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ON HOMOGENEOUS CONFORMALLY SYMMETRIC PSEUDO-RIEMANNIAN MANIFOLDS

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1. Introduction. An n-dimensional ($n \geqslant 4$) pseudo-Riemannian manifold (M, g) is called *conformally symmetric* [1] if $\nabla C = 0$, where ∇ denotes the Levi-Cività connection and C is the Weyl conformal curvature tensor of M given by

$$C_{hijk} = R_{hijk} - (n-2)^{-1}(g_{ij}S_{hk} + g_{hk}S_{ij} - g_{hj}S_{ik} - g_{ik}S_{hj}) + \\ + K(n-1)^{-1}(n-2)^{-1}(g_{ij}g_{hk} - g_{hj}g_{ik}),$$

R, S and $K = g^{ij}S_{ij} = g^{ij}g^{hk}R_{hijk}$ being the curvature tensor, the Ricci tensor and the scalar curvature of (M, g), respectively. The manifold (M, g) is said to be *essentially conformally symmetric* (shortly, e.c.s.) if it satisfies $\nabla C = 0$, but is neither *conformally flat* ($C = 0$) nor *locally symmetric* ($\nabla R = 0$). The existence of e.c.s. manifolds was proved first by Roter in [10]. The aim of this paper is to investigate homogeneous e.c.s. manifolds (examples of which can be found in [3] and [4]).

In Section 3 we define homogeneous e.c.s. manifolds $M^n_{E,r,F}$, $M^n_{H,r,F}$ and $M^n_{P,s,a,\varepsilon}$ which are universal in the sense that the pseudo-Riemannian universal coverings of their homogeneous open submanifolds exhaust, up to isometry, all simply connected homogeneous e.c.s. manifolds (Theorem 2). This, in particular, implies that a homogeneous e.c.s. manifold must not be geodesically complete (Theorem 3).

Section 4 is devoted to certain remarks on the global structure of simply connected homogeneous e.c.s. manifolds. We prove there (Theorem 4) that such a manifold is always diffeomorphic to a product $\boldsymbol{R}^4 \times M$ or $\boldsymbol{R}^2 \times M$, M being flat and homogeneous. Moreover, if the metric signature is $(- \; - \; + \ldots +)$, then, in some cases, such a manifold must be isometric to a universal model, hence topologically Euclidean (Corollary 1). In general, however, a simply connected homogeneous e.c.s. manifold need not be Euclidean (Theorem 5).

Throughout this paper, by a manifold we shall mean a connected paracompact manifold of class C^∞ or analytic. Concerning pseudo-Riemannian manifolds, we shall often write M instead of (M, g). The group of all isometries of M will be denoted by $I(M)$, while $I^0(M)$ will stand for its identity component.

2. The universal models. Given a pseudo-Riemannian manifold (M, g), by a *local isometry* of M we shall mean any isometry between open connected subsets of M.

LEMMA 1. *Let (M, g) and (M_0, g_0) be two homogeneous pseudo-Riemannian manifolds, locally isometric to each other. If M is simply connected and M_0 has the property*

(1) *any local isometry of M_0 can be extended to a global isometry of M_0 onto itself,*

then there exists an isometric immersion $f\colon M \to M_0$ such that

(i) *the image $f(M)$ is homogeneous (as an open submanifold of M_0),*

(ii) *$f\colon M \to f(M)$ is a covering.*

Proof. Fix $p \in M$, $p_0 \in M_0$ and their neighbourhoods U, U_0, respectively, together with an isometry $h\colon U \to U_0$. Since $I(M_0)$ acts on M_0 transitively, so does its identity component $G_0 = I^0(M_0)$; hence we may write $M_0 = G_0/H_0$, where H_0 is the isotropy subgroup of p_0 and the identification is given by $G_0/H_0 \ni aH_0 \mapsto ap_0 \in M_0$. Similarly, the universal covering group G of $I^0(M)$ acts transitively on M *via* the covering homomorphism $G \to I^0(M)$ and we have $M = G/H$, $H \subset G$ being the isotropy subgroup of p. Note that the elements of the Lie algebra G' of G (respectively, G_0' of G_0) can be viewed as complete Killing vector fields on M (respectively, on M_0) and (1) implies that any Killing field on U_0 can be extended to a unique complete Killing field on M_0. Therefore, the differential h_* of h defines a Lie algebra homomorphism $h_*\colon G' \to G_0'$ (as it assigns a Killing field on U_0 to a Killing field on U), which sends H' (complete Killing fields vanishing at p) into H_0'. Simple connectivity of G implies now that h_* is the differential of a Lie group homomorphism $\bar{h}\colon G \to G_0$. Moreover, H is connected, since it is the fibre of the bundle projection $G \to G/H = M$ with $\pi_1 M = 0$. Hence $\bar{h}(H) \subset H_0$, and so $\bar{h}$ defines a mapping $f\colon G/H \to G_0/H_0$. The $(0, 2)$ tensor fields g^*, g_0^* on G, G_0, respectively, induced from the metrics by the projections $G \to M$, $G_0 \to M_0$, are clearly left-invariant. Thus, since $g_e^*(X, Y) = g(X_p, Y_p)$ for Killing fields $X, Y \in T_eG = G'$ (e being the unit of G), we have $(\bar{h}^* g_0^*)_e = g_e^*$, and hence $\bar{h}^* g_0^* = g^*$ by left-invariance. This shows that f is an isometric immersion. Next, for any $\varphi \in I(M)$, there exists $\varphi_0 \in I(M_0)$ such that $f \circ \varphi = \varphi_0 \circ f$ (in fact, we can define such a φ_0 locally and then extend it by (1)). This, clearly, implies that $f(M)$ is homogeneous. Finally, since

$\bar{h}_*: G' \to G'_0$ induces the isomorphism

$$f_{*,p}: G'/H' \to G'_0/H'_0,$$

we have $\bar{h}_*^{-1}(H'_0) = H'$, which shows that H is the identity component of $\bar{h}^{-1}(H_0)$. It is now clear that the composite

$$G \xrightarrow{\bar{h}} G_0 \to G_0/H_0$$

defines a diffeomorphism $G/\bar{h}^{-1}(H_0) \to f(M)$, which makes the map $f: M \to f(M)$ correspond to the natural mapping

$$\pi: G/H \to G/\bar{h}^{-1}(H_0).$$

However, the discrete group $\bar{h}^{-1}(H_0)/H$ acts on G/H on the right by the formula

$$(aH, b \operatorname{mod} H) \mapsto abH, \qquad b \in \bar{h}^{-1}(H_0)$$

(this is well defined, since H is normal in $\bar{h}^{-1}(H_0)$). The action is clearly free and its orbits coincide with the fibres of π, hence π is a covering. This completes the proof.

As shown in [6] (Theorems 3 and 5), every e.c.s. manifold M satisfies the relations $\operatorname{rank} S \leqslant 2$ and $S_{ij}S_{hk} - S_{hj}S_{ik} = FC_{hijk}$ for some function F, called the *fundamental function* of M. It is obvious that F vanishes exactly at the points where $\operatorname{rank} S \leqslant 1$. If M is homogeneous, F is clearly constant and either $F = 0$ (M is then called *parabolic*) or F is a non-zero constant, i.e., $\operatorname{rank} S = 2$. In the latter case, M is called *elliptic* or *hyperbolic* according to whether S is semidefinite or is not at some (hence each) point (cf. [3], Lemma 1).

Let us also note that there exist e.c.s. manifolds which are Ricci-recurrent in the sense that $S_{hi} \nabla_j S_{kl} = S_{kl} \nabla_j S_{hi}$ ([10], Theorem 3), and that each of them is parabolic ([6], proof of Theorem 5).

