + \tilde{\mathcal{D}}$ , which leads to

$$(5.9) \quad \text{a homomorphism } \Gamma \ni \gamma \mapsto C \in \text{O}(V, \langle \cdot, \cdot \rangle) \text{ with } CAC^{-1} = q^2A,$$

$q$  being associated with  $\gamma$  as in in (5.3-b).

We will say that  $(M, \mathfrak{g})$  itself is *generic* if  $A$  in (5.8) is generic in the sense of Section 4. By (4.1),  $(M, \mathfrak{g})$  is always generic when  $n = 4$ .

## 6. The rank-one ECS models and their isometry groups

Rank-one ECS models are built from the following data, cf. [27]:

$$(6.1) \quad \begin{array}{l} \text{an integer } n \geq 4, \text{ a pseudo-Euclidean vector space} \\ (V, \langle \cdot, \cdot \rangle) \text{ of dimension } n - 2, \text{ a self-adjoint endomor-} \\ \text{phism } A \in \mathfrak{sl}(V) \setminus \{0\}, \text{ and a nonconstant smooth} \\ \text{function } f: I \rightarrow \mathbb{R} \text{ defined on an open interval } I \subseteq \mathbb{R}. \end{array}$$

Then, defining  $\kappa: I \times \mathbb{R} \times V \rightarrow \mathbb{R}$  by  $\kappa(t, s, v) = f(t)\langle v, v \rangle + \langle Av, v \rangle$  and regarding  $\langle \cdot, \cdot \rangle$  as a constant flat metric on  $V$ , we consider the simply connected  $n$ -dimensional pseudo-Riemannian manifold

$$(6.2) \quad (\widehat{M}, \widehat{\mathfrak{g}}) = (I \times \mathbb{R} \times V, \kappa dt^2 + dt ds + \langle \cdot, \cdot \rangle),$$

where we identify  $dt, ds$  and  $\langle \cdot, \cdot \rangle$  with their pull-backs to  $\widehat{M}$ .

By [10, Theorem 4.1]  $(\widehat{M}, \widehat{\mathfrak{g}})$  is a rank-one ECS manifold. Calling the manifolds (6.2) *models* is justified by their being locally *universal*:

$$(6.3) \quad \begin{array}{l} \text{every point of a rank-one ECS manifold of dimension } n \text{ has a neigh-} \\ \text{borhood isometric to an open subset of a manifold of type (6.2), with} \\ \text{one possible exception in (6.1): } f \text{ may be constant [10, Theorem 4.1].} \end{array}$$

Our two uses of the term ‘generic’ are mutually consistent:

$$(6.4) \quad \begin{array}{l} \text{genericity of } (\widehat{M}, \widehat{\mathfrak{g}}) \text{ – see the end of Section 5 – is equivalent} \\ \text{to that of the endomorphism } A \text{ in (6.1) as defined in Section 4.} \end{array}$$

Indeed, the Olszak distribution  $\widehat{\mathcal{D}}$  of  $(\widehat{M}, \widehat{\mathfrak{g}})$  – defined in the Introduction – is spanned by the null parallel coordinate vector field  $\partial_s$  [27, p. 93], so that the leaves of  $\widehat{\mathcal{D}}^\perp$  are the  $\mathbb{R} \times V$  factor submanifolds of  $\widehat{M}$  in (6.2). This allows us to isometrically identify  $(V, \langle \cdot, \cdot \rangle)$  with the space of parallel sections of  $\widehat{\mathcal{D}}^\perp/\widehat{\mathcal{D}}$ , which, as shown in [15, the lines following (7.3)], also identifies  $A$  in (6.1) with  $A$  in (5.8) (where one may set  $\mathbf{u} = 2\partial_t$ ).

Central to the discussion are: the  $2(n-2)$ -dimensional symplectic vector space

$$(6.5) \quad (\mathcal{E}, \Omega) \text{ consisting of all solutions } u: I \rightarrow V \text{ of the second-order equation } \ddot{u} = fu + Au, \text{ where } \Omega \text{ is defined by } \Omega(u, w) = \langle \dot{u}, w \rangle - \langle u, \dot{w} \rangle, \text{ and its associated Heisenberg group } H: \text{ the Cartesian product } \mathbb{R} \times \mathcal{E} \text{ with the operation defined by } (r, u)(\hat{r}, \hat{u}) = (r + \hat{r} - \Omega(u, \hat{u}), u + \hat{u}).$$

We also need

$$(6.6) \quad \text{the subgroup } S \text{ of } \text{Aff}(\mathbb{R}) \times O(V, \langle \cdot, \cdot \rangle) \text{ formed by all } (q, p, C) \text{ having } CAC^{-1} = q^2A, \text{ with } qt + p \in I \text{ and } f(t) = q^2f(qt + p) \text{ for all } t \in I.$$

Each of  $q$ ,  $(q, p)$ , and  $C$  depends homomorphically on  $\sigma = (q, p, C)$ , so that  $S$  acts (from the left) on  $I$ ,  $\mathbb{R}$ , and  $C^\infty(I, V)$  via, respectively,

$$(6.7) \quad \text{i) } \sigma t = qt + p, \quad \text{ii) } \sigma s = q^{-1}s, \quad \text{iii) } (\sigma u)(t) = Cu(q^{-1}(t - p)).$$

As the notations in (6.7-i) and (6.7-ii) are in conflict, we will adopt only the former and explicitly write  $q^{-1}s$  for the latter, always understanding that  $q$  is the first component of  $\sigma$ . The action of  $S$  on  $C^\infty(I, V)$ , obviously leaving  $\mathcal{E}$  in (6.5) invariant, restricts to an action on  $\mathcal{E}$  with  $\det \sigma = q^{2-n}$  on  $\mathcal{E}$  for all  $\sigma \in S$ , since

$$(6.8) \quad \sigma^* \Omega = q^{-1} \Omega, \text{ if } \sigma \text{ is regarded as an operator } \sigma: \mathcal{E} \rightarrow \mathcal{E}.$$

