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THE LOCAL STRUCTURE OF ESSENTIALLY CONFORMALLY SYMMETRIC MANIFOLDS WITH CONSTANT FUNDAMENTAL FUNCTION

I. THE ELLIPTIC CASE

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1. Introduction. A Riemannian manifold M with a (possibly indefinite) metric g is said to be *conformally symmetric* [1] if its Weyl conformal tensor

$$C_{hijk} = R_{hijk} - (g_{ij}R_{hk} + g_{hk}R_{ij} - g_{hj}R_{ik} - g_{ik}R_{hj})/(n-2) + \\ + R(g_{ij}g_{hk} - g_{hj}g_{ik})/(n-1)(n-2) \tag{1}$$

satisfies the condition

$$C_{hijk,l} = 0, \tag{2}$$

where $n = \dim M \geqslant 4$, and R_{hijk}, R_{ij}, R and the comma denote the curvature tensor, Ricci tensor, scalar curvature and covariant differentiation, respectively. Conformally symmetric manifolds which are neither conformally flat ($C_{hijk} = 0$) nor locally symmetric ($R_{hijk,l} = 0$) are called *essentially conformally symmetric* (e.c.s. in short). For existence remarks, see Section 2. Every e.c.s. manifold M satisfies ([7], Theorem 3) a relation of the form

$$R_{ij}R_{hk} - R_{hj}R_{ik} = FC_{hijk} \tag{3}$$

for some function F, called the *fundamental function* of M. In what follows we restrict our consideration to e.c.s. manifolds with $F = \text{const}$. By Lemma 1, such a manifold is either *elliptic* ($F \neq 0$, R_{ij} semidefinite everywhere) or *hyperbolic* ($F \neq 0$, R_{ij} semidefinite nowhere), or else *parabolic* ($F = 0$).

In the present paper we treat the elliptic case only (the hyperbolic and parabolic ones are dealt with by the forthcoming papers [3] and [4]). The main result of this paper (Theorem 4) gives a general local form for elliptic e.c.s. manifolds with $F = \text{const}$. Theorem 1 shows the existence of such manifolds which are also homogeneous. Moreover, we derive some topological and metric properties of homogeneous elliptic e.c.s. manifolds (Theorems 2, 3, 5 and Corollaries 1-4).

Throughout this paper, all manifolds are assumed to be connected, paracompact and of class C^∞. However, all results and arguments remain valid, *mutatis mutandis*, in the analytic category. Considering Riemannian manifolds, we shall identify contravariant and covariant vectors (by raising and lowering indices).

2. General remarks. The conformal curvature tensor satisfies the following well-known relations:

$$C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi}, \tag{4}$$

$$C_{hijk} + C_{hjki} + C_{hkij} = 0, \tag{5}$$

$$C^r{}_{rij} = C^r{}_{irj} = C^r{}_{ijr} = 0. \tag{6}$$

The metric of an e.c.s. manifold is never definite (see [12] and [5], Theorem 2). Moreover, every e.c.s. manifold satisfies

$$R_{ij,k} = R_{ik,j}, \tag{7}$$

$$R = 0, \tag{8}$$

$$R_{hi}C_{jklm} + R_{hj}C_{kilm} + R_{hk}C_{ijlm} = 0, \tag{9}$$

$$R_{ij,kl} = R_{ij,lk} \tag{10}$$

(see [6], Theorems 7, 9 and formula (6), and [7], Theorem 7).

Examples of e.c.s. manifolds can be found in [11] (Theorem 3), [2] and [7] (Theorem 6). It is easy to see that, given an e.c.s. manifold M and a point $p \in M$, the fundamental function F defined by (3) vanishes at p if and only if $\operatorname{rank} R_{ij}(p) \leqslant 1$. It may happen that $F = 0$ identically ([11] and [7], loc. cit.) or $F = \text{const} \neq 0$, or F is non-constant ([2], cases $c = 0$ and $c \neq 0$).

LEMMA 1. *Let M be an e.c.s. manifold whose fundamental function $F = \text{const}$. Then one of the following cases holds:*

(i) *$F \neq 0$, $\operatorname{rank} R_{ij} = 2$ and R_{ij} is semidefinite at each point of M;*

(ii) *$F \neq 0$, $\operatorname{rank} R_{ij} = 2$ everywhere and R_{ij} is semidefinite at no point of M;*

(iii) *$F = 0$ and $\operatorname{rank} R_{ij} \leqslant 1$ everywhere.*

Proof. We have $\operatorname{rank} R_{ij} \leqslant 2$ for every e.c.s. manifold ([7], Theorem 5). Therefore, $F = \text{const} \neq 0$ implies $\operatorname{rank} R_{ij} = 2$ everywhere and our assertion can be obtained from an elementary algebraic argument.

Given an e.c.s. manifold M and a point $p \in M$, we shall say that

(a) M is *elliptic at* p if $\operatorname{rank} R_{ij}(p) = 2$ and $R_{ij}(p)$ is semidefinite;

(b) M is *hyperbolic at* p if $\operatorname{rank} R_{ij}(p) = 2$ and $R_{ij}(p)$ is not semidefinite;

(c) M is *parabolic at* p if $\operatorname{rank} R_{ij}(p) \leqslant 1$ (i.e. $F(p) = 0$, F being the fundamental function of M).

The whole manifold M is called *elliptic* (respectively, *hyperbolic*, *parabolic*) if it is so at each point. Thus, Lemma 1 states that any e.c.s. manifold with constant fundamental function is either elliptic or hyperbolic, or parabolic.

Since the present paper is devoted to elliptic e.c.s. manifolds with constant fundamental function, it seems reasonable to start from establishing their existence. The example given in the sequel is homogeneous, more precisely — it is a Lie group with a left-invariant metric.

THEOREM 1. *Let G be the open subset of $\boldsymbol{R}^4$ defined by*

$$G = \{(u^1, u^2, u^3, u^4) \mid (u^1)^2 - 3(u^2)^2 \neq 0\}.$$

By identifying $(u^1, u^2, u^3, u^4) \in G$ with the matrix

$$\begin{bmatrix} u^1 & u^2 & 0 & 0 \\ 3u^2 & u^1 & 0 & 0 \\ u^3 & -u^4 & 1 & 0 \\ 3u^4 & -u^3 & 0 & 1 \end{bmatrix}$$

we introduce in G a Lie group structure such that G becomes isomorphic to a closed subgroup of $GL(4, \boldsymbol{R})$. Let M be the identity component of G. Denote by g the left-invariant metric on M determined at the unit element $e \in G$ by

$$\tag{11} \begin{gathered} g_e(d_1, d_2) = t, \quad g_e(d_2, d_2) = 2Q, \quad g_e(d_1, d_3) = g_e(d_2, d_4) = 1, \\ g_e(d_1, d_1) = g_e(d_1, d_4) = g_e(d_2, d_3) = 0, \\ g_e(d_3, d_3) = g_e(d_3, d_4) = g_e(d_4, d_4) = 0, \end{gathered}$$

$d_1, \ldots, d_4$ being the canonical frame of $\boldsymbol{R}^4$ at e, where t and Q are fixed real numbers with $Q \neq 0$.

Then (M, g) is an elliptic e.c.s. Riemannian manifold with fundamental function $F = 8Q^{-1}$.

