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Two-jets of conformal fields along their zero sets

Research Article

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Abstract: The connected components of the zero set of any conformal vector field v, in a pseudo-Riemannian manifold (M,g) of arbitrary signature, are of two types, which may be called 'essential' and 'nonessential'. The former consist of points at which v is essential, that is, cannot be turned into a Killing field by a local conformal change of the metric. In a component of the latter type, points at which v is nonessential form a relatively-open dense subset that is at the same time a totally umbilical submanifold of (M,g). An essential component is always a null totally geodesic submanifold of (M,g), and so is the set of those points in a nonessential component at which v is essential (unless this set, consisting precisely of all the singular points of the component, is empty). Both kinds of null totally geodesic submanifolds arising here carry a 1-form, defined up to multiplications by functions without zeros, which satisfies a projective version of the Killing equation. The conformal-equivalence type of the 2-jet of v is locally constant along the nonessential submanifold of a nonessential component, and along an essential component on which the distinguished 1-form is nonzero. The characteristic polynomial of the 1-jet of v is always locally constant along the zero set.

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1. Introduction

A vector field v on a pseudo-Riemannian manifold (M,g) of dimension $n \geq 2$ is called *conformal* if the Lie derivative $\pounds_v g$ equals a function times g, that is, if for some section A of $\mathfrak{so}(TM)$ and some function $\phi : M \to \mathrm{I\!R}$,

$$2\nabla v = A + \phi\,\mathrm{Id}. \quad \text{In coordinates: } v_{j,k} + v_{k,j} = \phi g_{jk} \text{ (notation of Section 2).} \tag{1}$$

The covariant derivative ∇v is treated here as the bundle morphism $TM \to TM$ sending any vector field w to $\nabla_w v$, and sections of $\mathfrak{so}(TM)$ are endomorphisms of TM, skew-adjoint at every point. Clearly, $\phi = (2/n)\,\mathrm{div}\, v$.

If $n \geq 3$, such v is known to be uniquely determined by its 2-jet at any given point. Determining how the 2-jet of v may vary along the zero set Z of v is thus an obvious initial step towards understanding the dynamics of v near Z.

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Theorem 4.1 of this paper, which is an easy consequence of some facts proved in [4], deals with the 1-jet of v, establishing a restriction on its variability: the characteristic polynomial of ∇v must be locally constant on Z.

A point $x \in Z$ is called *nonessential* [2] if some local conformal change of the metric at x turns v into a Killing field, and *essential* otherwise. A connected component of Z is either *essential* (meaning that it consists of essential points only) or *nonessential* (when it contains some nonessential points, possibly along with essential ones).

The next main result, Theorem 5.1, explores structural properties of components of Z. Every essential component turns out to be a null totally geodesic submanifold, and so is, when nonempty, the possibly-disconnected set Σ of essential points in any given nonessential component N. At the same time, Σ coincides with the set of singular points of N, while $N \smallsetminus \Sigma$ is a totally umbilical submanifold. The tangent spaces of the submanifolds just mentioned, at all points x, are explicitly described in terms of ∇v_x and $d\phi_x$.

For N and Σ as in the last paragraph, let the same symbol Σ also stand for an essential component of Z. Section 6 discusses geometric structures on $N \smallsetminus \Sigma$ and on both types of Σ, naturally induced by the underlying conformal structure of (M, g). They consist of a constant-rank, possibly-degenerate conformal structure on $N \smallsetminus \Sigma$ along with its nullspace distribution, a projective structure on Σ, and a 1-form ξ on Σ defined only up to multiplications by functions without zeros. Their basic properties are listed in Proposition 6.1.

Finally, Section 10 addresses the problem, mentioned above, of variability of the 2-jet of v along Z. The conformal-equivalence type of the 2-jet is proved to be locally constant in $N \smallsetminus \Sigma$ and, generically, in Σ. The word 'generically' means here *in any component of* Σ *on which* ξ *is not identically zero*. Examples show that, in the case of Σ, some form of the 'generic' assumption is necessary. On the other hand, if $\Sigma \subset N$ is nonempty, the equivalence types at points of Σ are always different from those realized in $N \smallsetminus \Sigma$.

The results of Section 10 leave unanswered the question of how the 2-jet of v varies along Σ (of either kind) in the non-generic case. It is not clear to the author what a plausible conjectured answer should sound like, other than providing some "upper bound" on how variability of the conformal-equivalence type may deviate from the picture encountered in conformally flat manifolds.

2. Preliminaries

Manifolds *need not be connected*. However, their connected components must all have the same dimension. Submanifolds are always endowed with the subset topology; this convention, although too restrictive for purposes such as studying orbits of flows, is well suited to the present situation, where submanifolds mainly arise as nonsigular parts of the fixed-point set of a flow.

All manifolds, mappings, bundles and their sections, including tensor fields and functions, are of class C^∞. The symbol ∇ denotes both the Levi-Civita connection of a given pseudo-Riemannian metric g on a manifold M, and the g-gradient. Thus, for a vector field u and a function τ on M, we have $d_u\tau = g(u,\nabla\tau)$. The coordinate versions of (1) and formula (3) below use the standard index raising/lowering and the comma notation: $v_j = g_{jk}v^k$, $\phi^{,l} = g^{lk}\phi_{,k} = g^{lk}\partial_k\phi$ and $w^j = v^j{}_{,k}u^k$ for functions ϕ, vector fields u, v, and the covariant derivative $w = \nabla_u v$.

Given a submanifold K of a manifold M, we denote by T_KM the restriction of TM to K. The *normal bundle* of K is defined, as usual, to be the quotient vector bundle T_KM/TK. Any fixed torsion-free connection ∇ on M gives rise to the *second fundamental form* of K, which is a section b of $[T^*M]^{\odot 2}\otimes T_KM/TK$ (so that, at every $x\in K$, the mapping $b_x : T_xK\times T_xK\to T_xM/T_xK$ is bilinear and symmetric). We have

$$b(\dot x, w) \;=\; \pi\nabla_{\dot x}w \tag{2}$$

whenever $t\mapsto w(t)$ is a vector field tangent to K along a curve $t\mapsto x(t)$ in K, with $\pi : T_KM\to T_KM/TK$ denoting the quotient projection. When $b=0$ identically, K is said to be *totally geodesic* relative to ∇. If ∇ is the Levi-Civita connection of a pseudo-Riemannian metric g and $b = g_K\otimes u$ for some section u of T_KM/TK, where g_K is the restriction of g to K, one calls K *totally umbilical* in (M,g). Since changing g to $e^\tau g$ causes b to be replaced by $b - g_K\otimes\pi\nabla\tau/2$, this last property is conformally invariant, and so is b itself for arbitrary *null* submanifolds. In particular, the class of null totally geodesic submanifolds depends only on the underlying conformal structure.