Rephrasing [14, Theorem 4.1], we have:

**THEOREM 6.1.** *The isometry group of a model  $(\widehat{M}, \widehat{\mathfrak{g}})$ , with (6.1)–(6.2), can be identified with the set  $S \times H$ , cf. (6.5)–(6.6), so that  $\Phi = (\sigma, r, u)$  acts on  $(\widehat{M}, \widehat{\mathfrak{g}})$  via*

$$\Phi(t, s, v) = (\sigma t, -\langle \dot{u}(\sigma t), 2\sigma v + u(\sigma t) \rangle + q^{-1}s + r, \sigma v + u(\sigma t)),$$

for every  $(t, s, v) \in \widehat{M}$ , and the group operation in  $S \times H$  becomes

$$(\sigma, r, u)(\widehat{\sigma}, \widehat{r}, \widehat{u}) = (\sigma\widehat{\sigma}, r + q^{-1}\widehat{r} - \Omega(u, \sigma\widehat{u}), u + \sigma\widehat{u}),$$

for  $(\sigma, r, u), (\widehat{\sigma}, \widehat{r}, \widehat{u}) \in S \times H$ . Thus,  $\text{Iso}(\widehat{M}, \widehat{\mathfrak{g}})$  is isomorphic to a semidirect product  $S \ltimes H$ , where the diagonal action of  $S$  on  $H$  is defined via (6.7):  $\sigma \cdot (r, u) = (q^{-1}r, \sigma u)$ .

**REMARK 6.2.** We identify  $H$  with the normal subgroup  $\{(1, 0, \text{Id})\} \times H$  of  $\text{Iso}(\widehat{M}, \widehat{\mathfrak{g}})$ , the kernel of the homomorphism  $\text{Iso}(\widehat{M}, \widehat{\mathfrak{g}}) \ni (\sigma, r, u) \mapsto \sigma \in S$ .

**REMARK 6.3.** In any rank-one ECS manifold, the leaves of  $\mathcal{D}^\perp$  are totally geodesic,  $\mathcal{D}^\perp$  being parallel. In addition, the resulting induced connection on each leaf is *flat*. Namely, (6.3) allows us to assume that the manifold has the form (6.1), with (6.2) except for nonconstancy of  $f$ . For  $i, j$  ranging over  $2, \dots, n-1$ , any linear coordinates  $x^i$  on  $V$  form, along with  $x^1 = t$  on  $I$  and  $x^n = s/2$  on  $\mathbb{R}$ , a coordinate system on  $I \times \mathbb{R} \times V$ , and then – see [15, the lines following formula

(7.2)] – the coordinate vector fields  $\partial_n$  and  $\partial_i$  span  $\mathcal{D}^\perp$ , while, according to [27, p. 93],  $\Gamma_{ij}^\bullet = \Gamma_{in}^\bullet = \Gamma_{nn}^\bullet = 0$ , where  $\bullet$  denotes any index.

REMARK 6.4. In a rank-one ECS model manifold  $(\widehat{M}, \widehat{g})$ , let there exist a subgroup  $\Gamma$  of  $\text{Iso}(\widehat{M}, \widehat{g})$  acting freely and properly discontinuously on  $(\widehat{M}, \widehat{g})$  with a compact isometric quotient  $M = \widehat{M}/\Gamma$ . Using (5.4) we also assume that  $q > 0$  whenever  $(q, p, C, r, u) \in \Gamma$ . If the resulting ECS manifold  $(M, g)$  is translational, then all  $(q, p, C, r, u) \in \Gamma$  have  $q = 1$ , due to [14, formula (3.5-a)]. Also,  $I = \mathbb{R}$  in (6.1): otherwise, all  $(1, p, C, r, u) \in \Gamma$  acting on  $\widehat{M}$  (see Theorem 6.1) would have  $p = 0$  (since  $t \mapsto t + p$  sends  $I$  onto itself), and so

$$(6.9) \quad \begin{array}{l} t \text{ would descend to a function without} \\ \text{critical points on the compact manifold } M. \end{array}$$

Finally, according to [13, formula (3.1)], such translational  $(M, g)$  is geodesically complete but not locally homogeneous.

## 7. Proofs of Theorems A, B, and C

In the first two proofs below,  $(\widetilde{M}, \widetilde{g})$  is the isometric universal covering manifold of the  $n$ -dimensional compact rank-one ECS manifold  $(M, g)$  in question. The objects  $\pi, \Gamma, \widetilde{\mathcal{D}}, \mathbf{w}, t, I, (V, \langle \cdot, \cdot \rangle)$ , and  $A$  are all defined as in Section 5.

Recall from the Introduction that  $(\widetilde{M}, \widetilde{g})$  is said to be  $\widetilde{\mathcal{D}}^\perp$ -complete if all the leaves of  $\widetilde{\mathcal{D}}^\perp$  are complete, while its maximal completeness means that every non-complete maximal geodesic intersects all leaves of  $\widetilde{\mathcal{D}}^\perp$ . As  $\widetilde{\mathcal{D}}^\perp$  and geodesics in  $(\widetilde{M}, \widetilde{g})$  are mapped under  $\pi$  onto their analogs in  $(M, g)$ , it suffices to prove Theorem B for  $(\widetilde{M}, \widetilde{g})$  rather than  $(M, g)$ .

PROOF OF THEOREM B. We apply Lemma 1.2 to any given leaf  $L$  of  $\widetilde{\mathcal{D}}^\perp$  with the flat connection induced on it (Remark 6.3), setting  $\mathcal{X} = \mathbb{R}\mathbf{w}$  and  $\mathcal{Y} = \bigoplus_{j=1}^m \mathbb{R}\widetilde{\mathbf{y}}_j$  for  $m = n - 2$  vector fields  $\widetilde{\mathbf{y}}_j$  on  $\widetilde{M}$  defined below, which, due to their  $\Gamma$ -invariance, will descend to the compact manifold  $M = \widetilde{M}/\Gamma$ , making each  $\mathbf{v} \in \mathcal{Y}$  complete.