Proof. The left-invariant vector fields $\bar{d}_i$ on M whose values at e are just d_i, $i = 1, \ldots, 4$, satisfy

$$\tag{12} \begin{gathered} [\bar{d}_1, \bar{d}_2] = 0, \quad [\bar{d}_1, \bar{d}_3] = -\bar{d}_3, \quad [\bar{d}_1, \bar{d}_4] = -\bar{d}_4, \\ [\bar{d}_2, \bar{d}_3] = \bar{d}_4, \quad [\bar{d}_2, \bar{d}_4] = 3\bar{d}_3, \quad [\bar{d}_3, \bar{d}_4] = 0. \end{gathered}$$

From now on we refer all tensor and connection components to the left-invariant frame field

$$(13)\qquad \begin{aligned} e_1 &= -\bar{d}_3, \quad e_2 = \bar{d}_4, \\ e_3 &= -\bar{d}_1, \quad e_4 = \bar{d}_2 - t\bar{d}_3 - Q\bar{d}_4. \end{aligned}$$

Thus, setting $g_{ij} = g(e_i, e_j)$, from (11) we obtain

$$(14)\qquad g_{ij} = g^{ij} = \begin{cases} 1 & \text{if } \{i,j\} = \{1,3\} \text{ or } \{i,j\} = \{2,4\}, \\ 0 & \text{otherwise.} \end{cases}$$

The components of the Riemannian connection D of g, given by $D_{e_i}e_j = \Gamma^k_{ij}e_k$, can be determined by the relations

$$D_{e_i}e_j - D_{e_j}e_i = [e_i, e_j] \quad \text{and} \quad g(D_{e_k}e_i, e_j) = -g(e_i, D_{e_k}e_j),$$

which may also be written in the form

$$(15)\qquad \Gamma^k_{ij} - \Gamma^k_{ji} = c^k_{ij}$$

and

$$(16)\qquad \Gamma^r_{ki}g_{jr} = -\Gamma^r_{kj}g_{ir},$$

c^k_{ij} being the structure constants defined by $[e_i, e_j] = c^k_{ij}e_k$. Adding up the equalities

$$g_{kr}\Gamma^r_{ij} - g_{kr}\Gamma^r_{ji} = g_{kr}c^r_{ij},$$

$$g_{jr}\Gamma^r_{ik} - g_{jr}\Gamma^r_{ki} = g_{jr}c^r_{ik}, \quad g_{ir}\Gamma^r_{kj} - g_{ir}\Gamma^r_{jk} = g_{ir}c^r_{kj},$$

which follow from (15), and using (16), we obtain

$$(17)\qquad \Gamma^k_{ij} = \frac{1}{2}g^{ks}(g_{ir}c^r_{sj} + g_{jr}c^r_{si}) + \frac{1}{2}c^k_{ij}.$$

From (12) and (13) we obtain the commutator expressions

$$(18)\qquad \begin{aligned} &[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1, \quad [e_1, e_4] = e_2, \\ &[e_2, e_3] = -e_2, \quad [e_2, e_4] = 3e_1, \quad [e_3, e_4] = te_1 - Qe_2, \end{aligned}$$

which yield, in view of (17) and (14), the following formulae for covariant derivatives:

$$(19)\qquad \begin{aligned} &D_{e_1}e_1 = D_{e_1}e_2 = D_{e_1}e_3 = D_{e_1}e_4 = D_{e_2}e_1 = D_{e_2}e_2 = 0, \\ &D_{e_2}e_3 = -2e_2, \quad D_{e_2}e_4 = 2e_1, \quad D_{e_3}e_1 = e_1, \\ &D_{e_3}e_2 = -e_2, \quad D_{e_3}e_3 = -te_2 - e_3, \quad D_{e_3}e_4 = te_1 + e_4, \\ &D_{e_4}e_1 = -e_2, \quad D_{e_4}e_2 = -e_1, \quad D_{e_4}e_3 = Qe_2 + e_4, \quad D_{e_4}e_4 = -Qe_1 + e_3. \end{aligned}$$

Using the formula

$$R_{hijk} = g(e_k, D_{e_h}D_{e_i}e_j - D_{e_i}D_{e_h}e_j - D_{[e_h, e_i]}e_j)$$

and relations (18), (19) and (1), it is now easy to see that the only non--zero components of the curvature tensor, Ricci tensor and Weyl conformal tensor are those related to

$$R_{1434} = 2, \quad R_{2334} = -2, \quad R_{3434} = -2Q,$$

and

(20) $$R_{33} = R_{44} = -4,$$

and

(21) $$C_{3434} = -2Q.$$

It follows now from the formula

$$C_{hijk,l} = -\Gamma^r_{lh} C_{rijk} - \Gamma^r_{li} C_{hrjk} - \Gamma^r_{lj} C_{hirk} - \Gamma^r_{lk} C_{hijr}$$

that (M, g) is conformally symmetric. Moreover, $R_{33,3} = -8$, whence M is e.c.s. The relation $F = 8Q^{-1}$ and ellipticity of M are immediate consequences of (3), (21) and (20). This completes the proof.

Remark 1. For any homogeneous e.c.s. manifold, constancy of the fundamental function follows immediately from homogeneity.

Remark 2. The Ricci tensor of the homogeneous e.c.s. manifold described in Theorem 1 is *negative* semidefinite. As we shall see later (Corollary 3) it cannot be positive semidefinite for any homogeneous e.c.s. manifold.

A Riemannian manifold is called *Ricci-recurrent* if it satisfies the relation $R_{hi}R_{jk,l} = R_{jk}R_{hi,l}$. An e.c.s. manifold may be Ricci-recurrent (see [11]) or not ([2] and [7], Theorem 6). If an e.c.s. manifold is Ricci--recurrent, then it is parabolic ([7], Theorem 5), but the converse statement fails in general ([7], Theorem 6).

LEMMA 2 ([7], Theorem 4). *If M is a non-Ricci-recurrent e.c.s. manifold, then*

(22) $$C_{hijk} = \delta \omega_{hi} \omega_{jk},$$

where $|\delta| = 1$ and ω is a parallel, absolute (i.e. determined at each point up to a sign) exterior 2-form on M such that $\operatorname{rank}\omega = 2$ *and* $\omega_{ir}\omega^r{}_j = 0$.

LEMMA 3. *Let M be a non-Ricci-recurrent e.c.s. manifold (e.g. an elliptic or hyperbolic one). Then*

(i) *The image* $\operatorname{im}\omega$ *of ω (the absolute 2-form defined by* (22)), *i.e. the set of all vectors u of the type $u_i = \pm \omega_{ij} v^j$, is a parallel field of totally isotropic (2-dimensional) planes, which contains all vectors u of the form*

(23) $$u_i = R_{ij} v^j.$$

(ii) *The orthogonal complement of* $\operatorname{im}\omega$ *coincides with the kernel* $\ker\omega$ *of ω (the set of all vectors v with $\pm\omega_{ij}v^j = 0$) and it is contained in* $\ker R_{ij}$.

(iii) *The* $(n-2)$*-plane field* $\ker\omega$ *is integrable (involutive) and its leaves (integral manifolds) are totally geodesic submanifolds of* M*, flat with respect to the symmetric connection inherited from* M (cf. [10], p. 56-59).

(iv) *The tensor fields* R_{ij} *and* $R_{ij,k}$ *are parallel along leaves of* $\ker\omega$.