Now we formulate certain important examples of e.c.s. manifolds. Fix an integer $n \geqslant 4$. From now on we adopt the convention that the final Roman letters x, y, z take on the values $3, \ldots, n-2$, while the Greek indices α, β, γ vary in the range $2, \ldots, n-1$. Suppose we are given real numbers $F, \varepsilon, \varepsilon_2, \ldots, \varepsilon_{n-1}$ with $F \neq 0 \neq \varepsilon$, $|\varepsilon_\alpha| = 1$, and a non-zero symmetric $(n-2) \times (n-2)$-matrix $a = [a_{\alpha\beta}]$ such that

$$\sum_\alpha \varepsilon_\alpha a_{\alpha\alpha} = 0. \tag{2}$$

We define the pseudo-Riemannian manifolds $(M^n_{E,r,F}, g^E)$, $(M^n_{H,r,F}, g^H)$ and $(M^n_{P,s,a,\varepsilon}, g^P)$ as

$$M^n_{E,r,F} = M^n_{H,r,F} = \mathbf{R}^n, \qquad M^n_{P,s,a,\varepsilon} = \mathbf{R}^n_+ = (0, \infty) \times \mathbf{R}^{n-1},$$

while the essential non-zero metric components (in the Cartesian coordinates $u^1, \ldots, u^n$) are given by

$$g_{11}^E = 2u^n e^{-\lambda} + 2e^{-4\lambda} \sum_x \varepsilon_x (u^x)^2 + (2F)^{-1} e^{8\lambda}, \quad g_{12}^E = -2u^{n-1} e^{-\lambda},$$

$$g_{22}^E = -2u^n e^{-\lambda} + 2e^{-4\lambda} \sum_x \varepsilon_x (u^x)^2 + (2F)^{-1} e^{8\lambda}, \quad g_{1n}^E = g_{2,n-1}^E = e^{\lambda},$$

$$g_{xx}^E = \varepsilon_x, \quad g_{11}^H = -2u^{n-1} e^{-\mu}, \quad g_{12}^H = -e^{-4\mu} \sum_x \varepsilon_x (u^x)^2 - F^{-1} e^{8\mu},$$

$$g_{22}^H = 2u^n e^{-\mu}, \quad g_{1n}^H = g_{2,n-1}^H = e^{\mu}, \quad g_{xx}^H = \varepsilon_x,$$

$$g_{11}^P = \varepsilon (u^1)^{-2} \sum_a \varepsilon_a (u^a)^2 + \sum_{a\beta} a_{a\beta} u^a u^\beta, \quad g_{1n}^P = 1, \quad g_{aa}^P = \varepsilon_a,$$

where $\lambda = \frac{1}{6} \log(2n-4)$, $\mu = \frac{1}{6}\log(n-2)$, and $r-2$ (respectively, $s-1$) is the number of minuses among the ε_x (respectively, among the ε_a). Thus $2 \leqslant r \leqslant n-2$ and $1 \leqslant s \leqslant n-1$. It is clear that r (respectively, s) is equal to the index of g^E or g^H (respectively, of g^P).

Note that g^P is a particular case of the general formula for e.c.s. Ricci-recurrent metrics given by Roter in [10].

LEMMA 2. *In the Cartesian coordinates, any local isometry $f = (f^1, \ldots, f^n)$ of $(M^n_{P,s,a,\varepsilon}, g^P)$ is of the form*

$$f^1 = Tu^1, \quad f^a = \sum_\beta H^a_\beta u^\beta + C^a(u^1),$$
$$(3)\qquad f^n = -T^{-1} \sum_a \varepsilon_a \dot{C}^a(u^1) \Big[\sum_\beta H^a_\beta u^\beta + \tfrac{1}{2} C^a(u^1)\Big] + T^{-1} u^n + z,$$

where $T > 0$ and z are real numbers, $H = [H^a_\beta]$ is a matrix such that

$$(4)\qquad \sum_\gamma \varepsilon_\gamma H^\gamma_a H^\gamma_\beta = \varepsilon_a \delta_{a\beta}, \quad \sum_{\gamma\delta} a_{\gamma\delta} H^\gamma_a H^\delta_\beta = T^{-2} a_{a\beta},$$

and the functions C^a of $u^1 > 0$ form a solution of the system

$$(5)\qquad \ddot{C}^a = \varepsilon (u^1)^{-2} C^a + T^2 \varepsilon_a \sum_\beta a_{a\beta} C^\beta.$$

Conversely, given $T > 0$, z and H^a_β, C^a satisfying (4) *and* (5), *formulae* (3) *define a global isometry of $M^n_{P,s,a,\varepsilon}$ onto itself.*

Proof. One can explicitly compute that (3) together with (4) and (5) define a global isometry. Conversely, let f be a local isometry of $(M, g) = (M^n_{P,s,a,\varepsilon}, g^P)$. Our assertion can now be obtained by proceeding as in [2] (proof of Theorem 2). We can sketch this argument as follows. The differential du^1 is the unique (up to a factor) parallel covariant vector field in M (cf. [10], p. 93). Hence $f^* du^1 = T du^1$ for some constant $T \neq 0$. Raising indices, we obtain $f_* \partial/\partial u^n = T^{-1} \partial/\partial u^n$ (i.e. $\partial_n f^n = T^{-1}$, $\partial_n f^i = 0$

for $i < n$). Moreover,

$$\nabla S = -2\varepsilon(n-2)(u^1)^{-3}(du^1)^3$$

(see [10], p. 93), so that the relation $f^*\nabla S = \nabla S$ yields $f^1 = u^1 \circ f = Tu^1$. As $u^1 > 0$, this implies $T > 0$. Next, f leaves invariant the orthogonal complement D of $\partial/\partial u^n$, which is an integrable codimension one distribution in M. Any leaf N of D is given by $u^1 = \text{const}$ and inherits from M a symmetric connection (as a totally geodesic submanifold), which is flat since $du^2, \ldots, du^n$ are parallel along N (so that $u^2, \ldots, u^n$ are affine coordinates for N). Our local isometry f, wherever defined, sends (local) leaves of D affinely into leaves. Thus, f^α and f^n are affine functions of $u^2, \ldots, u^n$, namely

$$f^\alpha = \sum_\beta f^\alpha_\beta(u^1)u^\beta + C^\alpha(u^1), \quad f^n = \sum_\alpha f^n_\alpha(u^1)u^\alpha + T^{-1}u^n + C^n(u^1),$$

with

(6) $$\det[f^\alpha_\beta(u^1)] \neq 0.$$

We have $g_{1\alpha} = (f^*g)_{1\alpha}$ and similarly for $g_{\alpha\beta}$ and g_{11}, which easily implies

$$\sum_\gamma \varepsilon_\gamma \dot{f}^\gamma_\alpha f^\gamma_\beta = 0$$

(hence f^α_β is constant by (6), say $f^\alpha_\beta(u^1) = H^\alpha_\beta$), and

(7) $$f^n_\alpha = -T^{-1}\sum_\gamma \varepsilon_\gamma \dot{C}^\gamma H^\gamma_\alpha,$$

$$\sum_\gamma \varepsilon_\gamma H^\gamma_\alpha H^\gamma_\beta = \varepsilon_\alpha \delta_{\alpha\beta}, \quad \sum_{\gamma\delta} a_{\gamma\delta} H^\gamma_\alpha H^\delta_\beta = T^{-2} a_{\alpha\beta},$$

(8) $$T\dot{f}^n_\alpha + T^2 \sum_{\beta\gamma} a_{\beta\gamma} H^\beta_\alpha C^\gamma + \varepsilon(u^1)^{-2}\sum_\beta \varepsilon_\beta H^\beta_\alpha C^\beta = 0,$$

and, finally,

(9) $$2T\dot{C}^n + \sum_\alpha \varepsilon_\alpha(\dot{C}^\alpha)^2 + \varepsilon(u^1)^{-2}\sum_\alpha \varepsilon_\alpha (C^\alpha)^2 + T^2\sum_{\alpha\beta} a_{\alpha\beta}C^\alpha C^\beta = 0.$$

Combining now (7) with (8), we obtain (5). From (9) and (5) it follows immediately that

$$d\left(C^n + \frac{1}{2}T^{-1}\sum_\gamma \varepsilon_\gamma \dot{C}^\gamma C^\gamma\right)\Big/ du^1 = 0,$$

whence

$$C^n = -\frac{1}{2}T^{-1}\sum_\gamma \varepsilon_\gamma \dot{C}^\gamma C^\gamma + z$$

for some real z. In view of (7), this completes the proof.