We begin by assuming (5.5) and choosing a basis  $(\mathbf{v}_1, \dots, \mathbf{v}_m)$  of  $V$ . Let  $K \subseteq (0, \infty)$  be the image of  $\Gamma$  under (5.3-b). By (5.5),  $K$  is either trivial, or infinite. In the former case, the image of (5.9) is finite due to genericity of  $A$ , and so its kernel has finite index in  $\Gamma$ . Thus, (5.4) allows us to further assume that  $\Gamma$  acts trivially on  $V$ , via (5.9), and  $(\mathbf{v}_1, \dots, \mathbf{v}_m)$  can be completely arbitrary. If  $K$  is infinite, (5.9) implies nilpotency of  $A$ , and we select a basis  $(\mathbf{v}_1, \dots, \mathbf{v}_m)$  of  $V$  for which (4.3)–(4.4) holds. As  $\delta$  in (4.4) depends homomorphically on  $\gamma$ , using (5.4) we may require that  $\delta = 1$  for every  $\gamma \in \Gamma$ .

In either case, we fix a Riemannian metric  $g^\circ$  on  $M$  and lift  $(\mathbf{v}_1, \dots, \mathbf{v}_m)$  to vector fields  $(\mathbf{y}_1, \dots, \mathbf{y}_m)$  tangent to leaves of  $\tilde{\mathcal{D}}^\perp$  which are  $\pi^*g^\circ$ -orthogonal to  $\mathbf{w}$ . For  $\psi$  as in (5.6-a), the vector fields  $\tilde{\mathbf{y}}_j = \psi^{2j-1-m}\mathbf{y}_j$ , are  $\Gamma$ -invariant (cf. (4.4) and (5.7)), which completes the proof. (When  $K = \{1\}$ , we may set  $\psi = 1$ .)  $\square$

PROOF OF THEOREM A. In view of (5.2) and  $\tilde{\mathcal{D}}^\perp$ -completeness of  $(\tilde{M}, \tilde{g})$  (due to Theorem B), every maximal geodesic of  $(\tilde{M}, \tilde{g})$  transverse to  $\tilde{\mathcal{D}}^\perp$  can be parametrized by  $t$ , as  $t: \tilde{M} \rightarrow I$  restricted to its image is a diffeomorphism onto a subinterval  $I' \subseteq I$ , cf. [9, Remark 7.2]. To show that  $I' = I$ , we invoke [9, Lemma 7.3] with minimal modifications in its proof, which never uses the assumption stated there that the metric under consideration is Lorentzian. See Appendix A for details.  $\square$

PROOF OF THEOREM C. Assume that  $(\tilde{M}, \tilde{g})$  is a  $n$ -dimensional simply connected and maximally complete rank-one ECS manifold, and again choose  $\pi, \Gamma, \tilde{\mathcal{D}}, \mathbf{w}, t, I, (V, \langle \cdot, \cdot \rangle)$ , and  $A$  as in Section 5. For the function  $f: I \rightarrow \mathbb{R}$  such that  $\tilde{\text{Ric}} = (2-n)f(t) dt \otimes dt$  (cf. [15, formula (6.6)]), we consider the model ECS manifold  $(\hat{M}, \hat{g})$  built from these ingredients as in (6.2). An isometry between  $(\hat{M}, \hat{g})$  and  $(\tilde{M}, \tilde{g})$  is defined as in the proof of [9, Theorem 7.1], with no significant changes. Details are given in Appendix B.  $\square$

## 8. The transitive-commutation property

For a group-subgroup pair  $(G, H)$ , consider the *transitive-commutation property*: the commutation relation on  $G \setminus H$  is transitive, that is, the equalities  $xy = yx$  and  $yz = zy$  for  $x, y, z \in G \setminus H$  imply  $xz = zx$ .

LEMMA 8.1. *Let  $(G, H)$  be a group-subgroup pair.*

- (a)  *$(G, H)$  has the transitive-commutation property if and only if there exists a family  $\mathcal{K}$  of Abelian subgroups of  $G$  such that  $\{K \setminus H\}_{K \in \mathcal{K}}$  is a partition of  $G \setminus H$ , and any two elements of  $G \setminus H$  which commute lie in the same  $K \in \mathcal{K}$ .*
- (b) *Whenever  $(G, H)$  has the transitive-commutation property and  $\mathcal{K}$  is as in (a), any Abelian subgroup of  $G$  is contained in  $H$ , or in a unique element of  $\mathcal{K}$ .*

PROOF. The ‘only if’ part of (a) is obvious. Conversely, since commutation is now an equivalence relation on  $G \setminus H$ , the subgroup  $K$  generated by any equivalence class  $E \subseteq G \setminus H$  of the commutation relation is Abelian and  $K \setminus H = E$ , the right-to-left inclusion being immediate, the other one clear as elements in  $K \setminus H$  commute with all of  $E$  and hence lie in  $E$ . This yields (a).

To prove (b), let  $N$  be an Abelian subgroup of  $G$ . If there exists an element  $x \in N \setminus H$ , we fix the unique subgroup  $K \in \mathcal{K}$  with  $N \setminus H \subseteq K$ . Then we must have  $N \subseteq K$ , for if  $y \in (N \cap H) \setminus K$ , then  $xy \in N \setminus H$  and so  $y = x^{-1}(xy) \in K$ .  $\square$

REMARK 8.2. One easily verifies the fact (not needed in our argument) that  $K$  associated with  $E$  in the above proof is both the unique Abelian subgroup of  $G$  containing  $E$ , and the centralizer of  $E$ .