Proof. By (9) and (22), any vector u of form (23) satisfies

$$u\wedge\omega=0, \tag{24}$$

i.e. lies in $\operatorname{im}\omega$. Therefore, Lemma 2 yields (i). Assertion (ii) follows now from skew-symmetry of ω and from symmetry of R_{ij}. As $\operatorname{im}\omega$ is parallel, so is $\ker\omega$, and hence it is integrable and its leaves are totally geodesic. By (1), (8) and (3), we have $R_{hij}{}^{k}v_1^h v_2^i v_3^j=0$ for any vectors v_1, v_2, v_3 of $\ker\omega$, which implies (iii) (cf. [10], p. 58). In view of (7) and (10), the tensor fields $R_{ij,k}$ and $R_{ij,kl}$ are symmetric in all indices. Therefore, differentiating (9) twice covariantly and using (22), we deduce from (24) that $\operatorname{im}\omega$ contains all vectors u of the form $u_k=R_{ij,k}v_1^iv_2^j$ or $u_l=R_{ij,kl}v_1^iv_2^jv_3^k$. Thus, if v is orthogonal to $\operatorname{im}\omega$, then $R_{ij,k}v^k=0$ and $R_{ij,kl}v^l=0$, which completes the proof.

The assertion of the following lemma reduces easily to showing the existence of a cross-section in a certain fibre bundle. However, it seems more convenient to prove it directly.

LEMMA 4. *Let* M *be a Riemannian manifold with an indefinite metric* g. *Suppose that* a, b *is a* C^∞*-field of 2-frames on* M *(i.e. a pair of* C^∞ *vector fields, linearly independent at each point of* M*) such that*

$$a_ia^i=a_ib^i=b_ib^i=0. \tag{25}$$

Then M *admits a* C^∞*-field* c, d *of 2-frames which is dual to* a, b *in the sense that*

$$a_id^i=b_ic^i=c_ic^i=c_id^i=d_id^i=0,\qquad c_ia^i=d_ib^i=1$$

and a, b, c, d *are linearly independent at each point of* M.

Proof. Choose a positive definite C^∞ Riemannian metric $\bar h$ for M. For $p\in M$, let $V(p)$ be the $(n-2)$-plane at p, $\bar h$-orthogonal to a and b. Define a new positive definite metric h by

$$h(a,a)=h(b,b)=1,$$

$$h(a,b)=h\big(a(p),V(p)\big)=h\big(b(p),V(p)\big)=0\quad\text{ and }\quad h=\bar h$$

on $V(p)$ for any $p\in M$. Now let $u^i=g^{ir}h_{rs}a^s$ and $v^i=g^{ir}h_{rs}b^s$. Clearly, we have $g(a,u)=g(b,v)=1$ and $g(a,v)=g(b,u)=0$. To obtain the required vector fields it is now sufficient to set

$$c=u-\frac{1}{2}g(u,u)a-\frac{1}{2}g(u,v)b,\qquad d=v-\frac{1}{2}g(u,v)a-\frac{1}{2}g(v,v)b.$$

Linear independence of a, b, c, d follows immediately from their inner product relations, which completes the proof.

Remark 3. The argument above works equally well in the analytic category. In fact, in virtue of Grauert's embedding theorem ([8], Theorem 3, p. 470), every analytic paracompact manifold admits a positive definite analytic Riemannian metric.

LEMMA 5. *Let M be a non-Ricci-recurrent e.c.s. manifold (e.g. an elliptic or hyperbolic one). Suppose that a, b are vectors at $p \in M$ (respectively, vector fields in a neighbourhood of $p \in M$) which span the plane* $\operatorname{im}\omega$ *at p (respectively, span the plane field* $\operatorname{im}\omega$ *in a neighbourhood of p). Then we have*

$$R_{ij,k} = Aa_ia_ja_k + Bb_ib_jb_k + C(a_ib_jb_k + b_ia_jb_k + b_ib_ja_k) + + D(a_ia_jb_k + a_ib_ja_k + b_ia_ja_k) \tag{26}$$

at p, where A, B, C, D are real numbers (respectively, in a neighbourhood of p, for certain C^∞-functions A, B, C, D).

Proof. By (7), $R_{ij,k}$ is symmetric in all indices. Thus, in view of (iv) of Lemma 3, any vector u of the form $u_i = R_{ij,k}v_1^jv_2^k$ is orthogonal to $\ker\omega$, and so it lies in $\operatorname{im}\omega$. By a simple algebraic argument, this implies that $R_{ij,k}$ is a combination of tensor products of powers of a and b. The particular relations among the coefficients occurring in (26) follow from (7), which completes the proof.

LEMMA 6. *Let M be a non-Ricci-recurrent e.c.s. manifold of dimension $n \geq 4$ and let $p \in M$. Suppose that a, b are two C^∞ vector fields in a neighbourhood of p, which span* $\operatorname{im}\omega$ *and are parallel along* $\ker\omega$ (cf. Lemma 3). *Then the 2-frame field a, b can be completed, in a sufficiently small neighbourhood of p, to a C^∞ n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ satisfying the relations*

$$g(a, e_x) = g(b, e_x) = g(c, e_x) = g(d, e_x) = 0, \tag{27}$$

$$g(a, d) = g(b, c) = g(c, c) = g(c, d) = g(d, d) = 0, \tag{28}$$

$$g(a, c) = g(b, d) = 1, \tag{29}$$

$$g(e_x, e_y) = 0 \text{ for } x \neq y, \quad g(e_x, e_x) = \varepsilon_x, \quad |\varepsilon_x| = 1, \tag{30}$$

$$D_ae_x = D_be_x = D_{e_y}e_x = 0, \tag{31}$$

$$D_ac = D_bc = D_{e_x}c = D_ad = D_bd = D_{e_x}d = 0, \tag{32}$$

where $x, y = 3, \ldots, n-2$ and D denotes the Riemannian covariant derivative. In other words, (31) *and* (32) *state that the whole frame field is to be parallel along* $\ker\omega$.

Proof. Choose a two-dimensional submanifold N of M, passing through p and transverse to $\ker\omega$ in a neighbourhood of p. It is easy to see (cf. Lemma 4) that we may choose C^∞ vector fields $c, d, e_3, \ldots, e_{n-2}$ *along* N, satisfying (27)-(30). In a neighbourhood of p, each (local) leaf of $\ker\omega$ meets N at a unique point, so that we can extend $c, d, e_3, \ldots, e_{n-2}$ to vector fields in a neighbourhood of p by displacing them parallel along geodesics in the leaves, starting radially from the intersection points. In view of our hypothesis, relations (27)-(30) remain valid in a neighbourhood of p. By (27), e_x are tangent to the leaves of $\ker\omega$, and hence (iii) of Lemma 3 yields (31) in virtue of our extension procedure. All we have to do now is to modify our frame field so as to obtain (32). Note that (27)-(31) keep holding if we replace c, d by $\bar{c} = c-hb$, $\bar{d} = d+ha$ for an arbitrary C^∞-function h. The equalities

$$\begin{aligned} D_a c &= g(D_a c, d)b, & D_b c &= g(D_b c, d)b, & D_{e_x} c &= g(D_{e_x} c, d)b, \\ D_a d &= -g(D_a c, d)a, & D_b d &= -g(D_b c, d)a, & D_{e_x} d &= -g(D_{e_x} c, d)a \end{aligned} \tag{33}$$

can be verified by inner multiplication of both sides by our frame vectors, applying the Leibniz rule. Thus, $\bar{c}, \bar{d}, e_3, \ldots, e_{n-2}, b, a$ will satisfy our assertion if h is a solution of the system

$$D_a h = g(D_a c, d), \qquad D_b h = g(D_b c, d), \qquad D_{e_x} h = g(D_{e_x} c, d). \tag{34}$$

Using the fact that a, b and e_x's mutually commute (as they are parallel along one another) and taking into account the formula

$$R_{hijk} = \frac{1}{F}(R_{ij}R_{hk} - R_{hj}R_{ik}) + \frac{1}{n-2}(g_{ij}R_{hk} + g_{hk}R_{ij} - g_{hj}R_{ik} - g_{ik}R_{hj}), \tag{35}$$

which follows immediately from (1), (3), (8), we can verify the integrability conditions of (34) as follows:

$$D_b\big(g(D_a c, d)\big) - D_a\big(g(D_b c, d)\big) = R_{hijk} b^h a^i c^j d^k = 0$$

(cf. (33)), and so on. By the existence theorem for systems of partial differential equations (cf. [13], Chapter IX, §1) we may choose a solution h of (34), which clearly completes the proof.