As shown by Weyl [10, p. 100], two torsionfree connections on a manifold M are *projectively equivalent*, in the sense of having the same re-parametrized geodesics, if and only if their difference E can be written as $E = \theta\odot\mathrm{Id}$ for some 1-form θ on M (in coordinates: $2E^l_{jk} = \theta_j\delta^l_k + \theta_k\delta^l_j$). On the other hand, given a pseudo-Riemannian metric g on M, with the Levi-Civita connection ∇, and a function $\tau : M\to \mathrm{I\!R}$, the conformally related metric $e^\tau g$ has the Levi-Civita connection $\nabla + E$, where $E = d\tau\odot\mathrm{Id} - g\otimes\nabla\tau/2$. Thus, if Σ is a null totally geodesic submanifold of M, the connections on Σ induced by the Levi-Civita connections of g and $e^\tau g$ are projectively equivalent.

For every conformal vector field v on a pseudo-Riemannian manifold (M,g) of dimension $n\ge 3$ and any vector fields u,w on M one has the well-known equalities of bundle morphisms $TM\to TM$ and functions $M\to\mathrm{I\!R}$:

$$\begin{aligned}&2\nabla_u\nabla v = 2R(v\wedge u) + d\phi\otimes u - g(u,\,\cdot\,)\otimes\nabla\phi + g(u,\nabla\phi)\mathrm{Id},\\ &(1-n/2)[\nabla d\phi](u,w) = S(u,\nabla_w v) + S(w,\nabla_u v) + [\nabla_v S](u,w),\end{aligned} \tag{3}$$

cf. [4, formula (22))], where R and S are the curvature and Schouten tensors. In coordinates, (3) reads $2v^l{}_{,kj} = 2R_{pjk}{}^l v^p + \phi_{,k}\delta^l_j - \phi^{,l}g_{jk} + \phi_{,j}\delta^l_k$ and $(1-n/2)\phi_{,jk} = S_{jp}v^p{}_{,k} + S_{kp}v^p{}_{,j} + S_{jk,p}v^p$.

Remark 2.1.
If a vector field v on a manifold M vanishes at a point x, the endomorphism ∇v_x of T_xM does not depend of the choice of the connection ∇, which is immediate from the local-coordinate formula for ∇v. One then also refers to ∇v_x as the *linear part* (or *Jacobian*, or *derivative*, or *differential*) of v at the zero x. At the same time, ∇v_x is the infinitesimal generator of the local flow of v acting in T_xM.

Remark 2.2.
For a conformal vector field v on a pseudo-Riemannian manifold (M,g) and any function $\tau : M \to \mathrm{I\!R}$, the conformally equivalent metric $e^\tau g$ satisfies, along with v, the analog of (1) in which the role of ϕ is played by $\phi + d_v\tau$. In fact, (1) is equivalent to $\pounds_v g = \phi g$, while $\pounds_v(e^\tau g) = e^\tau \pounds_v g + (d_v\tau)e^\tau g$. At a point x such that $v_x = 0$, switching from g to $e^\tau g$ thus results in replacing $d\phi_x$ by $d\phi_x + (d\tau_x)\nabla v_x$.

Remark 2.3.
A Killing field v and any vector field u on a pseudo-Riemannian manifold satisfy (3) with $\phi = 0$. Therefore, ∇v is parallel along any curve to which v is tangent, such as an integral curve of v or a curve of zeros of v.

3. The zero set Z of a conformal field v

In addition to the function $\phi = (2/n)\,\mathrm{div}\, v : M \to \mathrm{I\!R}$ appearing in (1), let us also consider

$$\text{the zero set } Z \text{ of a conformal vector field } v \text{ on a pseudo-Riemannian manifold } (M,g), \quad \dim M = n \geq 3. \tag{4}$$

If $x \in Z$, the *simultaneous kernel at* x of the differential $d\phi$ and the bundle morphism $\nabla v : TM \to TM$ is

$$\text{the space } \mathcal{H}_x \;=\; \mathrm{Ker}\,\nabla v_x \cap \mathrm{Ker}\, d\phi_x, \quad \text{often denoted simply by } H, \tag{5}$$

and depending on g only via the underlying conformal structure (see Remarks 2.1 – 2.2).

As in [2], we call $x \in Z$ a *nonessential* zero of v if v restricted to a suitable neighborhood of x is a Killing field for some metric conformal to g. When no such neighborhood and metric exist, the zero of v at x is said to be *essential*.

By a *nonsingular point* of Z we mean any $x \in Z$ such that, for some neighborhood U of x in M, the intersection $Z \cap U$ is a submanifold of M. Points of Z not having a neighborhood with this property will be called *singular*.

For $(M,g), v, Z$ as above, a point $x \in Z$, and the exponential mapping $\exp_x$ of g at x, we will repeatedly consider

$$\begin{array}{l}\text{any sufficiently small neighborhoods } U \text{ of } 0 \text{ in } T_xM \text{ and } U' \text{ of } x \text{ in } M \text{ such that}\\ U \text{ is a union of line segments emanating from } 0 \text{ and } \exp_x \text{ is a diffeomorphism } U \to U'.\end{array} \tag{6}$$

Theorem 3.1 (Kobayashi [7]).
For $(M,g), v, Z, U, U', x \in Z$ and $H = \mathcal{H}_x$ as in (4) – (6), let v also be a Killing field. Then

$$Z \cap U' \;=\; \exp_x[H \cap U], \qquad \textit{with} \quad H = \mathrm{Ker}\,\nabla v_x \quad \textit{since} \quad \phi = 0 \;\; \textit{in (1)}.$$

Thus, the connected components of Z are totally geodesic submanifolds of even codimensions in (M,g).

Even though in [7] Kobayashi only considered the case of Riemannian metrics g, his proof of Theorem 3.1 is valid for all metric signatures: $\exp_x$ sends short line segments emanating from 0 in T_xM onto g-geodesics, and so the local flow of v corresponds via $\exp_x$ to the linear local flow on a neighborhood of 0 in T_xM, generated by ∇v_x.

Theorem 3.2 (Beig [1, 3]).
Let Z be the zero set of a conformal vector field v on a pseudo-Riemannian manifold (M,g) of dimension $n \geq 3$. A point $x \in Z$ is nonessential if and only if

$$\phi(x) = 0 \quad \text{and} \quad \nabla\phi_x \in \nabla v_x(T_xM). \tag{7}$$

for the function $\phi = (2/n)\,\mathrm{div}\, v : M \to \mathrm{I\!R}$ appearing in (1). In other words, $x \in Z$ is essential if and only if

$$\text{either} \quad \phi(x) \neq 0\,, \quad \text{or} \quad \phi(x) = 0 \quad \text{and} \quad \nabla\phi_x \notin \nabla v_x(T_xM). \tag{8}$$

Theorem 3.3 (Derdzinski [4]).
Suppose that v is a conformal vector field on a pseudo-Riemannian manifold (M,g) of dimension $n \geq 3$. If Z is the zero set of v, while $x \in Z$ satisfies (8), and $C = \{w \in T_xM : g_x(w,w) = 0\}$ stands for the null cone, then, with U, U' as in (6) and $H = \mathrm{Ker}\,\nabla v_x \cap \mathrm{Ker}\, d\phi_x$,

$$Z \cap U' \;=\; \exp_x[C \cap H \cap U]. \tag{9}$$

In addition, $\phi = (2/n)\,\mathrm{div}\, v$ is constant along each connected component of Z.