### 9. Generic homogeneous models

By a *standard homogeneous rank-one ECS model manifold* (briefly, a *standard homogeneous model*), we mean  $(\widehat{M}, \widehat{\mathfrak{g}})$  as in (6.1)–(6.2) with

$$(9.1) \quad I = (0, \infty) \text{ and } f(t) = \frac{c^2 - 1/4}{t^2}, \text{ where } c \in [0, 1/2) \cup (1/2, \infty) \cup i(0, \infty).$$

All such  $(\widehat{M}, \widehat{\mathfrak{g}})$  are homogeneous, as pointed out in [7, Remark 1 on p. 172]. (This also follows from Theorem 6.1: one easily sees that the group of all elements  $(q, p, C, r, u) \in \text{Iso}(\widehat{M}, \widehat{\mathfrak{g}})$  having  $p = 0$  acts on  $\widehat{M}$  transitively.) The factor  $c^2 - 1/4$  is just any real constant  $h \neq 0$ , written so for later convenience.

LEMMA 9.1. *Every locally homogeneous rank-one ECS model manifold is isometric to an open submanifold  $(a, b) \times \mathbb{R} \times V$  of a standard homogeneous model (6.2).*

PROOF. By [13, formula (3.4)], in (6.1),  $f \neq 0$  everywhere and  $|f|^{-1/2}$  is a linear function of  $t$ , so that  $f(t) = h(t - t_0)^{-2}$  for some real  $h \neq 0$  and  $t_0$ . Under a suitable coordinate change  $(t, s) \mapsto (qt + p, q^{-1}s)$ , with  $(q, p) \in \text{Aff}(\mathbb{R})$ , (6.1)–(6.2) remain valid, allowing us to assume that  $t_0 = 0$  and  $I \subseteq (0, \infty)$ .  $\square$

As a trivial consequence of Lemma 9.1 and (6.3), we obtain:

COROLLARY 9.2. *All locally homogeneous rank-one ECS manifolds are locally isometric to standard homogeneous models.*

If a standard homogeneous model  $(\widehat{M}, \widehat{\mathfrak{g}})$  is also generic, cf. (6.4), then, with notations of (6.5) and Theorem 6.1, elements of

$$(9.2) \quad \text{the identity component } G_0 \text{ of } \text{Iso}(\widehat{M}, \widehat{\mathfrak{g}})$$

have the form  $(q, 0, C_q, r, u)$ , where  $q \in (0, \infty)$ ,  $(r, u) \in H$ , cf. (6.5) and Remark 6.2, while  $C_q: V \rightarrow V$  is as in (4.4) with  $\delta = 1$ . We also consider

$$(9.3) \quad \sigma_q: \mathcal{E} \rightarrow \mathcal{E} \text{ defined as in (6.7-iii): } (\sigma_q u)(t) = C_q u(t/q).$$

Abbreviating  $\Phi = (q, 0, C_q, r, u)$  simply to  $(q, r, u)$ , we may now

$$(9.4) \quad \text{identify } G_0 \text{ with } (0, \infty) \times H,$$

and, for  $\widehat{\Phi} \in G_0$  and  $(t, s, v) \in \widehat{M}$ , we obtain, from Theorem 6.1 that

$$(9.5) \quad \begin{aligned} \text{i) } & \Phi\widehat{\Phi} = (q\widehat{q}, r + q^{-1}\widehat{r} - \Omega(u, \sigma_q\widehat{u}), u + \sigma_q\widehat{u}), \\ \text{ii) } & \Phi^{-1} = (q^{-1}, -qr, -\sigma_q^{-1}u), \\ \text{iii) } & \text{the } \mathcal{E}\text{-component of the commutator } [\Phi, \widehat{\Phi}] \text{ is } (\sigma_q - 1)\widehat{u} - (\sigma_{\widehat{q}} - 1)u, \\ \text{iv) } & \Phi(t, s, v) = (qt, -\langle \dot{u}(qt), 2C_q v + u(qt) \rangle + q^{-1}s + r, C_q v + u(qt)) \end{aligned}$$

With 1 denoting the identity operator  $\mathcal{E} \rightarrow \mathcal{E}$ , conjugation by  $\Phi$  has the form

$$\Phi\widehat{\Phi}\Phi^{-1} = \left( \widehat{q}, q^{-1}\widehat{r} + (1 - \widehat{q}^{-1})r - \Omega((1 + \sigma_{\widehat{q}})u, \sigma_q\widehat{u}) + \Omega(u, \sigma_{\widehat{q}}u), (1 - \sigma_{\widehat{q}})u + \sigma_q\widehat{u} \right).$$

Let us now fix a generic homogeneous model  $(\widehat{M}, \widehat{g})$ .

By [14, Theorem 6.1], each  $\sigma_q: \mathcal{E} \rightarrow \mathcal{E}$  has the spectrum  $\lambda_j^\pm = q^{m+1-2j}\mu^\pm$ ,  $j = 1, \dots, m$ , for the eigenvalues  $\mu^\pm \in \mathbb{C}$  of the operator  $T$  with  $(Tu)(t) = u(t/q)$ , on the space  $\mathcal{W}$  of solutions  $y: (0, \infty) \rightarrow \mathbb{C}$  to the ordinary differential equation  $\dot{y} = fy$ . The expression for  $f$  in (9.1) gives  $\mu^\pm = q^{-\frac{1}{2} \mp c}$ , the corresponding  $T$ -diagonalizing (or,  $T$ -triangular) basis of  $\mathcal{W}$  being  $t \mapsto t^{\frac{1}{2} \pm c}$  if  $c \neq 0$  (or, respectively,  $t \mapsto t^{\frac{1}{2}}$  and  $t \mapsto t^{\frac{1}{2}} \log t$  when  $c = 0$ ). Hence

$$(9.6) \quad \text{the spectrum of } \sigma_q \text{ becomes } \lambda_j^\pm = q^{m+\frac{1}{2}-2j \mp c}, \text{ for } j = 1, \dots, m.$$