3. Local structure theorem. Throughout this section, we shall often assume the following hypothesis:

(36) (M, g) is an *n-dimensional* ($n \geqslant 4$) elliptic e.c.s. Riemannian manifold with fundamental function $F = \text{const} \neq 0$ whose Ricci tensor is ε-semidefinite (i.e. positive or negative semidefinite according as $\varepsilon = 1$ or $\varepsilon = -1$) and p is a point of M such that

$$R_{ij,k}(p) \neq 0. \tag{37}$$

Note that a point p satisfying (37) always exists, since M is not locally symmetric.

Under hypothesis (36), a 2-frame a, b at p is called *special* if the equalities

$$R_{ij} = \varepsilon(a_i a_j + b_i b_j) \tag{38}$$

and

$$R_{ij,k} = 2\varepsilon S(a_i a_j a_k - a_i b_j b_k - b_i a_j b_k - b_i b_j a_k) \tag{39}$$

are satisfied at p for some real number $S = S(p) > 0$.

The following lemma asserts nothing but the existence of local orthonormal frame fields in a certain 2-dimensional Riemannian vector bundle. Nevertheless, we give a direct proof for the sake of completeness.

LEMMA 7. *Suppose that M is an e.c.s. manifold, $p \in M$ and M is elliptic at p. Then there exists a C^∞-field a, b of 2-frames in a neighbourhood of p which satisfies* (38) *at each point, ε being* 1 *or* -1.

Proof. By (i) of Lemma 3 we may choose vector fields $\bar{a}, \bar{b}$ spanning $\operatorname{im}\omega = \operatorname{im} R_{ij}$ in a neighbourhood of p. Therefore,

$$\bar{a}_i \bar{a}^i = \bar{a}_i \bar{b}^i = \bar{b}_i \bar{b}^i = 0 \quad \text{and} \quad R_{ij} = X\bar{a}_i\bar{a}_j + Y(\bar{a}_i\bar{b}_j + \bar{b}_i\bar{a}_j) + Z\bar{b}_i\bar{b}_j$$

for some functions X, Y and Z. Let $\bar{c}, \bar{d}$ be a field of 2-frames dual to $\bar{a}, \bar{b}$ (cf. Lemma 4). We assert that $Z(p) \neq 0$. In fact, $Z(p) = 0$ would yield

$$\varepsilon R_{ij}(\bar{c}^i - t\bar{d}^i)(\bar{c}^j - t\bar{d}^j) = \varepsilon(X - 2tY) \geqslant 0$$

at p for each real t, whence $Y(p) = 0$ and $\operatorname{rank} R_{ij}(p) \leqslant 1$, a contradiction. Setting

$$a = |X - Y^2 Z^{-1}|^{1/2}\bar{a} \quad \text{and} \quad b = |Z|^{1/2}\bar{b} + \theta Y|Z|^{-1/2}\bar{a}$$

in a neighbourhood of p, we obtain $R_{ij} = \eta a_i a_j + \theta b_i b_j$, $|\eta| = |\theta| = 1$. Consequently, a, b are linearly independent, and ellipticity yields $\eta = \theta$, which completes the proof.

LEMMA 8. *Assume* (36). *Then*

(i) *There exist finitely many* (*at least one*) *special 2-frames at p.*

(ii) *Every special 2-frame at p can be extended to a unique C^∞-field of 2-frames in a neighbourhood of p, which is special at each point.*

(iii) *Every C^∞-field of special 2-frames in a neighbourhood of p is parallel along the* $(n-2)$*-plane field* $\ker\omega$ (cf. Lemma 3).

Proof. Choose a C^∞-field a, b of 2-frames in a neighbourhood of p, satisfying (38) at each point (Lemma 7). Clearly, a and b span $\operatorname{im} R_{ij}$; hence, by (i) of Lemma 3, they satisfy (25). Let c, d be a 2-frame field dual to a, b (Lemma 4). Transvecting the formula $R_{ij,m}R_{hk} + R_{ij}R_{hk,m}$

$= R_{hj,m}R_{ik} + R_{hj}R_{ik,m}$, which follows from (3) by covariant differentiation, with $d^i d^j c^h c^k c^m$ and with $c^i c^j d^h d^k d^m$, in terms of Lemma 5 we obtain $C+A = D+B = 0$. Thus, by Lemma 5 we have, in a neighbourhood of p,

$$(40) \qquad R_{ij,k} = A(a_i a_j a_k - a_i b_j b_k - b_i a_j b_k - b_i b_j a_k) + \\ + B(b_i b_j b_k - b_i a_j a_k - a_i b_j a_k - a_i a_j b_k)$$

for some C^∞-functions A and B. Now let $\bar{a}, \bar{b}$ be another field of 2-frames in a neighbourhood of p, satisfying (38) (hence spanning $\operatorname{im} R_{ij}$) and inducing, at each point sufficiently near to p, the same orientation of $\operatorname{im} R_{ij}$ as a, b does. Then $\bar{a}, \bar{b}$ arise from a, b by linear combinations with function coefficients. By (38) we have

$$\bar{a} = \cos\varphi \cdot a - \sin\varphi \cdot b, \qquad \bar{b} = \sin\varphi \cdot a + \cos\varphi \cdot b,$$

φ being a real function. A field $\bar{c}, \bar{d}$ of 2-frames, satisfying together with $\bar{a}, \bar{b}$ the duality relations (28) and (29), can be defined by

$$\bar{c} = \cos\varphi \cdot c - \sin\varphi \cdot d, \qquad \bar{d} = \sin\varphi \cdot c + \cos\varphi \cdot d.$$

The function $\bar{B}$, determined by $\bar{a}, \bar{b}$ according to (40), is now equal to $\bar{B} = R_{ij,k}\bar{d}^i \bar{d}^j \bar{d}^k$, i.e. to

$$(41) \quad f(A, B, \varphi) = A\sin^3\varphi - 3A\sin\varphi\cos^2\varphi + B\cos^3\varphi - 3B\cos\varphi\sin^2\varphi.$$

Clearly, the field $\bar{a}, \bar{b}$ consists of special 2-frames if and only if $\bar{B} = 0$ (note that the inequality $S > 0$ in (39) can be checked by changing signs of both $\bar{a}$ and $\bar{b}$, if necessary). The function $f = f(A, B, \varphi)$ of three real variables A, B, φ, which we define for $A^2+B^2 > 0$ (cf. (37)) by (41), is periodic in φ for fixed A and B and has finitely many (at least one) zeros in each period interval. Hence to either orientation of $\operatorname{im} R_{ij}(p)$ there corresponds a non-empty finite set of special 2-frames, which implies (i). Moreover, it is easy to verify that

$$\left.\frac{\partial f}{\partial \varphi}\right|_{f=0} \neq 0,$$

so that (ii) follows immediately from the implicit function theorem. Now let γ be a differentiable curve contained in a leaf of $\ker R_{ij}$. Given a special 2-frame at the origin of γ, its parallel displacement along γ consists of special 2-frames, which follows immediately from (38), (39) and (iv) of Lemma 3. This completes the proof.