The right-to-left inclusion in (9) was first proved by Lampe [8, Proposition 3.4.3]. See also [6, Chapter 11.4].

Given a conformal vector field v on a pseudo-Riemannian manifold (M,g) of dimension $n \geq 3$, and a parallel vector field $t \mapsto w(t) \in T_{y(t)}M$ along a geodesic $t \mapsto y(t)$ contained in the zero set Z of v, such that $g(\dot y, w) = 0$, replacing u in (3) with $\dot y$ we obtain

$$\text{(a)} \quad 2\nabla_{\dot y}\nabla_w v = g(w, \nabla\phi)\,\dot y\,, \qquad \text{(b)} \quad (1 - n/2)[g(w,\nabla\phi)]^{\cdot} = S(\dot y, \nabla_w v)\,. \tag{10}$$

(The other terms vanish since $v = \nabla_{\dot y} v = 0$ at $y(t)$, and $g(\dot y, \nabla\phi) = 0$ due to the final clause of Theorem 3.3.)

Remark 3.1.
In view of Theorems 3.1 – 3.3, Z in (4) is always locally pathwise connected. Thus, the connected components of Z are pathwise connected, closed subsets of M.

Remark 3.2.
Suppose that $x \in M$ is a zero of a conformal vector field v on a pseudo-Riemannian manifold (M,g), while ϕ and A denote the objects appearing in (1).

(a) Either $\mathrm{Ker}\,\nabla v_x = \{0\}$, or $\mathrm{Ker}\,\nabla v_x$ is the eigenspace of the skew-adjoint operator $A_x : T_xM \to T_xM$ for the eigenvalue $-\phi(x)$.

(b) If $\phi(x) = 0$, then $\nabla v_x : T_xM \to T_xM$ is skew-adjoint; consequently, $\mathrm{rank}\,\nabla v_x$ is even, and $\mathrm{Ker}\,\nabla v_x$ coincides with the orthogonal complement is the image $\nabla v_x(T_xM)$.

(c) If $\phi(x) \neq 0$, then $\mathrm{Ker}\,\nabla v_x$ is a null subspace of T_xM.

In fact, (a) and (b) are consequences of (1), while (c) follows from (a).

4. The characteristic polynomial of ∇v

Given a torsion-free connection ∇ on an n-dimensional manifold M, and a vector field v on M, we denote by $\mathcal{P}_n$ the space of all real polynomials in one variable of degrees not exceeding n, and by $\chi(\nabla v)$ the function $M \to \mathcal{P}_n$ assigning to each $x \in M$ the characteristic polynomial of the endomorphism $\nabla v_x : T_x M \to T_x M$.

Lemma 4.1 (Derdzinski [4], Lemma 12.2(b)–(iii)).
If a conformal vector field v on a pseudo-Riemannian manifold (M, g) is tangent to a null geodesic segment Γ, and ϕ appearing in (1) is constant along Γ, then $\chi(\nabla v)$ is constant along Γ as well.

Theorem 4.1.
Let Z be the zero set of a conformal vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$. Then $\chi(\nabla v) : M \to \mathcal{P}_n$ is constant on every connected component of Z and, consequently, so is $\phi = (2/n)\operatorname{div} v : M \to \mathbb{R}$, as $\operatorname{div} v = \operatorname{tr} \nabla v$.

Proof. We fix $x \in Z$ and show that $\chi(\nabla v)$, at zeros of v near x, is the same as at x, cf. Remark 3.1.

First, if x is a nonessential zero of v, changing the metric conformally near x, we may assume that v is a Killing field. By Theorem 3.1, the nearby zeros of v then form a submanifold K of M, while, according to Remark 2.3, ∇v is parallel along K. Since ∇v restricted to K is unaffected by the conformal change (Remark 2.1), this proves our assertion for nonessential zeros x.

Finally, let the zero of v at x be essential. Theorem 3.2 then gives (8). In view of Theorem 3.3, every nearby point of Z is joined to x by a null geodesic segment Γ contained in Z. Our claim about ϕ now follows from the final clause of Theorem 3.3. Constancy of $\chi(\nabla v)$ along Γ is therefore immediate from Lemma 4.1. □

5. Essential and nonessential components of Z

By the *components* of the set Z appearing in (4) we mean its (pathwise) connected components, cf. Remark 3.1.

A component of Z will be called *essential* if all of its points are essential zeros of v, as defined in Section 3. Otherwise, the component is said to be *nonessential*.

This definition allows a nonessential component N to contain some essential zeros of v. However, as shown in Theorem 5.1(v),(vii) below, the set of nonessential zeros in N is relatively open and dense.

Remark 5.1.
Unlike the components, submanifolds—such as Σ and $N \smallsetminus \Sigma$ in case (b) of Theorem 5.1—need not be connected, although all of their components are required to have the same dimension.

As usual, semidefiniteness of a bilinear form $\langle\,,\rangle$ on a real vector space V means that the function $V \ni v \mapsto \langle v, v\rangle$ cannot assume both positive and negative values.

Theorem 5.1.
Given a conformal vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$, suppose that

(a) *Σ is an essential component of Z, or*

(b) *Σ is the set of essential points in a nonessential component N of Z,*

where Z is the zero set of v. Then, with $\phi = (2/n)\operatorname{div} v : M \to \mathrm{I\!R}$ as in (1) and $\mathcal{H}_x = \operatorname{Ker} \nabla v_x \cap \operatorname{Ker} d\phi_x$,

(i) *Σ, if nonempty, is a null totally geodesic submanifold of (M, g), closed as a subset of M, cf. Remark 5.1,*

(ii) *$T_x\Sigma = \mathcal{H}_x \cap \mathcal{H}_x^\perp$ at every point x of Σ,*

(iii) *for any $x \in \Sigma$ the metric g_x restricted to $\mathcal{H}_x$ is semidefinite in case (a), non-semidefinite in case (b).*

Finally, in case (b), with singular points defined in Section 3, C as in Theorem 3.3, and $H = \mathcal{H}_x$,

(iv) *Σ consists of singular, $N \smallsetminus \Sigma$ of nonsingular zeros of v in N,*

(v) *$N \smallsetminus \Sigma$ is a totally umbilical submanifold of M, while the sign pattern of g restricted to $N \smallsetminus \Sigma$, including its rank r, is the same at all points and, if Σ is nonempty, $\dim(N \smallsetminus \Sigma) - \dim \Sigma = r + 1$,*

(vi) *whenever $y \in N \smallsetminus \Sigma$ one has $T_y(N \smallsetminus \Sigma) = \operatorname{Ker} \nabla v_y$, and $\operatorname{rank} \nabla v_y = 2 + \operatorname{rank} \nabla v_x$ if $x \in \Sigma$,*

(vii) *for $x \in \Sigma$ and sufficiently small U, U' in (6), $\Sigma \cap U' = \exp_x[H \cap H^\perp \cap U]$ and $N \cap U' = \exp_x[C \cap H \cap U]$.*

Proof. As a consequence of Theorem 3.2, for $x \in Z$ and $H = \mathcal{H}_x$ there are three possibilities:

(α) x is a nonessential zero of v, that is, (7) holds,

(β) x is essential and the metric g_x is semidefinite on H,

(γ) x is essential and g_x restricted to H is not semidefinite.