The assignment  $(0, \infty) \ni q \mapsto \sigma_q \in \text{GL}(\mathcal{E})$ , being a homomorphism, has an infinitesimal generator  $B \in \mathfrak{gl}(\mathcal{E})$ , with  $\sigma_q = \exp[(\log q)B]$ . By (3.1) and (9.6),

$$(9.7) \quad \text{the spectrum of } B \text{ is } \kappa_j^\pm = m + \frac{1}{2} - 2j \mp c, \text{ where } j = 1 \dots, m.$$

LEMMA 9.3. *For  $\mathcal{E}$  and  $B$  as above, let  $\mathcal{E}_0 = \ker B$  and  $\mathcal{E}_+ = B(\mathcal{E})$ . Then:*

- (a) *Either  $\mathcal{E}_0$  is trivial, or  $\dim \mathcal{E}_0 = 1$  and  $B$  is diagonalizable with  $2m$  distinct real eigenvalues. In both cases,  $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_+$  and  $(\sigma_q - 1)(\mathcal{E}_0) = \{0\}$  if  $q \in (0, \infty)$ .*
- (b) *For every  $q \in (0, \infty) \setminus \{1\}$  the operator  $\sigma_q - 1: \mathcal{E}_+ \rightarrow \mathcal{E}_+$  is an isomorphism.*

PROOF. If  $\det B = 0$ , some  $\kappa_j^\pm$  in (9.7) equals 0, so that  $2c = \pm(2m - 4j + 1)$  is an odd integer. Hence, the eigenvalues of  $B$  are real and mutually distinct:  $j \mapsto \kappa_j^\pm$  is injective for a fixed sign  $\pm$ , while  $2c$  would be the even integer  $2(i - j)$  if there existed  $i$  and  $j$  with  $\kappa_j^+ = \kappa_i^-$ . Since  $\sigma_q = \exp[(\log q)B]$ , this yields (a).

To prove (b), we use (9.6) and consider two cases. If  $c \in \mathbb{R}$ , the resulting injectivity of  $a \mapsto q^a$  on  $\mathbb{R}$  for  $q \neq 1$  gives  $\lambda_j^\pm \neq 1$  except – see the last paragraph – when  $\dim \mathcal{E}_0 = 1$  and  $(j, \pm)$  is the unique pair with  $\kappa_j^\pm = 0$ . If  $c \in i(0, \infty)$ ,  $|\lambda_j^\pm| = q^{m-2j+1/2}$ , being a half-integer power of  $q$ , cannot equal 1 unless  $q = 1$ , and hence  $\lambda_j^\pm \neq 1$  again.  $\square$

LEMMA 9.4. *For a generic standard homogeneous model, the group-subgroup pair  $(G_0, H)$  given by (9.2) and (6.5) has the transitive-commutation property of Section 8,*

and its equivalence classes generate subgroups of  $G_0$  acting freely on  $\widehat{M}$ , and isomorphic to  $\mathbb{R}^k$ , for  $k = \dim \mathcal{E}_0 + 1 \in \{1, 2\}$ .

PROOF. Define  $J: \mathbb{R} \times \mathcal{E}_+ \times (0, \infty) \times \mathcal{E}_0 \rightarrow G_0 \setminus H$  by

$$J(a, z, q, w) = \left( q, a(1 - q^{-1}) + \Omega(z, \sigma_q z + (1 + q^{-1})w), (\sigma_q - 1)z + w \right).$$

It is an obvious consequence of Lemma 9.3, (9.5-i) and (6.8) that

- i)  $J$  maps  $\mathbb{R} \times \mathcal{E}_+ \times [(0, \infty) \setminus \{1\}] \times \mathcal{E}_0$  bijectively onto  $G_0 \setminus H$ ,
- ii)  $J(a, z, \cdot, \cdot): (0, \infty) \times \mathcal{E}_0 \rightarrow G_0$  is an injective group homomorphism whenever  $(a, z) \in \mathbb{R} \times \mathcal{E}_+$ .

By (ii), the images  $K_{a,z}$  of  $J(a, z, \cdot, \cdot)$ , with  $(a, u) \in \mathbb{R} \times \mathcal{E}_+$ , are connected Abelian Lie subgroups of  $G_0$ , isomorphic to  $\mathbb{R}$  (if  $\mathcal{E}_0 = \{0\}$ ) or  $\mathbb{R}^2$  (when  $\dim \mathcal{E}_0 = 1$ ), while, due to (i), the family  $\{K_{a,z} \setminus H : (a, z) \in \mathbb{R} \times \mathcal{E}_+\}$  is a partition of  $G_0 \setminus H$ .

We now define  $F: G_0 \setminus H \rightarrow \mathbb{R} \times \mathcal{E}_+$  by  $F(J(a, z, q, w)) = (a, z)$  when  $q \neq 1$ , which makes sense according to (i). Thus,  $F$  associates with  $\Phi \in G_0 \setminus H$  the unique  $(a, z)$  for which  $\Phi \in K_{a,z}$ . It follows that

- iii)  $\Phi, \widehat{\Phi} \in G_0 \setminus H$  commute if and only if  $F(\Phi) = F(\widehat{\Phi})$ .