Remark 4. Under hypothesis (36), the number $S = S(p) > 0$ occurring in (39) is independent of the choice of the special 2-frame a, b. In fact, from (38) it follows easily that

$$(42) \qquad 2S(p) = \max\{R_{ij,k}u^i u^j u^k \mid u \in T_pM,\ R_{ij}u^i u^j = \varepsilon\}.$$

The non-negative real function $S: p \mapsto S(p)$, defined by (42) for each $p \in M$ (not necessarily satisfying (37)) will be called the *fundamental invariant* of M. Note that S is continuous on M and even C^∞ on the open set defined by $S > 0$.

In Theorem 1 we have established the existence of homogeneous elliptic e.c.s. manifolds. We shall now derive a certain topological property of such manifolds.

THEOREM 2. *Let M be an elliptic e.c.s. manifold with constant fundamental function such that $R_{ij,k} \neq 0$ everywhere (e.g. a homogeneous one). Then there exists a finite Riemannian covering $\pi: \bar{M} \to M$ such that $\bar{M}$ admits a global C^∞-field of special 2-frames.*

Proof. Using a standard covering construction, let us consider the set of all special 2-frames, which in view of Lemma 8 forms a (possibly disconnected) finite fibre subbundle of the 2-frame bundle of M. For any connected component $\bar{M}$ of this set, the bundle projection $\pi: \bar{M} \to M$ is a finite covering and induces an elliptic e.c.s. metric on $\bar{M}$. We can now define the desired 2-frame field by

$$\bar{M} \ni (a, b) \mapsto \left(\pi_{*,(a,b)}^{-1}(a),\ \pi_{*,(a,b)}^{-1}(b)\right),$$

which completes the proof.

COROLLARY 1. *For any elliptic e.c.s. manifold with constant fundamental function, satisfying $R_{ij,k} \neq 0$ everywhere (e.g. a homogeneous one), there exists a finite covering $\bar{M} \to M$ such that $\bar{M}$ admits a C^∞-field of 4-frames.*

Proof. By (38), any special 2-frame spans $\operatorname{im} R_{ij}$, and so its vectors a, b satisfy (25). Our assertion follows now from Lemma 4 combined with Theorem 2.

COROLLARY 2. *Every elliptic e.c.s. manifold with constant fundamental function, of dimension $n = 4$ or $n = 5$, satisfying $R_{ij,k} \neq 0$ everywhere, can be finitely covered by a parallelizable manifold.*

LEMMA 9. *Under hypothesis* (36), *let a, b be a C^∞-field of special 2-frames in a neighbourhood of p. Then*

$$a_{i,j} = Sa_i a_j + \sigma b_i b_j + \lambda b_i a_j, \tag{43}$$

$$b_{i,j} = -\lambda a_i a_j - Sb_i a_j - (2S + \sigma) a_i b_j, \tag{44}$$

S being the fundamental invariant of M (see Remark 4), *while σ and λ are certain C^∞-functions in a neighbourhood of p.*

Proof. The plane field $\operatorname{im} R_{ij} = \operatorname{im}\omega$ spanned by a, b is parallel in view of Lemma 3. On the other hand, by (iii) of Lemma 8, $a_{i,j}v^j = b_{i,j}v^j = 0$ for any vector v of $\ker\omega$. Hence each vector w of the form $w_j = a_{i,j}u^i$ or $w_j = b_{i,j}u^i$ is orthogonal to $\ker\omega$, and so it lies in $\operatorname{im}\omega$.

This implies that $a_{i,j}$ and $b_{i,j}$ are combinations of tensor squares and products of a and b. To determine the coefficient functions, it is sufficient to differentiate (38) covariantly and compare the resulting expression with (39), which completes the proof.

Assume now (36). A C^∞-field $c, d, e_3, \ldots, e_{n-2}, b, a$ of n-frames in a neighbourhood of p will be called *special* if a, b is a field of special 2-frames and the whole frame field satisfies (27)-(32), i.e. the assertion of Lemma 6.

Here and in the sequel the final letters x, y, z will assume values from the range $\{3, \ldots, n-2\}$ (empty for $n = 4$).

LEMMA 10. *Under hypothesis* (36), *every special* C^∞ *2-frame field* a, b *can be completed to a special n-frame field* $c, d, e_3, \ldots, e_{n-2}, b, a$ *in a neighbourhood of* p. *Moreover, any special n-frame field* $c, d, e_3, \ldots, e_{n-2}, b, a$ *satisfies the relations*

$$\begin{gathered} D_c c = \xi b - Sc + \lambda d - \sum_x \varepsilon_x A_x e_x, \\ D_c d = -\xi a - \lambda c + Sd - \sum_x \varepsilon_x B_x e_x, \\ D_c e_x = A_x a + B_x b + \sum_y C_{xy} e_y, \quad C_{yx} = -\varepsilon_x \varepsilon_y C_{xy}, \\ D_c b = -\lambda a - Sb, \quad D_c a = Sa + \lambda b, \\ D_d c = \psi b + (2S+\sigma) d - \sum_x \varepsilon_x E_x e_x, \quad D_d d = -\psi a - \sigma c - \sum_x \varepsilon_x F_x e_x, \\ D_d e_x = E_x a + F_x b + \sum_y G_{xy} e_y, \quad G_{yx} = -\varepsilon_x \varepsilon_y G_{xy}, \\ D_d b = -(2S+\sigma) a, \quad D_d a = \sigma b, \quad D_{e_x} \ldots = D_b \ldots = D_a \ldots = 0, \end{gathered} \tag{45}$$

where ... *stands for any frame vector*, $\varepsilon_x = g(e_x, e_x)$, S *is the fundamental invariant of* M (cf. Remark 4), λ *and* σ *are determined by* a, b *as in Lemma* 9, *and* $\xi, \psi, A_x, B_x, E_x, F_x, C_{xy}, G_{xy}$ *are certain* C^∞*-functions which satisfy the conditions*

$$D_a S = D_a C_{xy} = D_a G_{xy} = D_a A_x = D_a F_x = 0, \tag{46}$$

$$\begin{gathered} D_a B_x = D_a \lambda = D_a \sigma = D_a E_x = 0, \\ D_a \psi = -(n-2)^{-1} \varepsilon, \quad D_a \xi = 0, \end{gathered} \tag{47}$$

$$D_b S = D_b C_{xy} = D_b G_{xy} = D_b A_x = D_b F_x = 0, \tag{48}$$

$$D_b B_x = D_b \lambda = D_b \sigma = D_b E_x = D_b \psi = 0, \quad D_b \xi = (n-2)^{-1} \varepsilon, \tag{49}$$

(50) $$D_c\psi - D_d\xi + \lambda\xi + (S+\sigma)\psi + \sum_x \varepsilon_x(A_xF_x - B_xE_x) + \frac{1}{F} = 0,$$

(51)
$$D_dA_x - D_cE_x - (2S+\sigma)(B_x+E_x) + \lambda(F_x - A_x) + \sum_y (C_{xy}E_y - G_{xy}A_y) = 0,$$

(52) $$D_dB_x - D_cF_x + \sigma(A_x - F_x) - \lambda(B_x+E_x) + \sum_y (C_{xy}F_y - G_{xy}B_y) = 0,$$

(53) $$D_cG_{xy} - D_dC_{xy} + (S+\sigma)G_{xy} + \lambda C_{xy} + \sum_z (G_{xz}C_{zy} - C_{xz}G_{zy}) = 0,$$