Remark 1. By Lemma 2, homogeneity of $M^n_{P,s,a,\varepsilon}$ is obviously equivalent to the following algebraic condition on $a = [a_{\alpha\beta}]$:

(10) For each $T > 0$ there exists a matrix H satisfying (4).

It is clear that (by using powers of H) (10) may equivalently be formulated as follows:

"For each T sufficiently close to $1 \ldots$".

Therefore, local homogeneity of any particular open submanifold of $M^n_{P,s,a,\varepsilon}$ implies homogeneity of $M^n_{P,s,a,\varepsilon}$. An example for (10) can be formulated as follows:

Let $\varepsilon_2 = -\varepsilon_3 = 1$ ($\varepsilon_4, \ldots, \varepsilon_{n-1}$ of arbitrary signs). The matrix a with the non-zero entries $a_{22} = -a_{23} = -a_{32} = a_{33} = 1$ satisfies (10).

In fact, given $T > 0$, (4) holds for the matrix H whose non-zero components are

$$H_2^2 = H_3^3 = (2T)^{-1}(T^2+1), \quad H_2^3 = H_3^2 = (2T)^{-1}(T^2-1),$$

$$H_i^i = 1 \quad \text{for } 4 \leqslant i \leqslant n-1.$$

However, non-zero symmetric matrices a satisfying (10) are of a very exceptional type. For such an a, set $\bar{a}_\alpha^{\ \beta} = \varepsilon_\beta a_{\alpha\beta}$, so that (4) yields $H\bar{a}H^{-1} = T^{-2}\bar{a}$. It follows now easily that the characteristic polynomial $P(\xi) = \det(\bar{a} - \xi \cdot \mathrm{Id})$ of $\bar{a}$ is positively homogeneous of degree $n-2$. Hence $P(\xi) = (-1)^{n-2}\xi^{n-2}$. In particular, $\det\bar{a} = 0 = \det a$ and $\operatorname{tr}\bar{a} = 0$ (so that (2) is a consequence of (10)). Consequently, the existence of a matrix a satisfying (10) implies $2 \leqslant s \leqslant n-2$ (i.e., among the ε_α there must occur both plus and minus signs). In fact, in the case $s = 1$ (or $s = n-1$), the matrix $\bar{a} \neq 0$ is symmetric, and so it has a non-zero eigenvalue.

From now on, $M^n_{E,r,F}$, $M^n_{H,r,F}$ and those $M^n_{P,s,a,\varepsilon}$ for which the matrix a satisfies (10) will be referred to as (elliptic, hyperbolic or parabolic) *universal models*.

In the following theorem we use the notation introduced above.

THEOREM 1. (i) *$M^n_{E,r,F}$ (respectively, $M^n_{H,r,F}$) is an n-dimensional elliptic (respectively, hyperbolic) e.c.s. pseudo-Riemannian manifold with the metric of index r and the fundamental function F.*

(ii) *$M^n_{E,r,F}$ and $M^n_{H,r,F}$ are homogeneous. More precisely, they admit Lie group structures which make their metrics left-invariant.*

(iii) *$M^n_{P,s,a,\varepsilon}$ is Ricci-recurrent (hence parabolic) with the metric of index s. It is homogeneous if and only if a satisfies* (10).

(iv) *Every n-dimensional homogeneous elliptic (respectively, hyperbolic or parabolic) e.c.s. manifold M is locally isometric to $M^n_{E,r,F}$ (respectively, $M^n_{H,r,F}$ or $M^n_{P,s,a,\varepsilon}$) for some r and F or s, ε and a satisfying* (10).

Here r (respectively, s) is the metric index of M while (in the elliptic or hyperbolic case) F is its fundamental function.

Proof. For $M^n_{E,r,F}$ and $M^n_{H,r,F}$, (i) and (iv) are stated explicitly in [3] (Remark 5) and [4] (Remark 3). As for (ii), finding a Lie group structure on M compatible with the metric g is equivalent (provided that $\pi_1 M = 0$) to appointing a flat complete connection $\bar{\nabla}$ on M with parallel torsion and such that $\bar{\nabla} g = 0$ ([7] and [8], p. 6). This in turn can be reduced to constructing a $\bar{\nabla}$-parallel frame field on M, i.e., vector fields $e_1, \ldots, e_n$ on M ($n = \dim M$), linearly independent at each point, spanning an n-dimensional Lie algebra, satisfying $g(e_i, e_j) = \text{const}$ and such that their combinations (with constant coefficients) are complete vector fields. Thus, for $M = M^n_{E,r,F}$ we can put

$$e_1 = e^{-\lambda} d_1 + e^{-3\lambda} \sum_x u^x d_x + \left(\frac{3}{2}\sqrt{3} e^{-6\lambda} \sum_x (u^x)^2 - u^{n-1} e^{-3\lambda}\right) d_{n-1} -$$
$$- \left(u^n e^{-3\lambda} + \frac{3}{2} e^{-6\lambda} \sum_x \varepsilon_x (u^x)^2 + (4F)^{-1} e^{6\lambda}\right) d_n,$$
$$e_2 = e^{-\lambda} d_2 - \sqrt{3} e^{-3\lambda} \sum_x \varepsilon_x u^x d_x + \left(e^{-3\lambda} u^n - \frac{5}{2} e^{-6\lambda} \sum_x \varepsilon_x (u^x)^2 -\right.$$
$$\left. - (4F)^{-1} e^{6\lambda}\right) d_{n-1} + \left(3u^{n-1} e^{-3\lambda} - \frac{1}{2}\sqrt{3} e^{-6\lambda} \sum_x (u^x)^2\right) d_n,$$
$$e_x = d_x + \sqrt{3} e^{-3\lambda} u^x d_{n-1} - \varepsilon_x e^{-3\lambda} u^x d_n, \qquad e_{n-1} = d_{n-1}, \qquad e_n = d_n,$$

while for $M^n_{H,r,F}$ it is sufficient to set

$$e_1 = e^{-\mu} d_1 - e^{-3\mu} \sum_x u^x d_x + \left(e^{-6\mu} \sum_x \varepsilon_x (u^x)^2 - u^{n-1} e^{-3\mu} - u^n e^{-3\mu}\right) d_{n-1} +$$
$$+ \left(u^{n-1} e^{-3\mu} - \frac{1}{2} e^{-6\mu} \sum_x \varepsilon_x (u^x)^2\right) d_n,$$
$$e_2 = e^{-\mu} d_2 + e^{-3\mu} \sum_x u^x d_x - \left(u^n e^{-3\mu} + \frac{1}{2} e^{-6\mu} \sum_x \varepsilon_x (u^x)^2\right) d_{n-1} +$$
$$+ \left(F^{-1} e^{6\mu} + u^n e^{-3\mu} + u^{n-1} e^{-3\mu} + e^{-6\mu} \sum_x \varepsilon_x (u^x)^2\right) d_n,$$
$$e_x = d_x - \varepsilon_x e^{-3\mu} u^x d_{n-1} + \varepsilon_x e^{-3\mu} u^x d_n, \qquad e_{n-1} = d_{n-1}, \qquad e_n = d_n$$