For $\phi = (2/n)\operatorname{div} v$ and any $x \in Z$, it easily follows from (8) that

$$\text{in case } (\gamma)\text{: } \phi(x) = 0 \text{ and } \nabla\phi_x \notin \nabla v_x(T_xM), \tag{11}$$

since, if $\phi(x)$ were nonzero, g_x would be semidefinite on $H \subset \operatorname{Ker} \nabla v_x$ as a consequence of Remark 3.2(c).

Theorem 3.3 implies in turn that, in case (γ), x is singular; specifically, (11) yields (8) and so, by (9), the set of singular points in $Z \cap U'$ coincides with $\exp_x[H \cap H^\perp \cap U]$, as g_x is not semidefinite on H. On the other hand, x is nonsingular both in case (β), for exactly the same reason, and in case (α), due to Theorem 3.1. Thus,

$$x \text{ is nonsingular in cases } (\alpha) \text{ and } (\beta), \text{ but singular in case } (\gamma). \tag{12}$$

Let x satisfy (γ). For U, U' as in (6), $\Sigma' = \exp_x[H \cap H^\perp \cap U]$ and $N' = \exp_x[C \cap H \cap U] \smallsetminus \Sigma'$ are submanifolds of M which, by Theorem 3.3, consist precisely of all singular and, respectively, nonsingular points of $Z \cap U'$. One has

$$T_y N' \,=\, \mathrm{Ker}\,\nabla v_y \quad \text{and} \quad \mathrm{rank}\,\nabla v_y \,=\, 2 \,+\, \mathrm{rank}\,\nabla v_x \quad \text{for every} \quad y \in N', \tag{13}$$

with sufficiently small U and U'. We now prove (13). First, the final clause of Theorem 3.3 and (11) give

$$\phi \,=\, 0 \quad \text{on} \quad N' \cup \Sigma'. \tag{14}$$

Also, $T_y N' \subset \mathrm{Ker}\,\nabla v_y$ as $N' \subset Z$, while $\dim N' = \dim H - 1 = \dim \mathrm{Ker}\,\nabla v_x - 2$ due to our definition (5) of H, combined with (11) and the final clause of Remark 3.2(b). Hence $\dim \mathrm{Ker}\,\nabla v_x - 2 \le \dim \mathrm{Ker}\,\nabla v_y$ or, equivalently, $\mathrm{rank}\,\nabla v_x \le \mathrm{rank}\,\nabla v_y \le\, 2 + \mathrm{rank}\,\nabla v_x$ (where the first inequality, for y near x, follows from semicontinuity of the rank). The two inequalities cannot be both strict, as both ranks are even in view of Remark 3.2(b) and (14). All $y \in N'$ close to x must now have $\mathrm{rank}\,\nabla v_y \,=\, 2 + \mathrm{rank}\,\nabla v_x$ since, if they did not, there would be a sequence of points $y \in N'$ with $\mathrm{rank}\,\nabla v_y = \mathrm{rank}\,\nabla v_x$, converging to x. All but finitely many of its terms y would satisfy the condition in (11), as well as (γ), with x replaced by y. (In fact, $\phi(y) = 0$ by (14), while $\nabla\phi_y \notin \nabla v_y(T_y M)$ and g_y is not semidefinite on $\mathcal{H}_y$ for terms y close to x, as otherwise we could find a subsequence with $\nabla\phi_y \in \nabla v_y(T_y M)$, or one with g_y semidefinite on $\mathcal{H}_y$ and, passing to a further subsequence for which $\nabla v_y(T_y M) \to \nabla v_x(T_x M)$ in the total space of an appropriate Grassmannian bundle, we would obtain, in the limit, the relation $\nabla\phi_x \in \nabla v_x(T_x M)$, or semidefiniteness of g_x on $\mathcal{H}_x$, contrary to (11) and (γ); note that, for any $x \in Z$, the condition in (11) implies, by (8), that x is essential.) This leads to a contradiction, as the terms y would be singular by (12), yet at the same time nonsingular since, in view of Theorem 3.3, the submanifold N' of M, containing y, is a relatively open subset of Z. The proof of (13) is now complete: according to the two lines following (14), the just-established equality $\mathrm{rank}\,\nabla v_y = 2 + \mathrm{rank}\,\nabla v_x$ means that $T_y N'$ is a codimension-zero subspace of $\mathrm{Ker}\,\nabla v_y$.

Furthermore, for $x \in Z$ with (γ) and Σ', N' chosen as above, with sufficiently small U and U',

$$\text{points of } \Sigma' \text{ have property } (\gamma), \text{ while points of } N' \text{ satisfy } (\alpha). \tag{15}$$

To verify (15), recall that, as stated in the line preceding (13), Σ' and N' consist of singular and, respectively, nonsingular points of Z. Now the first claim in (15) is obvious from (12). As for the second one, its failure would—again by (12)—amount to (β) for some points $y \in N'$, arbitrarily close to x. Theorem 3.3, applied to y, now would give $T_y N' = \mathcal{H}_y \cap \mathcal{H}_y^\perp$. Since $T_y N' = \mathrm{Ker}\,\nabla v_y$, cf. (13), both inclusions

$$\mathcal{H}_y \cap \mathcal{H}_y^\perp \subset \mathcal{H}_y = \mathrm{Ker}\,\nabla v_y \cap \mathrm{Ker}\,d\phi_y \quad \text{and} \quad \mathcal{H}_y = \mathrm{Ker}\,\nabla v_y \cap \mathrm{Ker}\,d\phi_y \subset \mathrm{Ker}\,\nabla v_y$$

would actually be equalities; the second one would thus read $\mathrm{Ker}\,\nabla v_y \subset [\nabla\phi_y]^\perp$. Taking the orthogonal complements, and using Remark 3.2(b), we would obtain $\nabla\phi_y \in \nabla v_y(T_y M)$. (By (14), $\phi(y) = 0$.) Theorem 3.2 for y would now yield case (α) for y rather than (β). The ensuing contradiction proves the second part of (15).