In fact, the ‘if’ part is obvious as  $\Phi$  and  $\widehat{\Phi}$  then lie in the same Abelian subgroup of  $G_0$ . By (9.5-i), two elements  $(q, r, u), (\widehat{q}, \widehat{r}, \widehat{u}) \in G_0$  commute if and only if

$$(9.8) \quad r + q^{-1}\widehat{r} - \Omega(u, \sigma_q \widehat{u}) = \widehat{r} + \widehat{q}^{-1}r - \Omega(\widehat{u}, \sigma_{\widehat{q}} u) \quad \text{and} \quad (\sigma_q - 1)\widehat{u} = (\sigma_{\widehat{q}} - 1)u.$$

We now prove the ‘only if’ part of (iii) assuming – see (i) – that  $J(a, z, q, w)$  commutes with  $J(\widehat{a}, \widehat{z}, \widehat{q}, \widehat{w})$  and  $q \neq 1 \neq \widehat{q}$ . The second equality of (9.8) with  $(u, \widehat{u}) = ((\sigma_q - 1)z + w, (\sigma_{\widehat{q}} - 1)\widehat{z} + \widehat{w})$  yields  $z = \widehat{z}$ , since – by Lemma 9.3(b) – the operators  $\sigma_q - 1$  annihilate  $\mathcal{E}_0$ , and form a family of mutually commuting automorphisms when restricted to  $\mathcal{E}_+$  (the latter since  $q \mapsto \sigma_q$  is a homomorphism). As  $z = \widehat{z}$ , the first equality of (9.8), for  $(u, \widehat{u})$  chosen above, and  $(r, \widehat{r})$  replaced by the  $\mathbb{R}$ -components of  $J(a, z, q, w)$  and  $J(\widehat{a}, \widehat{z}, \widehat{q}, \widehat{w})$ , easily yields  $a = \widehat{a}$ , and (iii) follows. The conclusion is now immediate from Lemma 8.1(a).  $\square$

## 10. Proof of Theorem E

We fix a generic compact rank-one ECS manifold  $(M, \mathfrak{g})$ . By Corollary D, the universal covering of  $(M, \mathfrak{g})$  is isometrically identified with a model  $(\widehat{M}, \widehat{\mathfrak{g}})$  as in (6.1)–(6.2), and then  $\widehat{M}/\Gamma = M$ , as at the beginning of Section 5, where (5.4) also allows us to assume that  $\Gamma \subseteq G_0$ , cf. (9.2).

By Remark 6.4, the ‘translational’ conclusion about  $(M, \mathfrak{g})$ , which we prove in the subsequent paragraphs, implies the other two assertions of Theorem E.

Next, suppose that, on the contrary, our  $(M, \mathfrak{g})$  is not translational. The dichotomy (\*) in the Introduction and [14, Corollary D] imply that  $(\widehat{M}, \widehat{\mathfrak{g}})$  is locally homogeneous. Lemma 9.1 now allows us to write  $\widehat{M} = (a, b) \times \mathbb{R} \times V$ . Then,  $(a, b) = (0, \infty)$ , or else all elements  $\Phi = (q, r, u) \in \Gamma$ , acting on  $\widehat{M}$  via (9.5-iv), would have  $q = 1$  (as  $t \mapsto qt$  maps  $(a, b)$  onto itself), leading to (6.9).

The group  $\Sigma = \Gamma \cap H$ , cf. Remark 6.2, is the kernel of the homomorphism

$$(10.1) \quad \Gamma \ni (q, r, u) \mapsto q \in (0, \infty).$$

As an immediate consequence of [14, Lemma 3.2(b) and (f) in Section 4] and [14, Theorems A, B, and Lemma 3.1],

$$(10.2) \quad \Sigma \cap (\{1\} \times \mathbb{R} \times \{0\}) \text{ is trivial and the image of (10.1) is dense in } (0, \infty).$$

The last line in (6.5) now has three consequences. First,  $\Sigma \ni (1, r, u) \mapsto u \in \mathcal{E}$  is a homomorphism, and its injectivity due to (10.2) implies that  $\Sigma$  is Abelian. Secondly, the image  $\Lambda$  of this last homomorphism spans a subspace  $\mathcal{L}$  of  $(\mathcal{E}, \Omega)$  which is isotropic, in the sense that  $\Omega = 0$  on it. Finally,  $\mathbb{R} \times \mathcal{L}$  is an Abelian subgroup of  $H$  (see Remark 6.2) containing  $\Sigma$ , and its group operation coincides with the addition in the vector space  $\mathbb{R} \times \mathcal{L}$ . Applying Remark 2.1 to  $\Sigma$  rather than  $\Gamma$  and the subspace  $\mathcal{Z}$  of  $\mathbb{R} \times \mathcal{L}$  spanned by  $\Sigma$ , we see that

$$(10.3) \quad \begin{aligned} &\Sigma \text{ is a lattice in } \mathcal{Z} \text{ and either } \mathcal{Z} = \mathbb{R} \times \mathcal{L}, \text{ or } \mathcal{Z} \text{ is} \\ &\text{a hyperplane in } \mathbb{R} \times \mathcal{L} \text{ transverse to } \mathbb{R} \times \{0\}. \end{aligned}$$

Any  $(q, r, u) \in \Gamma$  leads to the conjugation mapping  $C_{q,r,u}: H \rightarrow H$  given by

$$(10.4) \quad C_{q,r,u}(\widehat{r}, \widehat{u}) = \left( q^{-1}\widehat{r} - 2\Omega(u, \sigma_q \widehat{u}), \sigma_q \widehat{u} \right),$$

cf. Remark 6.2 and the lines after (9.5) with  $\widehat{q} = 1$ . This makes  $C_{q,r,u}$  a linear endomorphism of  $\mathbb{R} \times \mathcal{E}$  and, by (9.6), for  $\Lambda$  and  $\mathcal{Z}$  as in the lines before (10.3),

$$(10.5) \quad \begin{aligned} &\text{the spectrum of } C_{q,r,u} \text{ consists of } q^{-1} \text{ and the spectrum of } \sigma_q, \\ &\text{while } C_{q,r,u}(\Sigma) = \Sigma, \text{ so that } \sigma_q(\Lambda) = \Lambda \text{ and } C_{q,r,u}(\mathcal{Z}) = \mathcal{Z}. \end{aligned}$$

From (10.5), due to (9.4), (9.3), and the denseness conclusion in (10.2),

$$(10.6) \quad \sigma_q(\Lambda) = \Lambda \text{ and } \sigma_q(\mathcal{L}) = \mathcal{L} \text{ for every } q \in (0, \infty).$$