(where $d_i = \partial / \partial u^i$). Finally, $M^n_{P,s,a,\varepsilon}$ is Ricci-recurrent (hence parabolic) by Theorem 3 of [10], and (iii) is immediate from Remark 1. Consider now an arbitrary homogeneous parabolic e.c.s. manifold (M, g), $\dim M = n$, and fix $p \in M$. By homogeneity, $S_p \neq 0$ and $\nabla S_p \neq 0$. Moreover, M is Ricci-recurrent ([5], Proposition 1). Hence, in view of Theorem 3 of [10], there exists a coordinate system $u^1, \ldots, u^n$ at p such that

$$g = g_{11}(du^1)^2 + \sum_a \varepsilon_a (du^a)^2 + du^1 \otimes du^n + du^n \otimes du^1$$

with

$$|\varepsilon_a| = 1 \quad \text{and} \quad g_{11} = A(u^1) \sum_a \varepsilon_a (u^a)^2 + \sum_{a\beta} a_{a\beta} u^a u^\beta,$$

where A is a non-constant function of u^1 and $a = [a_{a\beta}]$ is a non-zero symmetric matrix satisfying (2). Since

$$S = (n-2)A(du^1)^2 \quad \text{and} \quad \nabla S = (n-2)A'(du^1)^3$$

([10], p. 93), the codirectional covariant vector fields

$$v = [(n-2)|A|]^{1/2} du^1 \quad \text{and} \quad w = [(n-2)A']^{1/3} du^1$$

are invariant under all isometries of M (as they are determined by $S = \pm v \otimes v$, $\nabla S = w \otimes w \otimes w$), and hence their proportionality coefficient $(n-2)^{-1/6}(A')^{1/3}|A|^{-1/2}$ is constant. This clearly yields $A(u^1) = \varepsilon(u^1 + b)^{-2}$ for some $\varepsilon \neq 0$ and b. By an obvious change of coordinates (translation of u^1 followed by reflection of u^1 and u^n, if necessary), we obtain for g an expression of type g^P with $u^1 > 0$, i.e., a neighbourhood of p is isometric to an open subset of some $M^n_{P,s,a,\varepsilon}$. This subset is locally homogeneous and so, by Remark 1, a satisfies (10), which completes the proof.

Going on to a further study of the universal models, let us investigate the structure of their local isometries.

LEMMA 3. *Any local isometry of $M^n_{E,r,F}$ or of $M^n_{H,r,F}$ is of the form $B \circ h$, where $h = (h^1, \ldots, h^n)$ is given by*

$$(11) \quad \begin{cases} h^1 = u^1, \quad h^2 = u^2, \quad h^x = \sum_y h^x_y u^y + D^x(u^1, u^2), \\ h^{n-1} = u^{n-1} + \sum_y D^{n-1}_y(u^1, u^2) u^y + G^{n-1}(u^1, u^2), \\ h^n = u^n + \sum_y D^n_y(u^1, u^2) u^y + G^n(u^1, u^2), \end{cases}$$

h^x_y being a constant matrix such that

$$(12) \quad \sum_z \varepsilon_z h^z_x h^z_y = \varepsilon_x \delta_{xy},$$

and the following conditions are satisfied:

(i) *For* $M^n_{E,r,F}$,

(13) $$D^x(u^1,u^2) = \zeta_x\exp(-2u^1e^{-2\lambda}) + P_x\exp(u^1e^{-2\lambda}+u^2\sqrt{3}e^{-2\lambda}) + \\ + Q_x\exp(u^1e^{-2\lambda}-u^2\sqrt{3}e^{-2\lambda}),$$

(14) $$D^{n-1}_x = -e^{-\lambda}\sum_z \varepsilon_z h^z_x\partial_2 D^z, \qquad D^n_x = -e^{-\lambda}\sum_z \varepsilon_z h^z_x \partial_1 D^z,$$

$$\begin{aligned} G^{n-1} = {} & \varkappa_0\exp(-u^1e^{-2\lambda}+u^2\sqrt{3}e^{-2\lambda}) + \sigma_0\exp(-u^1e^{-2\lambda}-u^2\sqrt{3}e^{-2\lambda}) - \\ & -\frac{1}{2}\sqrt{3}e^{-3\lambda}\sum_x \varepsilon_x P_x^2\exp(2u^1e^{-2\lambda}+2u^2\sqrt{3}e^{-2\lambda}) + \\ & +\frac{1}{2}\sqrt{3}e^{-3\lambda}\sum_x \varepsilon_x Q_x^2\exp(2u^1e^{-2\lambda}-2u^2\sqrt{3}e^{-2\lambda}) - \\ & -\frac{2}{3}\sqrt{3}e^{-3\lambda}\sum_x \varepsilon_x\zeta_x P_x\exp(-u^1e^{-2\lambda}+u^2\sqrt{3}e^{-2\lambda}) + \\ & +\frac{2}{3}\sqrt{3}e^{-3\lambda}\sum_x \varepsilon_x\zeta_x Q_x\exp(-u^1e^{-2\lambda}-u^2\sqrt{3}e^{-2\lambda}) + \tau_0\exp(2u^1e^{-2\lambda}), \end{aligned}$$

$$\begin{aligned} G^{n} = {} & \varkappa_0\sqrt{3}\exp(-u^1e^{-2\lambda}+u^2\sqrt{3}e^{-2\lambda}) - \sigma_0\sqrt{3}\exp(-u^1e^{-2\lambda}-u^2\sqrt{3}e^{-2\lambda}) - \\ & -\frac{1}{2}e^{-3\lambda}\sum_x \varepsilon_x P_x^2\exp(2u^1e^{-2\lambda}+2u^2\sqrt{3}e^{-2\lambda}) - \\ & -\frac{1}{2}e^{-3\lambda}\sum_x \varepsilon_x Q_x^2\exp(2u^1e^{-2\lambda}-2u^2\sqrt{3}e^{-2\lambda}) - \\ & -e^{-3\lambda}\sum_x \varepsilon_x P_x Q_x\exp(2u^1e^{-2\lambda}) + e^{-3\lambda}\sum_x \varepsilon_x\zeta_x^2\exp(-4u^1e^{-2\lambda}), \end{aligned}$$

and

(15) $$B^1 = \alpha u^1+\delta\beta u^2+c^1, \quad B^2 = -\beta u^1+\delta\alpha u^2+c^2, \quad B^x = u^x, \\ B^{n-1} = \delta\alpha u^{n-1}-\beta u^n, \quad B^n = \delta\beta u^{n-1}+\alpha u^n,$$

where $\alpha = \cos(2k\pi/3)$, $\beta = \sin(2k\pi/3)$ *for an integer* k, $|\delta| = 1$, *and* $c^1, c^2, \zeta_x, P_x, Q_x$ $(x = 3, \ldots, n-2)$, $\varkappa_0, \sigma_0, \tau_0$ *are real parameters.*