Let $\varPi_\alpha$ (or $\varPi_\beta$, or $\varPi_\gamma$) denote the subset of a given component N of Z formed by all points $x \in N$ with (α) (or (β) or, respectively, (γ)). According to [4, Remark 17.1], $\varPi_\alpha$ and $\varPi_\beta$ are relatively open in N. So is, by Theorem 3.3, the set $N' \cup \varSigma' = \exp_x[C \cap H \cap U]$ appearing in (14) and, consequently, the union $\varPi_\alpha \cup \varPi_\gamma$ (in view of (15)). Thus, due to connectedness of N, either $N = \varPi_\beta$ (in which case N is essential, and we denote it by the symbol $\varSigma$), or $N = \varPi_\alpha \cup \varPi_\gamma$ is a nonessential component (and we let $\varSigma$ stand for the set of its essential points, so that $\varSigma = \varPi_\gamma$). In other words, since $\varPi_\alpha, \varPi_\beta$ and $\varPi_\gamma$ are pairwise disjoint, we have

$$(*) \quad \varSigma = \varPi_\beta \quad \text{in case (a)}, \qquad (**) \quad \varSigma = \varPi_\gamma \quad \text{and} \quad N \smallsetminus \varSigma = \varPi_\alpha \quad \text{in case (b)}. \tag{16}$$

Assertions (i) – (iii), in both cases (a) and (b), are now immediate (with one exception): for any $x \in \varSigma$, Theorem 3.3 and (15) imply that $\varSigma \cap U' = \varSigma'$, with $\varSigma'$ as above and sufficiently small U, U'. The exception—the possibility that, in case (b), $\varSigma$ might have components of different dimensions—will only be excluded at the very end of the proof.

From now on we assume case (b). As an obvious consequence of (12) and (16)-($**$), we obtain (iv), while (vi) and (vii) follow from (13), (15) and (16)-($**$). Next, according to (iv) and [4, Theorem 1.1], the connected components of $N \smallsetminus \varSigma$ are totally umbilical submanifolds of (M, g), which will yield the first claim in (v) once we have verified that the nonsingular subset of any component of the zero set Z is a submanifold (in other words, its own components are all of the same dimension). This is, however, immediate from Theorems 3.1 – 3.3: under their hypotheses, if the set $\varXi$ of all singular points in $Z \cap U'$ is nonempty, then (8) holds, the metric g_x restricted to H is not semidefinite and, by (9), $\varXi = \exp_x[H \cap H^\perp \cap U]$, while all components of $(C \smallsetminus H^\perp) \cap H$ are clearly of dimension $\dim H - 1$.

To prove the remainder of (v), first note that the claim about the sign pattern is true locally: a local conformal change of the metric allows us to treat v as a Killing field and use the final clause of Theorem 3.1, which implies that the the tangent spaces of $N \smallsetminus \varSigma$ are invariant under parallel transports along $N \smallsetminus \varSigma$. The corresponding global claim could fail only if some connected component of $\varSigma$ would locally disconnect N, leading to different sign patterns on the resulting new components. This, however, cannot happen since, for any $x \in \varSigma$, any $\varepsilon \in (0, \infty)$, and any null geodesic $(-\varepsilon, \varepsilon) \ni t \mapsto y(t)$ with $y(0) = x$ which lies in $N \smallsetminus \varSigma$ except at $t = 0$, the family of tangent spaces $T_{y(t)}(N \smallsetminus \varSigma)$, for $t \neq 0$, is parallel along the geodesic. Namely, whenever $t \mapsto w(t) \in T_{y(t)}M$ is a parallel vector field and $g(\dot y, w) = 0$, relations (10) form a system of first-order linear homogeneous ordinary differential equations with the unknowns $\nabla_w v$ and $g(w, \nabla\phi)$. Therefore, if we choose a parallel field w satisfying at some fixed $t \neq 0$ the condition $w(t) \in T_{y(t)}(N \smallsetminus \varSigma)$ (so that, by (vi) and (14) – (16), $\nabla_w v$ and $g(w, \nabla\phi)$ both vanish at t), uniqueness of solutions gives $\nabla_w v = 0$ and $g(w, \nabla\phi) = 0$ at every t. Now (vi) yields $w(t) \in T_{y(t)}(N \smallsetminus \varSigma)$ for all $t \neq 0$, as required.

By (vii) the limit as $t \to 0$ of the above parallel family $t \mapsto T_{y(t)}(N \smallsetminus \varSigma)$ is $u^\perp \cap H$, where $u = \dot y(0) \in T_xM$, so that $u \in (C \cap H) \smallsetminus H^\perp$. Clearly, $\dim(N \smallsetminus \varSigma) = \dim(u^\perp \cap H)$. Letting $\varSigma$ temporarily stand for the connected component of Σ which contains x, we have, by (ii), $\dim \Sigma = \dim(H \cap H^\perp)$. Note that $H \cap H^\perp$ is the null-space of the symmetric bilinear form $\langle\,,\rangle$ in H obtained by restricting the metric g, and r in (v) is the rank of the restriction of $\langle\,,\rangle$ to $u^\perp \cap H$. Since $u \in (C \cap H) \smallsetminus H^\perp$, the sign pattern of the latter restriction arises from that of $\langle\,,\rangle$ in H by replacing a plus-minus pair with a zero. Consequently, $\langle\,,\rangle$ has the rank $r+2$, and $\dim(N \smallsetminus \Sigma) - \dim \Sigma = \dim(u^\perp \cap H) - \dim(H \cap H^\perp) = (\dim H - 1) - [\dim H - (r+2)] = r+1$. Now the dimension formula in (v) follows. This in turn shows that all connected components of Σ have the same dimension, completing the proof. □

6. Induced structures on Σ and $N \smallsetminus \Sigma$

Again, let Σ now be either an essential component, or the set of essential points in a nonessential component N of the zero set Z of a conformal vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$.

Both Σ and $N \smallsetminus \Sigma$ carry geometric structures naturally induced by the underlying conformal structure of (M, g). Fixing our metric g within the conformal structure allows us in turn to represent the induced structures by more concrete geometric objects, as explained below.

First, according to Theorem 5.1(v), g (or, the conformal structure), restricted to $N \smallsetminus \Sigma$, is a symmetric 2-tensor field having the same sign pattern at all points (or, respectively, a class of such tensor fields, arising from one another via multiplications by functions without zeros). We refer to it as the *possibly-degenerate metric* (or, *possibly-degenerate conformal structure*) of $N \smallsetminus \Sigma$. If $\Sigma \subset N$ is nonempty, the metric/structure must actually be degenerate—see the next paragraph—while the equality in Theorem 5.1(v) shows that this is the zero metric/structure (with $r = 0$) only in the case where $\dim \Sigma = \dim(N \smallsetminus \Sigma) - 1$.

A further natural structure on $N \smallsetminus \Sigma$ is the *nullspace distribution* $\mathcal{P}$ of the restriction of the metric g (or conformal structure) to $N \smallsetminus \Sigma$. Due to the equality in Theorem 5.1(v), if Σ is nonempty, $\mathcal{P}$ has the positive dimension $\dim \Sigma + 1$, so that the restricted metric is degenerate. Its degeneracy can also be derived from the fact that, by the Gauss lemma, short null geodesic segments emanating from Σ into $N \smallsetminus \Sigma$ are all tangent to $\mathcal{P}$. (The Gauss lemma and its standard proof in the Riemannian case [9, Lemma 10.5] remain valid for indefinite metrics.)