It follows that  $\text{rank } \Sigma \leq 1$ . Namely, by (10.5) and (10.4), each  $C_{q,r,u} \in \text{gl}(\mathbb{R} \times \mathcal{E})$  leaves invariant the subspaces  $\mathcal{Z}$  and  $\mathbb{R} \times \{0\}$ , as well as  $\mathcal{Z}' = \mathcal{Z} \cap (\mathbb{R} \times \{0\})$ , so that it descends to a linear endomorphism of the quotient  $\mathcal{Z}/\mathcal{Z}'$  and, in view of (10.4), the isomorphism  $\mathcal{Z}/\mathcal{Z}' \rightarrow \mathcal{L}$  induced by the projection  $(r, u) \mapsto u$  makes the latter endomorphism correspond to  $\zeta_q: \mathcal{L} \rightarrow \mathcal{L}$  arising as the restriction of  $\sigma_q$  to  $\mathcal{L}$ . The spectrum of  $C_{q,r,u}$  acting on  $\mathcal{Z}$  thus equals the spectrum of  $\zeta_q$  in  $\mathcal{L}$ , augmented – only when  $\mathcal{Z} = \mathbb{R} \times \mathcal{L}$  in (10.3) – by the eigenvalue  $q^{-1}$ .



At the same time, the infinitesimal generator of the Lie-group homomorphism  $(0, \infty) \ni q \mapsto \zeta_q \in \text{GL}(\mathcal{L})$ , cf. (10.4), being the restriction to  $\mathcal{L}$  of  $B$  appearing in (9.7), has the spectrum  $\alpha(1), \dots, \alpha(k)$  which is a part of that in (9.7). For reasons stated three lines after (3.1),  $\zeta_q$  then must have spectrum  $q^{\alpha(1)}, \dots, q^{\alpha(k)}$ . Thus, the spectrum of  $C_{q,r,u} : \mathcal{Z} \rightarrow \mathcal{Z}$  consists of

$$(10.7) \quad q^{\alpha(1)}, \dots, q^{\alpha(k)} \text{ and, possibly, } q^{-1},$$

which are complex powers of  $q$  with exponents not depending on  $q$ , so that, as  $C_{q,r,u}(\Sigma) = \Sigma$  in (10.5), the trace of  $C_{q,r,u} : \mathcal{Z} \rightarrow \mathcal{Z}$ , being integer-valued and continuous in  $q$ , must be constant. By (3.2), the eigenvalues (10.7) are all equal to 1, which excludes  $q^{-1}$  and, due to continuity in  $q$ , gives  $\alpha(1) = \dots = \alpha(k) = 0$ . From Lemma 9.3(a), it now follows that  $k = \text{rank } \Sigma \leq 1$ .

However, the conclusion that  $\text{rank } \Sigma \leq 1$  further implies that  $\Gamma$  is Abelian. Indeed, when  $\Sigma$  is trivial, injectivity of (10.1) makes  $\Gamma$  isomorphic to a subgroup of  $(0, \infty)$ , while if  $\text{rank } \Sigma = 1$ , we fix a generator  $(1, b, w)$  of  $\Sigma$ , noting that (10.2) gives  $w \neq 0$  and  $\sigma_q w = w$  for every  $q \in (0, \infty)$ . Thus, the commutator  $[\gamma, \hat{\gamma}] \in \Sigma$  of any two elements  $\gamma = (q, r, u)$  and  $\hat{\gamma} = (\hat{q}, \hat{r}, \hat{u})$  in  $\Gamma$  equals  $(1, \ell b, \ell w)$ , for some  $\ell \in \mathbb{Z}$ . By (9.5-iii),  $(\sigma_q - 1)\hat{u} - (\sigma_{\hat{q}} - 1)u = \ell w \in \mathcal{E}_0 \cap \mathcal{E}_+ = \{0\}$ , cf. Lemma 9.3(a). Consequently,  $\ell = 0$  and  $[\gamma, \hat{\gamma}] = (1, 0, 0)$  as required.

Now comes the contradiction: if  $\Gamma$  were Abelian, Lemmas 8.1(b) and 9.4 would give  $\Gamma \subseteq H$  or  $\Gamma \subseteq K$  for some subgroup  $K$  of  $G_0$  isomorphic to  $\mathbb{R}^k$ ,  $k \in \{1, 2\}$ , acting freely on  $\hat{M}$ . The former case leads to (6.9), while the latter contradicts Lemma 2.2 as  $\dim \hat{M} > 2$ .

## 11. The four-dimensional case: proof of Corollary F

Assume, on the contrary, that there exists a four-dimensional compact rank-one ECS manifold  $(M, g)$ , and let  $\Gamma$  be its fundamental group. As  $(M, g)$  is generic – see the very end of Section 5 – it must be translational by Theorem E, while Corollary D allows us to identify its universal covering with a model  $(\hat{M}, \hat{g})$  as in (6.1)–(6.2), so that  $\Gamma \subseteq \text{Iso}(\hat{M}, \hat{g})$  and  $\hat{M}/\Gamma = M$ . Applying (5.4) if necessary, we use Remark 6.4 to conclude that  $I = \mathbb{R}$  in (6.1) and that every  $\gamma \in \Gamma$  has the form  $\gamma = (1, p, \text{Id}, r, u)$ , with  $p \in \mathbb{R}$  and  $(r, u) \in H$  (cf. Theorem 6.1). Here, the  $O(V)$ -component of  $\gamma$  is assumed to be trivial due to genericity combined with (5.4). The image  $P$  of the homomorphism  $\Gamma \ni \gamma \mapsto p \in \mathbb{R}$  is infinite cyclic as its being dense (or, trivial) would imply constancy of  $f$  via the last condition in (6.6) (or, lead to (6.9)). While  $G = \{(1, p, \text{Id}, r, u) \in \text{Iso}(\hat{M}, \hat{g}) : p \in P \text{ and } (r, u) \in H\}$  contains  $\Gamma$  as a subgroup, no subgroup of  $G$  can act on  $\hat{M}$  freely and properly discontinuously with a compact quotient, as shown in [11, Theorem 8.1].