(ii) *For* $M^n_{H,r,F}$,

(16) $$D^x(u^1,u^2) = \zeta_x\exp(-u^1e^{-2\mu}+u^2e^{-2\mu}) + \left[P_x\sin\left(\frac{1}{2}\sqrt{3}e^{-2\mu}(u^1+u^2)\right) + \right. \\ \left. + Q_x\cos\left(\frac{1}{2}\sqrt{3}e^{-2\mu}(u^1+u^2)\right)\right]\exp\left(\frac{1}{2}e^{-2\mu}(u^1-u^2)\right),$$

(17) $$D^{n-1}_x = -e^{-\mu}\sum_z \varepsilon_z h^z_x\partial_2 D^z, \qquad D^n_x = -e^{-\mu}\sum_z \varepsilon_z h^z_x\partial_1 D^z,$$

and $G^{n-1} = G_0^{n-1}+g^{n-1}$, $G^n = G_0^n+g^n$, *where*

(18)
$$g^{n-1} = \tau_0 \exp\left(e^{-2\mu}(u^1-u^2)\right) - \frac{1}{2}\left[(\varkappa_0-\sqrt{3}\sigma_0)\cos\left(\frac{1}{2}\sqrt{3}e^{-2\mu}(u^1+u^2)\right) + \right.$$
$$\left. +(\sigma_0+\sqrt{3}\varkappa_0)\sin\left(\frac{1}{2}\sqrt{3}e^{-2\mu}(u^1+u^2)\right)\right]\exp\left(\frac{1}{2}e^{-2\mu}(u^2-u^1)\right),$$
$$g^{n} = \tau_0 \exp\left(e^{-2\mu}(u^1-u^2)\right) + \left[\varkappa_0\cos\left(\frac{1}{2}\sqrt{3}e^{-2\mu}(u^1+u^2)\right) + \right.$$
$$\left. +\sigma_0\sin\left(\frac{1}{2}\sqrt{3}e^{-2\mu}(u^1+u^2)\right)\right]\exp\left(\frac{1}{2}e^{-2\mu}(u^2-u^1)\right),$$

(19)
$$G_0^{n-1} = -\frac{1}{2}e^{-3\mu}\sum_x \varepsilon_x\zeta_x^2\exp(-2u^1e^{-2\mu}+2u^2e^{-2\mu}) +$$
$$+\frac{1}{8}e^{-3\mu}\left(\exp\left(e^{-2\mu}(u^1-u^2)\right)\right)\left[\sum_x \varepsilon_x(Q_x^2-P_x^2-2\sqrt{3}P_xQ_x)\times\right.$$
$$\times\cos\left(\sqrt{3}e^{-2\mu}(u^1+u^2)\right)+$$
$$+\sum_x \varepsilon_x(2P_xQ_x-\sqrt{3}P_x^2+\sqrt{3}Q_x^2)\sin\left(\sqrt{3}e^{-2\mu}(u^1+u^2)\right)+$$
$$\left.+2\sum_x \varepsilon_x(P_x^2+Q_x^2)\right]+$$
$$+\frac{1}{2}e^{-3\mu}\left(\exp\left(\frac{1}{2}e^{-2\mu}(u^2-u^1)\right)\right)\left[\sum_x \varepsilon_x(\sqrt{3}\zeta_xQ_x-\zeta_xP_x)\times\right.$$
$$\times\sin\left(\frac{1}{2}\sqrt{3}e^{-2\mu}(u^1+u^2)\right)-$$
$$\left.-\sum_x \varepsilon_x(\sqrt{3}\zeta_xP_x+\zeta_xQ_x)\cos\left(\frac{1}{2}\sqrt{3}e^{-2\mu}(u^1+u^2)\right)\right],$$

(20)
$$G_0^{n} = \frac{1}{2}e^{-3\mu}\sum_x \varepsilon_x\zeta_x^2\exp(-2u^1e^{-2\mu}+2u^2e^{-2\mu}) +$$
$$+\frac{1}{8}e^{-3\mu}\left(\exp\left(e^{-2\mu}(u^1-u^2)\right)\right)\left[\sum_x \varepsilon_x(\sqrt{3}Q_x^2-\sqrt{3}P_x^2-2P_xQ_x)\times\right.$$
$$\times\sin\left(\sqrt{3}e^{-2\mu}(u^1+u^2)\right)+$$
$$\left.+\sum_x \varepsilon_x(P_x^2-Q_x^2-2\sqrt{3}P_xQ_x)\cos\left(\sqrt{3}e^{-2\mu}(u^1+u^2)\right)\right],$$

and

$$(21)\quad \begin{aligned} &B^1 = \zeta u^1+(\zeta-1)u^2+c^1, \quad B^2 = (\zeta-1)u^1+\zeta u^2+c^2, \quad B^x = u^x, \\ &B^{n-1} = \zeta u^{n-1}+(\zeta-1)u^n, \quad B^n = (\zeta-1)u^{n-1}+\zeta u^n, \end{aligned}$$

where $\zeta = 0$ *or* $\zeta = 1$, *and* $c^1, c^2, \zeta_x, P_x, Q_x$ $(x = 3, \ldots, n-2)$, $\varkappa_0, \sigma_0, \tau_0$ *are real parameters.*

Conversely, the transformations B *and* h *of the above type (for any* $c^1, c^2, \zeta_x, P_x, Q_x, \varkappa_0, \sigma_0, \tau_0$ *with* $|\delta| = 1$, k *an integer or* $\zeta = 0$ *or* $\zeta = 1$) *are global isometries of* $M^n_{E,r,F}$ *or of* $M^n_{H,r,F}$, *respectively, onto itself.*

Proof. The connection and Ricci tensor components for $M_E = M^n_{E,r,F}$ and $M_H = M^n_{H,r,F}$ are calculated in [3] (proof of Theorem 4) and [4] (proof of Theorem 3). Namely, the essential components of S and ∇S are $S_{11} = S_{22} = -e^{2\lambda}$, $\nabla_1 S_{11} = -2$, $\nabla_2 S_{12} = \nabla_1 S_{22} = 2$ for M_E and $S_{12} = e^{2\mu}$, $\nabla_1 S_{11} = -2$, $\nabla_2 S_{22} = 2$ for M_H. Thus, $\mathrm{Lin}(d_{n-1}, d_n) = \mathrm{im} S$ is invariant under any local isometry $f = (f^1, \ldots, f^n)$ of M_E or M_H, and so is

$$\mathrm{Lin}(d_3, \ldots, d_n) = \ker S \quad (d_i = \partial/\partial u^i).$$

However, $f_* d_i|_{f(u)} = (\partial_i f^1, \ldots, \partial_i f^n)_u$, so that we obtain

$$(22)\quad \partial_{n-1}f^1 = \partial_{n-1}f^2 = \partial_{n-1}f^x = \partial_n f^1 = \partial_n f^2 = \partial_n f^x = \partial_x f^1 = \partial_x f^2 = 0.$$

Fix a point p in the domain of f. The relation $f^* S = S$ implies

$$(\partial_1 f^1)^2+(\partial_1 f^2)^2 = 1 = (\partial_2 f^1)^2+(\partial_2 f^2)^2, \quad \partial_1 f^1\cdot\partial_2 f^1+\partial_1 f^2\cdot\partial_2 f^2 = 0,$$

whence

$$\partial_1 f^1 = \cos\varphi, \quad \partial_2 f^1 = \delta\sin\varphi, \quad \partial_1 f^2 = -\sin\varphi, \quad \partial_2 f^2 = \delta\cos\varphi, \quad |\delta| = 1,$$