From now on Σ is assumed nonempty. In view of Theorem 5.1(i), g gives rise to an obvious torsion-free connection D on Σ, while the conformal structure of g induces on Σ a natural *projective structure*, that is, a class of torsion-free connections having the same family of nonparametrized geodesics. See the text preceding formula (3).

In addition, g naturally leads to a 1-form ξ on Σ. (Using the conformal structure instead of g, we obtain a 1-form

ξ defined only up to multiplications by functions without zeros.) To describe ξ, we consider two cases, noting that $\phi = (2/n)\,\mathrm{div}\, v$ is constant on $\varSigma$ and, in fact, on every component of Z, cf. the final clause of Theorem 3.3. Specifically, if $\phi = 0$ on $\varSigma$, then $\varSigma \ni x \mapsto \mathcal{H}_x = \mathrm{Ker}\,\nabla v_x \cap \mathrm{Ker}\, d\phi_x$ is, in both cases, a parallel subbundle of $T_\varSigma M$ contained in $\mathrm{Ker}\,\nabla v$ as a codimension-one subbundle [4, Lemma 13.1(b),(d)], and we set $\xi = g(u,\,\cdot\,)$, on $\varSigma$, for any section u of $\mathrm{Ker}\,\nabla v$ over $\varSigma$ with $g(u, \nabla\phi) = 1$. If $\phi \neq 0$ on $\varSigma$, we declare that $\xi = 0$.

Proposition 6.1.
Under the assumptions made at the beginning of this section, for $\mathcal{P}, \mathrm{D}$ and ξ defined above,

(i) *$\mathcal{P}$ is integrable and its leaves are null totally geodesic submanifolds of (M, g),*

(ii) *if $\varGamma \subset \varSigma$ is a geodesic segment and $T_x\varGamma \subset \mathrm{Ker}\,\xi_x$ for some $x \in \varGamma$, then $T_x\varGamma \subset \mathrm{Ker}\,\xi_x$ for every $x \in \varGamma$,*

(iii) *in the open subset $\varSigma' \subset \varSigma$ on which $\xi \neq 0$,*

$$\mathrm{sym}\,\mathrm{D}\xi \;=\; \mu \odot \xi, \quad \text{that is,} \quad \xi_{j,k} + \xi_{k,j} = \mu_j\xi_k + \mu_k\xi_j \quad \text{for some 1-form } \mu \text{ on } \varSigma', \tag{17}$$

(iv) *ξ has the following unique continuation property: if $\xi = 0$ at all points of some codimension-one connected submanifold Δ of $\varSigma$, then $\xi = 0$ everywhere in the connected component of $\varSigma$ containing Δ.*

Proof. For any sections w, w' of $\mathcal{P}$ and any curve $t \mapsto y(t)$ in the totally umbilical submanifold $K = N \smallsetminus \varSigma$, (2) gives $\pi\nabla_{\dot x} w = 0$, that is, $\nabla_{\dot x} w$ is tangent to K. Hence so is $\nabla_w w'$ and, for any vector field u tangent to K we have $g(\nabla_w w', u) = -g(\nabla_w u, w') = 0$, as one sees applying (2), this time, to w' instead of w and an integral curve $t \mapsto y(t)$ of w. Thus, $\nabla_w w'$ is a section of $\mathcal{P}$, and (i) follows.

In (ii) – (iv) we may assume that $\phi = 0$ on $\varSigma$, since otherwise $\xi = 0$. For $\varGamma$ and x as in (ii), let $t \mapsto w(t) \in T_{y(t)}M$ be a parallel vector field along a geodesic parametrization of $t \mapsto y(t)$ of $\varGamma$ such that $y(0) = x$ and $\nabla_{w(0)} v = \dot y(0)$. (That $\dot y(0) \in \nabla v_x(T_xM)$ is clear: as $\dot y(0)$ lies in $T_x\varGamma \subset \mathrm{Ker}\,\xi_x \subset T_x\varSigma$, it is orthogonal not just to $\mathcal{H}_x = \mathrm{Ker}\,\nabla v_x \cap \mathrm{Ker}\, d\phi_x$, cf. Theorem 5.1(ii), but also to the whole space $\mathrm{Ker}\,\nabla v_x$, while $\nabla v_x(T_xM) = [\mathrm{Ker}\,\nabla v_x]^\perp$ by Remark 3.2(b).) Choosing a function $t \mapsto \kappa(t)$ with $2\dot\kappa = g(w, \nabla\phi)$ and $\kappa(0) = 1$, then integrating (10)-(a), we get $\nabla_{w(t)} v = \kappa(t)\dot y(t)$, and hence $\dot y(t) \in \nabla v_{y(t)}(T_xM) = [\mathrm{Ker}\,\nabla v_{y(t)}]^\perp$ for all t near 0, which yields (ii).

Assertion (iii) is in turn a consequence of (ii): if $x \in \varSigma$ and $w \in \mathrm{Ker}\,\xi_x$, setting $y(t) = \exp_x tw$ and differentiating the resulting equality $\xi(\dot y) = 0$, we obtain $[\nabla_{\dot y}\xi](\dot y) = 0$, so that $[\nabla_w\xi](w) = 0$. Thus, $\mathrm{sym}\,\mathrm{D}\xi_x$ treated as a polynomial function on $T_x\varSigma$ vanishes on the zero set of the linear function ξ_x, which is well-known to imply divisibility of the former by the latter, cf. [5, Lemma 17.1(i)]. This proves (iii).

Finally, (iv) follows from (ii) since under the hypothesis of (iv), ξ must vanish on an open set containing Δ, namely, the set of points at which an open set of tangent directions is realized by geodesics intersecting Δ. $\square$

Note that condition (17) involves D only through its underlying projective structure, and remains valid after ξ has been multiplied by a function without zeros.

7. One-jets of v along components of Z

As before, Z stands for the zero set of a conformal vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$. For $x \in Z$, the endomorphism ∇v_x of T_xM, independent of the choice of the connection ∇ (see Remark 2.1), may be identified with the 1-jet of v at x.

Given $x, y \in Z$, we say that the 1-jets of v at x and y are *conformally equivalent* if, for some vertical-arrow conformal isomorphism $T_xM \to T_yM$, the following diagram commutes:

$$\begin{array}{ccc} T_xM & \xrightarrow{\nabla v_x} & T_xM \\ \downarrow & & \downarrow \\ T_yM & \xrightarrow{\nabla v_y} & T_yM \end{array}$$

By *conformal isomorphisms* we mean here nonzero scalar multiples of linear isometries.