(54) $$D_{e_x}S = D_{e_z}C_{xy} = D_{e_z}G_{xy} = 0, \quad D_{e_y}A_x = D_{e_y}F_x = -\varepsilon\varepsilon_x(n-2)^{-1}\delta_{xy},$$

(55) $$D_{e_y}B_x = D_{e_x}\lambda = D_{e_x}\sigma = 0,$$

(56) $$D_{e_y}E_x = D_{e_x}\psi = D_{e_x}\xi = 0,$$

(57) $$D_d\lambda - D_c\sigma + S\sigma - \lambda^2 - \sigma^2 - (n-2)^{-1}\varepsilon = 0,$$

(58) $$D_dS - 3S\lambda = 0, \quad D_cS + 3S^2 + 3S\sigma = 0.$$

Proof. Our existence statement follows immediately from Lemma 6 and (iii) of Lemma 8. Consider now a special n-frame field. Formulae (45) are then immediate consequences of (25)-(32), (43) and (44) together with the Leibniz rule and (iii) of Lemma 8. Using (25)-(30), (35) and (38) it is easy to verify that the only non-zero components of the curvature tensor in our frame field are those related to

(59) $$\begin{aligned} &R_{hijk}c^hd^ic^jd^k = -F^{-1}, \quad R_{hijk}c^he_x^ic^je_x^k = R_{hijk}d^he_x^id^je_x^k = -\varepsilon\varepsilon_x(n-2)^{-1}, \\ &R_{hijk}c^hd^ic^jb^k = -\varepsilon(n-2)^{-1}, \quad R_{hijk}c^hd^id^ja^k = \varepsilon(n-2)^{-1}. \end{aligned}$$

We can now obtain relations (46)-(58) by using the Jacobi identity and comparing (59) with the curvature components obtained from

(60) $$R_{hijk}u^hv^iu_1^jv_1^k = g(v_1, D_uD_vu_1 - D_vD_uu_1 - D_{[u,v]}u_1), \quad [u,v] = D_uv - D_vu,$$

by means of (45). Thus, (46) follows from (60) applied to $R_{acac} = R_{hijk}a^hc^ia^jc^k$, $R_{acxy} = R_{hijk}a^hc^ie_x^je_y^k$, R_{adxy}, R_{accx}, R_{addx}, (47) from R_{acdx}, R_{acad}, R_{adad}, R_{adcx}, R_{adcd}, R_{accd} with (46), (48) from R_{bcbd}, R_{bcxy}, R_{bdxy}, R_{bccx}, R_{bddx}, (49) from R_{bcdx}, R_{bcbc} and $[[a,b],d]+\ldots=0$ with (46) and (47), R_{bdcx}, R_{bdcd}, R_{bccd}, (50) from R_{cdcd}, (51) from R_{cdcx}, (52) from R_{cddx}, (53) from R_{cdxy}, (54) and (55) from $[[a,c],e_x]+\ldots=0$ with (46) and (47), R_{cxyz}, R_{dxyz}, R_{cxdy}, R_{cxcy}, R_{dxdy}, $[[a,d],e_x]+\ldots=0$ with (46) and (47). Finally, (56) follows from $[[c,d],e_x]+\ldots=0$ with (51)-(55), while (57) and (58) are consequences of $[[a,c],d]+\ldots=0$ and $[[b,c],d]+ +\ldots=0$ together with (46)-(49). This completes the proof.

LEMMA 11. *Under hypothesis* (36), *let* $c, d, e_3, \ldots, e_{n-2}, b, a$ *be a special* n*-frame field in a neighbourhood of* p. *Then the systems of differential equations*

$$D_c u^1 = e^{-T}, \quad D_d u^1 = D_{e_x} u^1 = D_b u^1 = D_a u^1 = 0 \tag{61}$$

and

$$D_d u^2 = e^{-T}, \quad D_c u^2 = D_{e_x} u^2 = D_b u^2 = D_a u^2 = 0, \tag{62}$$

where, in the notation of Lemma 10,

$$T = -\frac{1}{3} \log S, \tag{63}$$

are completely integrable and, therefore, define the functions u^1, u^2 *uniquely up to additive constants. If we set*

$$u^x = -(n-2)\varepsilon\varepsilon_x A_x, \qquad u^{n-1} = (n-2)\varepsilon\xi, \qquad u^n = -(n-2)\varepsilon\psi,$$

then $u^1, u^2, \ldots, u^n$ *is a coordinate system in a neighbourhood of* p. *The corresponding basic vector fields are given by*

$$\begin{aligned}
\frac{\partial}{\partial u^1} &= (n-2)\varepsilon e^T D_c\psi \cdot a - (n-2)\varepsilon e^T D_c \xi \cdot b + (n-2)\varepsilon e^T \sum_x \varepsilon_x D_c A_x \cdot e_x + e^T c,\\
\frac{\partial}{\partial u^2} &= (n-2)\varepsilon e^T D_d\psi \cdot a - (n-2)\varepsilon e^T D_d \xi \cdot b + (n-2)\varepsilon e^T \sum_x \varepsilon_x D_d A_x \cdot e_x + e^T d,
\end{aligned}$$
$$\frac{\partial}{\partial u^x} = e_x, \qquad \frac{\partial}{\partial u^{n-1}} = b, \qquad \frac{\partial}{\partial u^n} = a. \tag{64}$$

Proof. Integrability conditions for (61) and (62) follow immediately from (63), (58), (46)-(54). In fact, we have

$$\begin{aligned}
D_c D_d u^1 - D_d D_c u^1 - D_{[c,d]} u^1 &= -D_d S^{1/3} + \lambda D_c u^1\\
&= -\frac{1}{3} S^{-2/3}(D_d S - 3S\lambda) = 0,
\end{aligned}$$

etc. The vector fields $\partial/\partial u^i$ defined by (64) clearly satisfy

$$du^i\left(\frac{\partial}{\partial u^j}\right) = \delta^i_j, \qquad i, j = 1, \ldots, n,$$

which completes the proof.

LEMMA 12. (i) *Under hypothesis* (36), *every* C^∞*-field* a, b *of special* 2*-frames in a neighbourhood of* p *can be completed to a special* n*-frame field*

$c, d, e_3, \ldots, e_{n-2}, b, a$ in a neighbourhood of p such that (in the notation of Lemma 10)

$$C_{xy} = G_{xy} = 0, \tag{65}$$

$$B_x = E_x = 0, \quad F_x = A_x, \quad D_c A_x = D_d A_x = 0, \tag{66}$$

$$D_c \psi + S\psi + \lambda\xi + \frac{1}{2}\sum_x \varepsilon_x A_x^2 + \frac{1}{2F} = 0, \tag{67}$$

$$D_d \psi + (2S + \sigma)\xi = 0, \tag{68}$$

$$D_c \xi - \lambda\psi - S\xi = 0, \tag{69}$$

$$D_d \xi - \sigma\psi - \frac{1}{2}\sum_x \varepsilon_x A_x^2 - \frac{1}{2F} = 0. \tag{70}$$

(ii) *Moreover, the above-defined n-frame field may be chosen so that*

$$\psi(p) = \xi(p) = A_x(p) = 0.$$

Proof. First step. By Lemma 10 we may complete a, b to a special n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ in a neighbourhood of p. In view of (46), (48) and (53) the system of differential equations

$$D_c \tau_{xy} = -\sum_z \tau_{xz} C_{zy}, \quad D_d \tau_{xy} = -\sum_z \tau_{xz} G_{zy},$$

$$D_{e_z} \tau_{xy} = D_b \tau_{xy} = D_a \tau_{xy} = 0,$$

with unknown functions τ_{xy}, is completely integrable. Let τ_{xy} be its solution with the initial value $\tau_{xy}(p) = \delta_{xy}$. By differentiation we can verify that

$$\sum_z \varepsilon_z \tau_{xz} \tau_{yz} = \varepsilon_x \delta_{xy},$$

which implies that $c, d, \bar{e}_3, \ldots, \bar{e}_{n-2}, b, a$, where

$$\bar{e}_x = \sum_y \tau_{xy} e_y,$$

is a special n-frame field. Applying to it the obvious formulae

$$\bar{C}_{xy} = \varepsilon_y g(D_c \bar{e}_x, \bar{e}_y), \quad \bar{G}_{xy} = \varepsilon_y g(D_d \bar{e}_x, \bar{e}_y)$$

(the barred coefficients being the ones corresponding to the new frame), we see that it satisfies (65).