$\varphi = \varphi(u^1, u^2)$ being defined close to p (for M_E), while for M_H the corresponding relations are

$$\partial_1 f^1\cdot\partial_1 f^2 = 0 = \partial_2 f^1\cdot\partial_2 f^2, \quad \partial_1 f^1\cdot\partial_2 f^2+\partial_1 f^2\cdot\partial_2 f^1 = 1,$$

which implies

$$\partial_1 f^1 = \cos(m\pi/2)e^\theta, \quad \partial_2 f^1 = \sin(m\pi/2)e^{-\theta},$$
$$\partial_1 f^2 = \sin(m\pi/2)e^\theta, \quad \partial_2 f^2 = \cos(m\pi/2)e^{-\theta},$$

m being an integer, and $\theta = \theta(u^1, u^2)$. Now $f^*\nabla S = \nabla S$ yields, for M_E,

$$\sin\varphi(\sin^2\varphi-3\cos^2\varphi) = 0 \quad \text{and} \quad \cos\varphi(\cos^2\varphi-3\sin^2\varphi) = 1,$$

so that $\varphi = 2k\pi/3$, k — an integer, and, for M_H,

$$\big(\cos^3(m\pi/2)-\sin^3(m\pi/2)\big)e^{3\theta} = 1,$$

whence $\theta = 0$ and $m \equiv 0$ or $m \equiv 3 \pmod 4$, i.e.,

$$\partial_1 f^1 = \zeta, \quad \partial_2 f^1 = \zeta - 1 = \partial_1 f^2, \quad \partial_2 f^2 = \zeta,$$

where $\zeta = 0$ or $\zeta = 1$. In view of (22), this says that, for M_E,

$$f^1 = u^1 \cos(2k\pi/3) + \delta u^2 \sin(2k\pi/3) + c^1,$$

$$f^2 = -u^1 \sin(2k\pi/3) + \delta u^2 \cos(2k\pi/3) + c^2,$$

and, for M_H,

$$f^1 = \zeta u^1 + (\zeta - 1) u^2 + c^1, \quad f^2 = (\zeta - 1) u^1 + \zeta u^2 + c^2$$

(in a neighbourhood of p, c^1, c^2 being real constants). Writing now f in the form $f = B \circ h$, where the global isometry B is given by (15) or (21), respectively, for the local isometry h thus defined we obtain $h^1 = u^1$ and $h^2 = u^2$. Evaluating the equality

$$h^* g = g \tag{23}$$

for the components $g_{1n}, g_{1,n-1}, g_{2,n-1}, g_{2n}$ ($g = g^E$ or $g = g^H$) and using (22), we deduce

$$\partial_{n-1} h^{n-1} = \partial_n h^n = 1, \quad \partial_{n-1} h^n = \partial_n h^{n-1} = 0. \tag{24}$$

Thus, in view of (22) and (23),

$$h_* d_1 = d_1 + \{d_x, d_{n-1}, d_n\}, \quad h_* d_2 = d_2 + \{d_x, d_{n-1}, d_n\},$$

$$h_* d_x = \sum_y (\partial_x h^y \circ h^{-1}) d_y + (\partial_x h^{n-1} \circ h^{-1}) d_{n-1} + (\partial_x h^n \circ h^{-1}) d_n,$$

$$h_* d_{n-1} = d_{n-1}, \quad h_* d_n = d_n,$$

where the bracket is to be read "a combination of". Using the explicit form of the connection components ([3] and [4], loc. cit.), we can observe that $\nabla_{d_i} d_j$ is a combination of d_{n-1}, d_n if $i > 2$ or $j > 2$ and that it vanishes if both $i, j > 2$. Analogous relations hold for the fields $h_* d_i$. Thus, computing $\nabla_{h_* d_x} h_* d_y$ and using (22) and $\det[\partial_x h^y] \neq 0$, we obtain

$$\partial_x \partial_y h^z = \partial_x \partial_y h^{n-1} = \partial_x \partial_y h^n = 0.$$

Similarly, the d_z-coefficient of $\nabla_{h_* d_i} h_* d_x$ vanishes for $i \leqslant 2$, which implies $\partial_1 \partial_x h^y = \partial_2 \partial_x h^y = 0$. Hence, by (22), $h^y_x = \partial_x h^y$ is constant. From the relations above it follows immediately that h is of form (11). Relation (23) applied to the component g_{xy} implies (12).

Evaluating now (23) for $g_{1x}, g_{2x}, g_{11}, g_{12}, g_{22}$, we obtain certain equalities between polynomials in variables $u^3, \ldots, u^n$ and by comparing their coefficients we get, for M_E, (14) and

$$(25)\qquad \begin{cases} D^n_x + 2e^{-3\lambda}\sum_y \varepsilon_y h^y_x D^y + e^{2\lambda}\partial_1 D^n_x = 0, \\ 2D^{n-1}_x - e^{2\lambda}\partial_2 D^n_x - e^{2\lambda}\partial_1 D^{n-1}_x = 0, \\ D^n_x - 2e^{-3\lambda}\sum_y \varepsilon_y h^y_x D^y - e^{2\lambda}\partial_2 D^{n-1}_x = 0, \end{cases}$$

$$(26)\qquad \begin{cases} 2\partial_1 G^n + e^{-\lambda}\sum_x \varepsilon_x(\partial_1 D^x)^2 + 2e^{-5\lambda}\sum_x \varepsilon_x (D^x)^2 + 2e^{-2\lambda}G^n = 0, \\ 2e^{-\lambda}G^{n-1} - \sum_x \varepsilon_x(\partial_1 D^x)\partial_2 D^x - e^{\lambda}\partial_2 G^n - e^{\lambda}\partial_1 G^{n-1} = 0, \\ 2e^{-\lambda}G^n - 2e^{-4\lambda}\sum_x \varepsilon_x (D^x)^2 - \sum_x \varepsilon_x(\partial_2 D^x)^2 - 2e^{\lambda}\partial_2 G^{n-1} = 0, \end{cases}$$

while the corresponding equalities for M_H are (17) and

$$(27)\qquad \begin{gathered} D^{n-1}_x - e^{2\mu}\partial_1 D^n_x = 0, \quad D^n_x + e^{2\mu}\partial_2 D^{n-1}_x = 0, \\ \partial_1 D^{n-1}_x + \partial_2 D^n_x - 2e^{-5\mu}\sum_y \varepsilon_y h^y_x D^y = 0, \end{gathered}$$

$$(28)\qquad \begin{cases} 2\partial_1 G^n + e^{-\mu}\sum_x \varepsilon_x(\partial_1 D^x)^2 - 2e^{-2\mu}G^{n-1} = 0, \\ \partial_1 G^{n-1} + \partial_2 G^n + e^{-\mu}\sum_x \varepsilon_x(\partial_1 D^x)\partial_2 D^x - e^{-5\mu}\sum_x \varepsilon_x (D^x)^2 = 0, \\ 2e^{-\mu}G^n + \sum_x \varepsilon_x(\partial_2 D^x)^2 + 2e^{\mu}\partial_2 G^{n-1} = 0. \end{cases}$$

Let us now treat M_E and M_H separately.

(i) For M_E, (14) turns (25) into

$$\partial_1\partial_1 D^x = -e^{-2\lambda}\partial_1 D^x + 2e^{-4\lambda}D^x,$$
$$\partial_1\partial_2 D^x = e^{-2\lambda}\partial_2 D^x, \qquad \partial_2\partial_2 D^x = e^{-2\lambda}\partial_1 D^x + 2e^{-4\lambda}D^x.$$

It is easy to verify that the general solution D^x of this system is given by (13). Substituting (13) into (26), we obtain a system of partial differential equations with indeterminates G^{n-1} and G^n. For a fixed u^2, the first equation becomes an ordinary linear differential equation for G^n whose general solution is parametrized by a function of u^2. Inserting this solution into the third equation of (26), we can express G^{n-1} by two parameter functions, one depending on u^1 and the latter — on u^2. Finally, the second equation allows us to determine the parameter functions, which leads to the desired formulae for G^{n-1} and G^n.