Proposition 7.1.
Under the assumptions of Theorem 5.1, with ξ defined in Section 6,

(i) *in case* (b) *of Theorem 5.1, for any connected component N' of $N \smallsetminus \Sigma$, the 1-jets of v at all points of N' are conformally equivalent to one another, but not to the 1-jet of v at any point of Σ,*

(ii) *in both cases* (a) – (b) *of Theorem 5.1, if ξ is not identically zero on a connected component Σ' of Σ, then the 1-jets of v at any two points of Σ' are conformally equivalent.*

Proof. At any $y \in N'$, a local conformal change of the metric turns v into a Killing field. Remark 2.3 now yields (i), as ∇v becomes parallel along a neighborhood of y in N'. The claim about Σ is obvious from Theorem 5.1(vi).

In (ii), the definition of ξ (see Section 6) implies that $\phi = 0$ on Σ'. Thus, by Theorem 3.2, $\nabla\phi_x \notin \nabla v_x(T_xM)$ at every point $x \in \Sigma'$, and so (5) gives $\mathcal{H}_x^\perp = \nabla v_x(T_xM) \oplus \mathrm{I\!R}\nabla\phi_x$. (Note that $\nabla v_x(T_xM) = [\operatorname{Ker}\nabla v_x]^\perp$ by Remark 3.2(b).) Hence, in view of Theorem 5.1(ii), $T_x\Sigma' \subset \nabla v_x(T_xM) \oplus \mathrm{I\!R}\nabla\phi_x$, while the vectors tangent to Σ' at x and lying in the summand $\nabla v_x(T_xM)$ form precisely the subspace $\operatorname{Ker}\xi_x$, which has codimension one in $T_x\Sigma'$ for all points x of a dense open subset of Σ' (Proposition 6.1(iv)). We will now show that the conformal equivalence type of the 1-jets of v is constant along any geodesic segment Γ in Σ' with a parametrization $t \mapsto y(t)$ satisfying the condition $\dot y(t) \notin \nabla v_x(T_xM)$ at each $x = y(t)$. (As any two points of Σ' can be joined by piecewise smooth curves made up from such geodesic segments, in view of the denseness and openness property just mentioned, (ii) will then clearly follow.)

Specifically, our assumption about $\dot y(t)$ yields $\nabla\phi = \rho\dot y + \nabla_w v$ for some function $t \mapsto \rho(t)$ and a vector field $t \mapsto w(t) \in T_{y(t)}M$ along the geodesic; since $\operatorname{rank}\nabla v$ is constant on Σ' by [4, Lemma 13.1(d)], w may be chosen differentiable. As $\phi = 0$ on Σ', (3) gives $2\nabla_{\dot y}\nabla v = g(\nabla\phi, \cdot\,) \otimes \dot y - g(\dot y, \cdot\,) \otimes \nabla\phi$, and ∇v is easily verified to be

D-parallel for the new metric connection D in $T_\Gamma M$ given by $2\mathrm{D}_{\dot y} = 2\nabla_{\dot y} + g(w,\,\cdot\,)\otimes \dot y - g(\dot y,\,\cdot\,)\otimes w$. (Note that $\Gamma \subset \Sigma'$ consists of zeros of v, so that, for $x = y(t)$, the vector $\dot y(t)$ lies in $\mathrm{Ker}\,\nabla v_x$, and hence is orthogonal to $\nabla v_x(T_xM)$, while $\nabla v_x : T_xM \to T_xM$ is skew-adjoint, cf. Remark 3.2(b).) □

8. The associated quintuples

The symbol $[\eta]$ stands for the *homothety class* of a pseudo-Euclidean inner product η on a finite-dimensional vector space $\mathcal{T}$, that is, the set of all nonzero scalar multiples of η. The underlying conformal structure $[g]$ of a pseudo-Riemannian manifold (M, g) may thus be identified with the assignment $M \ni x \mapsto [g_x]$. Let us consider quintuples

$$(\mathcal{T},\,[\eta],\,B,\,\lambda,\,\delta) \tag{18}$$

formed by a pseudo-Euclidean vector space $\mathcal{T}$, the homothety class of its inner product η, a skew-adjoint endomorphism $B \in \mathfrak{so}(\mathcal{T})$, a real number λ, and a linear functional $\delta \in [\mathrm{Ker}\,(B+\lambda)]^*$ on the subspace $\mathrm{Ker}\,(B+\lambda)$ of $\mathcal{T}$ (which, if nontrivial, is the eigenspace of B for the eigenvalue $-\lambda$).

We call (18) *algebraically equivalent* to another such quintuple $(\mathcal{T}',\,[\eta'],\,B',\,\lambda',\,\delta')$ if $\lambda' = \lambda$ and some linear isomorphism $\mathcal{T} \to \mathcal{T}'$ sends $[\eta], B, \delta$ to $[\eta']$, B' and δ'.

Examples of quintuples (18) arise as follows. Given a conformal vector field v on a pseudo-Riemannian manifold (M, g) and a point $x \in M$ at which $v_x = 0$, we define the *quintuple associated with* v *and* x to be $(\mathcal{T},\,[\eta],\,B,\,\lambda,\,\delta) = (T_xM,\,[g_x],\,A_x,\,\phi(x),\delta)$, where A and ϕ are determined by v as in (1), and δ is the restriction of $d\phi_x$ to the subspace $\mathrm{Ker}\,(B+\lambda) = \mathrm{Ker}\,\nabla v_x$. In other words, B equals twice the skew-adjoint part of $\nabla v_x : T_xM \to T_xM$ (the value at x of the morphism $\nabla v : TM \to TM$), and λ is $2/n$ times $\mathrm{tr}\,\nabla v_x$, where $n = \dim M$.

The associated quintuple $(\mathcal{T},\,[\eta],\,B,\,\lambda,\,\delta)$ depends, besides v and x, only on the underlying conformal structure $[g]$, rather than the metric g. This is obvious for $\mathcal{T}, [\eta], B$ and $\lambda = (2/n)\,\mathrm{tr}\,\nabla v_x$, cf. the beginning of Section 7. Similarly, as $\mathrm{Ker}\,(B+\lambda) = \mathrm{Ker}\,\nabla v_x$, the last line in Remark 2.2 yields the claim about δ.

9. Conformal equivalence of two-jets

Let v and w be conformal vector fields on pseudo-Riemannian manifolds (M, g) and, respectively, (N, h), such that v vanishes at a point $x \in M$, and w at $y \in N$. We say that the 2-jet of v at x is *conformally equivalent* to the 2-jet of w at y if some diffeomorphism F between a neighborhood U of x in M and one of y in N, with $F(x) = y$, sends the former 2-jet to the latter, while, at the same time, for some function $\tau : U \to \mathrm{I\!R}$, the metrics F^*h and $e^\tau g$ have the same 1-jet at x.

As v and w vanish at x and y, the above condition on F involves F only through its 2-jet at x.

Lemma 9.1.
For M, g, v, x and N, h, w, y as in the last two paragraphs, the 2-jets of v at x and of w at y are conformally equivalent if and only if the quintuple $(\mathcal{T}, [\eta], B, \lambda, \delta)$ associated with v and x is algebraically equivalent, in the sense of Section 8, to the analogous quintuple $(\mathcal{T}', [\eta'], B', \lambda', \delta')$ for w and y.