Second step. Choose a special n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ satisfying (65). Consider the system of differential equations

$$(71)\quad \begin{gathered} D_c\zeta_x = \lambda\iota_x - S\zeta_x - A_x + \varkappa_x, \qquad D_d\zeta_x = (2S+\sigma)\iota_x - E_x, \\ D_{e_y}\zeta_x = D_b\zeta_x = D_a\zeta_x = 0, \\ D_c\iota_x = S\iota_x - \lambda\zeta_x - B_x, \qquad D_d\iota_x = -\sigma\zeta_x - F_x + \varkappa_x, \\ D_{e_y}\iota_x = D_b\iota_x = D_a\iota_x = 0, \\ D_c\varkappa_x = -(n-2)^{-1}\varepsilon\zeta_x, \qquad D_d\varkappa_x = -(n-2)^{-1}\varepsilon\iota_x, \\ D_{e_y}\varkappa_x = -(n-2)^{-1}\varepsilon\varepsilon_x\delta_{xy}, \qquad D_b\varkappa_x = D_a\varkappa_x = 0 \end{gathered}$$

with unknown functions $\zeta_x, \iota_x, \varkappa_x$. Its integrability conditions follow immediately from (45)-(49), (51), (52), (54)-(58) and (65). Choosing a solution $\zeta_x, \iota_x, \varkappa_x$ of (71) and setting

$$\bar{c} = c - \sum_x \varepsilon_x\zeta_x e_x - \frac{1}{2}\sum_x \varepsilon_x\zeta_x^2 a,$$

$$\bar{d} = d - \sum_x \varepsilon_x\iota_x e_x - \frac{1}{2}\sum_x \varepsilon_x\iota_x^2 b - \sum_x \varepsilon_x\zeta_x\iota_x a, \qquad \bar{e}_x = e_x + \zeta_x a + \iota_x b,$$

it is easy to verify, using (71) and Lemma 10, that $\bar{c}, \bar{d}, \bar{e}_3, \ldots, \bar{e}_{n-2}, b, a$ is a special n-frame field satisfying (65) and (66). To make the barred coefficients $\bar{A}_x$ satisfy $\bar{A}_x(p) = 0$, it is sufficient, in view of $\bar{A}_x = g(D_c\bar{e}_x, c) = \varkappa_x$, to choose a solution of (71) with $\varkappa_x(p) = 0$.

Third step. Let $c, d, e_3, \ldots, e_{n-2}, b, a$ be any special n-frame field satisfying (65) and (66) and such that $A_x(p) = 0$. The system of differential equations

$$(72)\quad \begin{gathered} D_c h = a, \quad D_d h = \beta, \quad D_{e_x} h = D_b h = D_a h = 0, \\ D_c a = Sa + \lambda\beta - \frac{1}{n-2}\varepsilon h + D_c\xi - \lambda\psi - S\xi, \\ D_d a = \sigma\beta + D_c\psi + S\psi + \lambda\xi + \frac{1}{2}\sum_x \varepsilon_x A_x^2 + \frac{1}{2F} \quad \text{(cf. (50))}, \\ D_{e_x} a = D_b a = D_a a = 0, \\ D_c\beta = D_c\psi + S\psi + \lambda\xi + \frac{1}{2}\sum_x \varepsilon_x A_x^2 + \frac{1}{2F} - \lambda a - S\beta, \\ D_d\beta = D_d\psi + (2S+\sigma)(\xi - a) - \frac{1}{n-2}\varepsilon h, \qquad D_{e_x}\beta = D_b\beta = D_a\beta = 0 \end{gathered}$$

with unknown functions h, a, β is completely integrable in view of Lemma 10. For instance, the commutator relation

$$D_d D_c \xi = D_c D_d \xi + \frac{1}{n-2} \varepsilon\psi + \lambda D_c \xi + (S+\sigma) D_d \xi$$

yields, by (66),

$$D_c D_d a - D_d D_c a - D_{[c,d]} a = (\psi - \beta)\left(D_d \lambda - D_c \sigma + S\sigma - \lambda^2 - \sigma^2 - \frac{1}{n-2}\varepsilon\right) +$$
$$+ (\xi - a)(D_d S - 3S\lambda) + \sigma\left(D_c \psi - D_d \xi + \lambda\xi + (S+\sigma)\psi + \sum_x \varepsilon_x A_x^2 + \frac{1}{F}\right) +$$
$$+ D_c\left[D_c \psi - D_d \xi + \lambda\xi + (S+\sigma)\psi + \sum_x \varepsilon_x A_x^2 + \frac{1}{F}\right],$$

which vanishes in view of (50), (57), (58) and (66), etc. Therefore, we may choose a solution h, a, β of (72). Setting $\bar{c} = c - hb$, $\bar{d} = d + ha$ we can easily verify that $\bar{c}$, $\bar{d}$, $e_3, \ldots, e_{n-2}$, b, a is a special n-frame field satisfying our assertion, which completes the proof of (i).

The barred coefficients $\bar{\psi}$, $\bar{\xi}$, $\bar{A}_x$ corresponding to the new frame are given by

$$\bar{\psi} = g(D_{\bar{d}}\bar{c}, \bar{d}) = \psi - \beta, \quad \bar{\xi} = g(D_{\bar{c}}\bar{c}, \bar{d}) = \xi - a, \quad \bar{A}_x = g(D_{\bar{c}} e_x, \bar{c}) = A_x,$$

whence, in order to check (ii), it is sufficient to take a solution of (72) with $a(p) = \xi(p)$, $\beta(p) = \psi(p)$. This completes the proof.

Under hypothesis (36), by a *distinguished* n-frame field in a neighbourhood of p we shall mean a special n-frame field satisfying (65)-(70), i.e. the assertion of (i) of Lemma 12.

LEMMA 13. *Assume* (36). *If* c, d, $e_3, \ldots, e_{n-2}$, b, a *and* $\bar{c}$, $\bar{d}$, $\bar{e}_3, \ldots, \bar{e}_{n-2}$, b, a *are two distinguished n-frame fields in a connected neighbourhood of* p, *then*

$$\bar{e}_x = \sum_y H_{xy} e_y + K_x a + L_x b, \tag{73}$$

where K_x, L_x *are certain* C^∞-*functions, and* H_{xy} *are constants satisfying*

$$\sum_z \varepsilon_z H_{xz} H_{yz} = \varepsilon_x \delta_{xy}. \tag{74}$$

Proof. Since $e_3, \ldots, e_{n-2}$, b, a span $\ker\omega$, we obtain (73) for some C^∞-functions H_{xy}. Relation (74) follows immediately from (30). Expressing $\bar{c}$ as a combination of the unbarred fields and applying (29), we obtain a relation of the form

$$\bar{c} = c + \sum_x U_x e_x + Vb + Wa.$$

Formulae (31), (45), (65) and (66), applied to the barred frame, yield

$$D_a H_{xy} = D_b H_{xy} = D_{e_z} H_{xy} = 0,$$
$$D_a \bar{e}_x = D_b \bar{e}_x = D_{e_y} \bar{e}_x = 0.$$

By (45), (65) and (67), $D_{\bar{c}} \bar{e}_x = D_c \bar{e}_x$ is a multiple of a and $D_{\bar{d}} \bar{e}_x = D_d \bar{e}_x$ is a multiple of b, which clearly implies $D_c H_{xy} = D_d H_{xy} = 0$. This completes the proof.