(ii) For M_H, (17) and (27) yield

$$\partial_1\partial_1 D^x = e^{-2\mu}\partial_2 D^x, \qquad \partial_1\partial_2 D^x = -e^{-4\mu}D^x, \qquad \partial_2\partial_2 D^x = -e^{-2\mu}\partial_1 D^x.$$

The solutions D^x of this system are given by (16). Substituting (16) into (28) we obtain a system of inhomogeneous linear partial differential equations for G^{n-1} and G^n. Given any particular solution G_0^{n-1}, G_0^n (which can be defined, e.g., by (19), (20)), every solution is of the form $G^{n-1} = G_0^{n-1} + g^{n-1}$, $G^n = G_0^n + g^n$, where g^{n-1}, g^n satisfy the associated homogeneous system

$$\partial_1 g^n = e^{-2\mu} g^{n-1}, \quad \partial_1 g^{n-1} + \partial_2 g^n = 0, \quad \partial_2 g^{n-1} = -e^{-2\mu} g^n.$$

The latter is clearly equivalent to the completely integrable one

$$\partial_1 g^{n-1} = \psi, \quad \partial_2 g^{n-1} = -e^{-2\mu} g^n, \quad \partial_1 g^n = e^{-2\mu} g^{n-1}, \quad \partial_2 g^n = -\psi,$$
$$\partial_1 \psi = e^{-4\mu} g^n, \quad \partial_2 \psi = -e^{-4\mu} g^{n-1},$$

whose general solution is given by (18) and $\psi = \partial_1 g^{n-1}$. This proves one implication of the lemma.

As for the inverse one, we can immediately verify that (15) and (11) (respectively, (21) and (11)) with the above-given data define global isometries of M_E (respectively, of M_H) onto itself. This completes the proof.

The name "universal models" for $M^n_{E,r,F}$, $M^n_{H,r,F}$ and $M^n_{P,s,a,\varepsilon}$ (with a satisfying (10)) can now be justified as follows.

THEOREM 2. *Any simply connected homogeneous e.c.s. manifold is isometric to the pseudo-Riemannian universal covering of an open homogeneous submanifold of some universal model.*

Proof. Any homogeneous e.c.s. manifold M with $\pi_1 M = 0$ is locally isometric to a universal model M_0 (by (iv) of Theorem 1). Lemmas 2 and 3 imply clearly that M_0 has property (1). By Lemma 1, there exists an isometric immersion $f\colon M \to f(M) \subset M_0$ which is nothing but the universal covering projection. This completes the proof.

As an immediate consequence, we obtain

THEOREM 3. *A homogeneous e.c.s. manifold M is never geodesically complete.*

Proof. If M were complete, so would be its universal covering $\tilde{M}$, which in turn covers, by Theorem 2, an open homogeneous submanifold U of a universal model M_0. Hence U must be complete, and so is M_0 in view of homogeneity. Thus, we must show that none of the universal models is complete (by appointing an "incomplete" geodesic in each model). For $M^n_{E,r,F}$ it is sufficient to use the connection components ([3], proof of Theorem 4) to observe that $u^1(t) = -e^{2\lambda}\log t$ and $u^2(t) = 0$ are the first two components of a geodesic. Similarly, $u^1(t) = -e^{2\mu}\log t$ and $u^2(t) = e^{2\mu}\log t$ are the first two projections of a geodesic in $M^n_{H,r,F}$ (cf. [4], proof of Theorem 3). Finally, $t \mapsto (t, 0, \ldots, 0)$ defines a geodesic in $M^n_{P,s,a,\varepsilon}$

(for the connection components, see [10], p. 93). All these geodesics cannot be extended beyond the interval $(0, \infty)$ (as $u^1 > 0$ for the parabolic model), which shows that no universal model is complete. This completes the proof.

3. Global structure remarks. We start with the following auxiliary fact:

LEMMA 4. *Given real vector spaces V and W with a (not necessarily definite) inner product in W, let G be a Lie group of transformations of $V\times W\times V$, each of which is of the form*

$$(29)\qquad (v_1, w, v_2)\mapsto \big(v_1+c,\ Aw+D(v_1),\ e^{\gamma(c)}v_2+P(v_1)w+S(v_1)\big)$$

for some linear isometry A of W, a linear functional γ on V, $c\in V$ and C^∞-mappings $D\colon V\to W$, $P\colon V\to L(W, V)$ and $S\colon V\to V$.

Then

(i) *any open orbit U of G is diffeomorphic to the product $V\times U_0\times V$, U_0 being an open subset of W on which a certain group G_0 of affine isometries acts transitively;*

(ii) *any open orbit of G coincides with $V\times W\times V$ whenever the inner product in W is definite.*

Proof. Since U is open, the group homomorphism $\xi\colon G\to V$, assigning $c\in V$ to transformation (29), sends G onto a neighbourhood of 0, and hence is surjective. Therefore, the projection π_1 of $V\times W\times V$ onto the first factor V satisfies $\pi_1(U) = V$. Let ψ be a right inverse of ξ, defined in a neighbourhood Y of $0\in V$. We assert that $\pi_1\colon U\to V$ is a locally trivial bundle projection. In fact, given $v_0\in V$, the local triviality map

$$(v_0+Y)\times\pi_1^{-1}(v_0)\to\pi_1^{-1}(v_0+Y)$$

over v_0+Y can be defined by

$$\big(v, (v_0, w, v_1)\big)\mapsto\big(\psi(v-v_0)\big)(v_0, w, v_1).$$

As V is contractible, we have a diffeomorphism $U\approx V\times Q$, where Q (any fibre of π_1) has the following property:

It is an open subset of the product $W\times V$, homogeneous with respect to the group H (a subgroup of G keeping Q invariant) of transformations, obtained by fixing v_1 in (29),

$$(30)\qquad (w, v)\mapsto(Zw, v+Lw),$$

Z being an affine isometry of W, and L an affine mapping $W\to V$.

The set of all Z occurring in transformations (30) of H forms a group G_0 of affine isometries of W, acting transitively on $U_0 = \pi(Q)$ (π being the projection $W\times V\to V$). Fix $w_0\in U_0$ so that $(w_0, v_0)\in Q$ for some v_0. Transformations of H can now be written as quadruples

$$(K, p, N, q)\colon (w, v) \mapsto (Kw+p, v+Nw+q),$$

where $Nw_0 = 0, Kw_0 = w_0$. The set Φ of all $q \in V$ such that $(K, 0, N, q) \in H$ for some K and N is an additive group (since H is a group) and contains a neighbourhood of 0 (since, for q close to 0, $(w_0, v_0+q) \in Q$ is congruent to (w_0, v_0) modulo H). Hence $\Phi = V$, which clearly implies that the H-orbit of (w_0, v_0) (i.e., Q) must contain $w_0 \times V$. Since $w_0 \in U_0$ was arbitrary, we have $Q = U_0 \times V$. This yields (i).

Now, if the inner product in W is definite, then $U_0 = W$ (as an open homogeneous, hence complete, submanifold of the Riemannian manifold W), which implies that any fibre Q of $\pi_1\colon U \to V$ is equal to $W \times V$, so that $U = V \times W \times V$. This completes the proof.