Proof. The 'only if' part of our claim is obvious from functoriality of the associated quintuple. To prove the 'if' part, we fix local coordinates x^j for M at x and y^a for N at y such that the corresponding Christoffel symbols of g, or h vanish at x, or y. We also set $v_j^k = \partial_j v^k$, $F_j{}^a = \partial_j F^a$, $F_{jk}^a = \partial_j \partial_k F^a$, $\tau_j = \partial_j \tau$, $w_c^a = \partial_c w^a$, where all the partial derivatives stand for their values at x or y (and those involving F or τ are treated as unknowns).

It now suffices to show that, if $(\mathcal{T}, [\eta], B, \lambda, \delta)$ and $(\mathcal{T}', [\eta'], B', \lambda', \delta')$ are equivalent, the system

$$\begin{aligned} &\text{(i)} \quad w_c^a F_j{}^c = F_k{}^a v_j^k, \\ &\text{(ii)} \quad F_l{}^a \,\partial_j \partial_k v^l + F_{jl}^a v_k^l + F_{kl}^a v_j^l = F_j{}^b F_k{}^c \,\partial_b \partial_c w^a + F_{jk}^c w_c^a, \\ &\text{(iii)} \quad h_{ac} F_j{}^a F_k{}^c = e^\tau g_{jk}, \qquad \text{iv)} \quad h_{ac}(F_j{}^a F_{kl}^c + F_{jl}^a F_k{}^c) = e^\tau \tau_l \, g_{jk}, \end{aligned} \tag{19}$$

where the values of $g_{jk}, \partial_j \partial_k v^l, h_{ac}$ and $\partial_b \partial_c w^a$ are taken at x or y, has a solution consisting of a real number τ and some quantities $F_j{}^a$, F_{jk}^a, τ_j with $F_{jk}^a = F_{kj}^a$.

As the first part of such a solution we choose a matrix $F_j{}^a$ which, when treated as a linear isomorphism $T_x M \to T_y N$ (with the aid of our fixed coordinates x^j and y^a), realizes the equivalence of the two quintuples. As $\lambda' = \lambda$ and the isomorphism in question sends $([\eta], B, \delta)$ to $([\eta'], B', \delta')$, we now clearly have (19)-(iii) for some $\tau \in \mathrm{I\!R}$, (19)-(i), and there exists a 1-form $\sigma \in T_x^* M$ with $\phi_{,j} - \psi_{,a} F_j{}^a = 2 v_j^k \sigma_k$, where ϕ is determined by v as in (1), ψ is its analog for w, and the components of $d\phi$ and $d\psi$ are evaluated at x or y. (That σ exists is obvious since our isomorphism sends δ to δ', and so $u^j(\phi_{,j} - \psi_{,a} F_j{}^a) = 0$ whenever $u \in T_x M$ and $u^j v_j^k = 0$.) It follows that

$$\text{(a)} \;\; g^{jk} F_j{}^a F_k{}^c = e^\tau h^{ac}, \quad \text{(b)} \;\; v_k^j \sigma^k = \phi \sigma^j - g^{jl} v_l^k \sigma_k, \;\; \text{where} \;\; \sigma^j = g^{jk} \sigma_k. \tag{20}$$

In fact, (19)-(iii) states that the matrix $e^{-\tau/2} F_j{}^a$, as a linear isomorphism $T_x M \to T_y N$, sends the metric g_x to h_y, and so the reciprocal metrics of g_x and h_y correspond to each other under the dual isomorphism $T_y^* N \to T_x^* M$. This amounts to (20)-(a), while (20)-(b) is a trivial consequence of (1).

Finally, let us set $F_{jk}^a = F_l{}^a \sigma^l g_{jk} - \sigma_j F_k{}^a - \sigma_k F_j{}^a$ and $\tau_j = -2\sigma_j$. Then (19)-(iii) implies (19)-(iv). Next, at x and y, in our special coordinates, the coordinate form of (3) yields $2 \partial_j \partial_k v^l = \phi_{,k} \delta_j^l - \phi^{,l} g_{jk} + \phi_{,j} \delta_k^l$ and, analogously, $2 \partial_b \partial_c w^a = \psi_{,c} \delta_b^a - \psi^{,a} h_{bc} + \psi_{,b} \delta_c^a$. Replacing $\phi_{,j}$ here with $\psi_{,a} F_j{}^a + 2 v_j^k \sigma_k$ (cf. our choice of σ in the lines preceding

(20)), we see that (19)-(ii) follows from (19)-(i), (19)-(iii) and (20)-(a) since, by (1), $v_j^l F_p^a \sigma^p g_{kl} + v_k^l F_p^a \sigma^p g_{jl} = \phi F_p^a \sigma^p g_{jk}$, while (19)-(i) and (20)-(b) give $w_c^a F_l^c \sigma^l = F_p^a v_l^p \sigma^l = \phi F_p^a \sigma^p - g^{pq} F_p^a v_q^l \sigma_l$. □

10. Two-jets of v along components of Z

Proposition 7.1 remains true if the word 1-jet(s) is replaced everywhere with 2-jet(s).

For both assertions (i) and (ii), this is a direct consequence of Lemma 9.1. Specifically, in the case of (i), the invariant δ vanishes at every point of $N \smallsetminus \Sigma$ by Theorem 3.2, while the remaining four objects in the associated quintuple (18) represent the 1-jet of v at the point in question. The algebraic-equivalence type of the quintuple is thus locally constant on $N \smallsetminus \Sigma$ as a consequence of Proposition 7.1.

Similarly, for the new metric connection D in $T_\Gamma M$ used to prove Proposition 7.1(ii), the second formula in (3) shows that the restriction of $d\phi$ to $\mathrm{Ker}\,\nabla v$ is D-parallel as well, as long as one chooses w with $g(w, \nabla\phi) = 0$. Since $\nabla\phi_x \notin \nabla v_x(T_x M) = [\mathrm{Ker}\,\nabla v_x]^\perp$, whenever $x \in \Sigma'$, such a choice is always possible.

The following example shows that the assumption about ξ in Proposition 7.1(ii) cannot in general be removed. On a pseudo-Euclidean space $(V, \langle\,,\rangle)$ of dimension n we may define a conformal vector field v by

$$v_x = w + Bx + cx + 2\langle u, x\rangle x - \langle x, x\rangle u,$$

using any fixed vectors $w, u \in V$, any skew-adjoint endomorphism B, and any scalar $c \in \mathrm{I\!R}$. Let us now choose n to be even, $\langle\,,\rangle$ to have the neutral signature, B with two null n-dimensional eigenspaces for the nonzero eigenvalues $c, -c$, and u which does not lie in the $-c$ eigenspace, along with $w = 0$. Then $\dim \mathrm{Ker}\,\nabla v_x$ is easily verified to decrease when one replaces $x = 0$ by any nearby $x \neq 0$ orthogonal to u and lying in the $-c$ eigenspace of B.

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