### Appendix A. How [9, Lemma 7.3] leads to Theorem A

With  $u$  in [9, Lemma 7.3] being  $\mathbf{w}$  in (5.1), we fix  $y: I \rightarrow \tilde{M}$  parametrized by  $t$  (see [9, Remark 7.2]) and consider the differential equation

$$(A.1) \quad \nabla_{\dot{y}} \nabla_{\dot{y}} \mathbf{z} + R(\dot{y}, \mathbf{z})\dot{y} + \nabla_{\dot{y}} \dot{y} = -\frac{Q(\mathbf{z})\mathbf{w}}{4},$$

where  $Q(\mathbf{z}) = 3\langle A(\mathbf{z} + \tilde{\mathcal{D}}), \mathbf{z} + \tilde{\mathcal{D}} \rangle + 3f\tilde{g}(\mathbf{z}, \mathbf{z}) + 2\dot{f}\tilde{g}(\mathbf{z}, \mathbf{z})$ , imposed on sections  $\mathbf{z}$  of  $\tilde{\mathcal{D}}^\perp$  along  $y$ . Here  $\langle \cdot, \cdot \rangle$  stands for the inner product on the vector space  $V$  of parallel sections of  $\tilde{\mathcal{D}}^\perp/\tilde{\mathcal{D}}$ . For a maximal solution  $\mathbf{z}$  of (A.1), we set  $x(t, s) = \exp_{y(t)} s\mathbf{z}(t)$ , and observe that  $\mathbf{z}$  and  $x$  are now defined on  $I$  and  $I \times \mathbb{R}$  instead of  $\mathbb{R}$  and  $\mathbb{R}^2$  as in [9]: namely, a solution  $\mathbf{z}_0$  of  $\nabla_{\dot{y}} \nabla_{\dot{y}} \mathbf{z} + R(\dot{y}, \mathbf{z})\dot{y} + \nabla_{\dot{y}} \dot{y} = 0$  is clearly defined on all of  $I$ , while for a function  $\mu: I \rightarrow \mathbb{R}$  with  $\ddot{\mu} = Q(\mathbf{z}_0)/4$ ,  $\mathbf{z} = \mathbf{z}_0 - \mu\mathbf{w}$  is a solution to (A.1). With the subscript convention referred to in (1.2), the vector field  $\mathbf{v}$  along  $x$  having  $\mathbf{v}_s = 0$  for all  $(t, s)$  and  $\mathbf{v} = \nabla_{\dot{y}} \dot{y}$  for  $s = 0$  satisfies the conditions  $x_{tt} + (s-1)(\mathbf{v} - Q(x_s)\mathbf{w}/4) = 0$  and  $[Q(x_s)]_s = 0$ : they are precisely [9, formula (9)], being established here by the argument given there repeated *verbatim*. It follows that  $I \ni t \mapsto x(t, 1) \in \tilde{M}$  is a maximal geodesic, which can be chosen to realize any  $t$ -normalized initial velocity.

### Appendix B. From [9, Theorem 7.1] to Theorem C

Due to maximal completeness and (5.2), we may fix a maximal null geodesic  $x: I \rightarrow \tilde{M}$  parametrized by  $t$  (cf. [9, Remark 7.2]), and define a parallel field  $\Pi$  of Lorentzian planes along  $x$  by  $\Pi_t = \mathbb{R}\dot{x}(t) \oplus \tilde{\mathcal{D}}_{x(t)}$ . As  $\tilde{\mathcal{D}}_{x(t)}^\perp = \Pi_t^\perp \oplus \tilde{\mathcal{D}}_{x(t)}$  and  $V$  is the space of parallel sections of the quotient bundle  $\tilde{\mathcal{D}}^\perp/\tilde{\mathcal{D}}$ , for every  $t \in I$  we have an isomorphism  $V \rightarrow \tilde{\mathcal{D}}_{x(t)}^\perp/\tilde{\mathcal{D}}_{x(t)} \rightarrow \Pi_t^\perp$ , the image of  $v \in V$  under which will be denoted by  $v(t)$ . Therefore, for each  $t \in I$  and  $\mathbf{w}$  as in (5.1),

$$(B.1) \quad \mathbb{R} \times V \ni (s, v) \mapsto v(t) + \frac{s\mathbf{w}_{x(t)}}{2} \in \tilde{\mathcal{D}}_{x(t)}^\perp \text{ is an obvious isomorphism.}$$

By [10, Lemma 5.1],  $F: \hat{M} \rightarrow \tilde{M}$  given by  $F(t, s, v) = \exp_{x(t)}(v(t) + s\mathbf{w}_{x(t)}/2)$  has  $F^*\tilde{g} = \hat{g}$ , and so, due to the italicized statement following (5.2),  $F_*\hat{\mathcal{D}} = \tilde{\mathcal{D}}$ . As the leaves of  $\tilde{\mathcal{D}}^\perp$  are simply connected – see [15, Theorem B] – and their induced connections are flat (Remark 6.3), each slice  $\{t\} \times \mathbb{R} \times V$  is diffeomorphically mapped under  $F$  onto the leaf of  $\tilde{\mathcal{D}}^\perp$  passing through  $x(t)$ , so that injectivity of (B.1) yields the one of  $F$ . Finally, the definition of maximal completeness and surjectivity of (B.1) imply that  $F$  is surjective as well. Hence,  $F$  is a global isometry.

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DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS, OH 43210, USA

Email address: `andrzej@math.ohio-state.edu`

Email address: `terekcouth.1@osu.edu`