Using Lemma 4, we can obtain some information about the structure of homogeneous e.c.s. manifolds.

THEOREM 4. *Let (M, g) be an n-dimensional simply connected homogeneous e.c.s. manifold. Then M is diffeomorphic to $\boldsymbol{R}^2 \times M_1$ (if it is parabolic) or to $\boldsymbol{R}^4 \times M_2$ (if it is elliptic or hyperbolic), where M_i $(i = 1, 2)$ is a simply connected homogeneous flat pseudo-Riemannian manifold with a metric of index k, $k = \operatorname{index} g - i$.*

Proof. By Theorem 2, M is the universal covering of an open homogeneous submanifold U of a universal model M_0. Let G be a group of isometries acting on U transitively. Without loss of generality, G may be assumed to be connected. We are now in the conditions of Lemma 4. In fact, we have the natural decomposition $M_0 = \boldsymbol{R}^i \times \boldsymbol{R}^{n-2i} \times \boldsymbol{R}^i$ (if M_0 is parabolic, we set $i = 1$ and identify $(0, \infty)$ with $\boldsymbol{R}$ using the logarithmic function; otherwise, $i = 2$) with the inner product $\sum\limits_a \varepsilon_a (du^a)^2$ or $\sum\limits_x \varepsilon_x (du^x)^2$ in $\boldsymbol{R}^{n-2i}$, and, by Lemmas 2 and 3, the transformations of G are of form (29) (as G is connected, we suppress the "discrete parameters", i.e., in the notation of Lemma 3, $k = 0$ and $\delta = 1$ or $\zeta = 1$). Thus, by Lemma 4, U is diffeomorphic to $\boldsymbol{R}^{2i} \times U_0$, U_0 being a flat homogeneous manifold with a metric of index equal to $(\operatorname{index} g - i)$. Hence M is diffeomorphic to $\boldsymbol{R}^{2i} \times M_i$, M_i being the universal covering of U_0. This completes the proof.

Combining the argument above with (ii) of Lemma 4, we easily obtain

COROLLARY 1. *Let M be a simply connected homogeneous e.c.s. manifold with a metric of index r. If M is not parabolic and $r = 2$ or $r = \dim M - 2$, then M is isometric to a universal model.*

We are now going to deliver examples of homotopically non-trivial simply connected homogeneous e.c.s. manifolds by using suitable non-contractible simply connected flat homogeneous manifolds.

LEMMA 5. *Let W be a real vector space with an indefinite inner product and suppose that we have an orthogonal direct sum decomposition*

$$W = W_0^{k_0} \times W_1^{2k_1} \times \ldots \times W_m^{2k_m}$$

(dimensions marked by superscripts) with $k_0 \geqslant 0$ and $k_i \geqslant 1$ for $i \geqslant 1$. If the index of the inner product restricted to W_i is k_i and V_i is a totally isotropic k_i-dimensional subspace of W_i ($i = 1, \ldots, m$), then the open subset

$$U_0 = W_0 \times (W_1 \setminus V_1) \times \ldots \times (W_m \setminus V_m)$$

of W is homogeneous with respect to a certain group G of affine isometries of W. Moreover, the group G may be chosen as a semidirect product T_0G_0 of a linear group G_0 with a vector space T_0 of translations.

Proof. Consider first a single space W of dimension $2k$ with an inner product (denoted in the sequel by a dot) of index k ($k \geqslant 1$) and let V be any totally isotropic k-dimensional subspace of W. We assert that the semidirect product VG_0 of the group G_0 of all linear isometries leaving V invariant with the additive group V acts on $W \setminus V$ transitively. In fact, choose a basis $X_1, \ldots, X_k$ for V. Given $X, Y \in W \setminus V = W \setminus V^{\perp}$, we have

$$\sum_i |X \cdot X_i| \neq 0 \neq \sum_i |Y \cdot X_i|,$$

so that there exists a regular ($k \times k$)-matrix C_{ij} such that

$$\sum_j C_{ij}(Y \cdot X_j) = X \cdot X_i.$$

Setting

$$Y_i = \sum_j C_{ij} X_j,$$

we obtain a new basis $Y_1, \ldots, Y_k$ of V and, by Witt's theorem (see [9], Chapter XIV, § 5), there exists a linear isometry Z of W such that $ZX_i = Y_i$ ($i = 1, \ldots, k$). Thus

$$(Y - ZX) \cdot Y_i = 0, \quad \text{i.e.,} \quad p = Y - ZX \in V^{\perp} = V.$$

Hence $Y = ZX + p$, $ZV = V$, $p \in V$, as required. Going now to the general case, it is sufficient to observe that the Cartesian product of transitive group actions is again transitive. This completes the proof.

THEOREM 5. *Let $n, r, p_1, \ldots, p_m$ ($m \geqslant 1$) be non-negative integers such that $n \geqslant 4$, $2 \leqslant r \leqslant n-2$. If*

$$m + p_1 + \ldots + p_m \leqslant \min(r-2, n-r-2), \tag{31}$$

then every connected component of the product

$$Q = R^{n-p_1-\cdots-p_m} \times S^{p_1} \times \ldots \times S^{p_m}$$

admits an elliptic (as well as hyperbolic) homogeneous e.c.s. pseudo-Riemannian metric of index r.

Proof. For a universal model $M = M^n_{E,r,F}$ or $M^n_{H,r,F}$, the underlying set can be written as $\boldsymbol{R}^n = \boldsymbol{R}^2 \times \boldsymbol{R}^{n-4} \times \boldsymbol{R}^2$ and we can endow the middle factor $W = \boldsymbol{R}^{n-4}$ with the indefinite inner product $\sum_x \varepsilon_x (du^x)^2$ of index $r-2$. Now (31) implies that W admits a totally isotropic vector subspace which can be decomposed into a direct sum $V_1^{p_1+1} + \ldots + V_m^{p_m+1}$ (with superscripts denoting dimensions). Choosing a suitable basis for this subspace and completing it appropriately (using, e.g., Witt's theorem), we get the hypothesis of Lemma 5 (with $k_i = p_i + 1$ and $k_0 = n - 2p_1 - \ldots - 2p_m - 2m - 4$). Lemma 5 implies now the existence of an open subset U_0 of W, homogeneous with respect to a group $T_0 G_0$, T_0 being a vector subspace of $W = \boldsymbol{R}^{n-4}$. The subset $U = \boldsymbol{R}^2 \times U_0 \times \boldsymbol{R}^2$ of M is clearly diffeomorphic to Q. To prove that U is homogeneous, fix $u, \bar{u} \in U$ and choose $h = [h^x_y] \in G_0$, $(\zeta_3, \ldots, \zeta_{n-2}) \in T_0$ such that

$$\sum_y h^x_y u^y + D^x(u^1, u^2) = \bar{u}^x$$

(D^x defined by (13) or (16) with $P_x = Q_x = 0$). It is now clear how to choose the auxiliary parameters $c^1, c^2, \varkappa_0, \sigma_0, \tau_0$ (notation of Lemma 3) which together with $k = 0$, $\delta = 1$ or $\zeta = 1$ should define an isometry f of M sending u to $\bar{u}$. We have $f(U) = U$ in view of (13) or (16), since T_0 is a vector space and $T_0 G_0$ leaves U_0 invariant. This completes the proof.

Remark 2. According to (ii) of Theorem 1, certain simply connected homogeneous e.c.s. manifolds admit Lie group structures compatible with their metrics. However, it is not so in general. In fact, by Theorem 5 such manifolds may have the homotopy type of spheres, which almost never admit homotopically group-like structures (see, e.g., [11], Chapter 5, § 8).

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