THEOREM 3. *Let (M, g) be an n-dimensional $(n \geqslant 4)$ simply connected elliptic e.c.s. Riemannian manifold with constant fundamental function, such that $R_{ij,k} \neq 0$ at each point (this is, e.g., the case where M is homogeneous). Then M is parallelizable.*

Proof. By Lemma 3, $\operatorname{im}\omega$ and $\ker\omega$ are vector subbundles of TM such that $\operatorname{im}\omega \subset \ker\omega$ and the former is the orthogonal complement of the latter. Therefore, g induces an indefinite Riemannian metric $\bar{g}$ in the quotient bundle $\ker\omega/\operatorname{im}\omega$, and the set of bases $v_3, \ldots, v_{n-2}$ of quotient fibres which are orthonormal, i.e. satisfy $\bar{g}(v_x, v_y) = \varepsilon_x \delta_{xy}$, forms a principal fibre bundle P over M whose structure group H consists of all matrices H_{xy} satisfying (74). Using Theorem 2 we may fix a global C^∞-field a, b of special 2-frames on M. It is clear that any completion of a, b to a local distinguished n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ determines a local section of P by $p \mapsto \{e_x + \operatorname{im}\omega_p\}_{x=3,\ldots,n-2}$, $e_x + \operatorname{im}\omega_p$ being the coset of $\operatorname{im}\omega_p$ through $e_x(p)$ in $\ker\omega_p$. Such sections may clearly be viewed as submanifolds of P. The family of n-dimensional submanifolds thus defined is clearly invariant under the action of H, hence it covers P and, by Lemma 13, if two manifolds of the family meet at a point, they must have equal tangent spaces. Therefore, the tangent spaces of our submanifolds form a distribution on P which is a connection in the sense of [9], p. 63. The existence of integral manifolds through any point, i.e. involutivity of the distribution, means that the connection is flat. Therefore, the principal bundle P is trivial ([9], p. 92, Corollary 9.2), i.e. it admits a global C^∞-section. As $\ker\omega/\operatorname{im}\omega$ is isomorphic to a subbundle of $\ker\omega$, transverse to $\operatorname{im}\omega$, the last statement implies that M admits C^∞ vector fields $e_3, \ldots, e_{n-2}$ orthogonal to a, b and satisfying (30). Choosing fields c, d on M dual to a, b as in Lemma 4, we obtain a global n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$. This completes the proof.

We are now in a position to prove the local structure theorem for elliptic e.c.s. manifolds with constant fundamental function.

THEOREM 4. (i) *Let (M, g) be an n-dimensional $(n \geqslant 4)$ elliptic e.c.s. manifold with a non-zero constant fundamental function F. Given a point $p \in M$ at which $R_{ij,k}(p) \neq 0$, there exists a coordinate system $u^1, \ldots, u^n$ in a neighbourhood of p such that $u^1(p) = u^2(p) = \ldots = u^n(p) = 0$ and the components of g are the following functions of coordinates:*

$$
\begin{aligned}
(75)\quad & g_{11} = 2u^n e^{-T} + 2u^{n-1} e^T \partial_2 T - (n-2)^{-1} \varepsilon e^{2T} \sum_x \varepsilon_x (u^x)^2 - (n-2)\varepsilon F^{-1} e^{2T}, \\
& g_{22} = 2u^n (e^T \partial_1 T - e^{-T}) - (n-2)^{-1} \varepsilon e^{2T} \sum_x \varepsilon_x (u^x)^2 - (n-2)\varepsilon F^{-1} e^{2T}, \\
& g_{12} = -u^n e^T \partial_2 T - u^{n-1}(e^T \partial_1 T + 2e^{-T}), \quad g_{1n} = g_{2,n-1} = e^T, \\
& g_{1,n-1} = g_{2n} = g_{n-1,n-1} = g_{n-1,n} = g_{nn} = 0, \\
& g_{1x} = g_{2x} = g_{x,n-1} = g_{xn} = 0, \\
& g_{xy} = 0 \textit{ for } x \neq y, \quad g_{xx} = \varepsilon_x, \quad |\varepsilon_x| = 1,
\end{aligned}
$$

where $\varepsilon = \pm 1$ *is the sign of* R_{ij}, *and* T *is a function of the first two variables* u^1, u^2 *(related by* (63) *to the fundamental invariant* S *of* M *given by* (42)), *satisfying the quasi-linear elliptic partial differential equation*

$$
(76)\qquad \partial_1 \partial_1 T + \partial_2 \partial_2 T + 2e^{-4T} + (n-2)^{-1} \varepsilon e^{2T} = 0.
$$

(ii) *Conversely, given real numbers* $F \neq 0$, ε, ε_x *with* $|\varepsilon| = |\varepsilon_x| = 1$ *and a function* T *of two real variables* u^1, u^2 *satisfying* (76), *formulae* (75) *define an e.c.s. Riemannian metric with fundamental function equal to* F, *which is elliptic (namely, its Ricci tensor is* ε*-semidefinite). The fundamental invariant of this metric* (cf. Remark 4) *is given by* $S = e^{-3T}$ *(hence* $R_{ij,k} \neq 0$ *everywhere).*

Proof. (i) By Lemmas 8 and 12 there exists a distinguished n-frame field $c, d, e_3, \ldots, e_{n-2}, b, a$ in a neighbourhood of p, which, according to Lemma 11, determines a local coordinate system $u^1, \ldots, u^n$. Choosing the solutions u^1, u^2 of (61) and (62) with $u^1(p) = u^2(p) = 0$ and using (ii) of Lemma 12, we may claim $u^i(p) = 0$, $i = 1, \ldots, n$. In the notation of Lemma 10, we have clearly

$$
(77)\quad \xi = (n-2)^{-1} \varepsilon u^{n-1}, \qquad \psi = -(n-2)^{-1} \varepsilon u^n, \qquad A_x = -(n-2)^{-1} \varepsilon \varepsilon_x u^x.
$$

Defining the function T by (63), from (58) and (64) we obtain

$$
(78)\quad \partial_x T = \partial_{n-1} T = \partial_n T = 0, \qquad \lambda = -e^{-T} \partial_2 T, \qquad \sigma = e^{-T} \partial_1 T - e^{-3T}.
$$

The components of g in our chart are given by

$$
g_{ij} = g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right),
$$

whence, by (64), (25), (27)-(30) and (66), we have $g_{11} = 2(n-2)\varepsilon e^{2T} D_c \psi$, etc., which yields (75) in virtue of (67)-(70), (77) and (78). Relation (76) follows now immediately from (57) together with (78), which proves (i).

(ii) The only non-zero contravariant components of our metric are related to

$$
g^{1n} = g^{2,n-1} = e^{-T}, \quad g^{xx} = \varepsilon_x, \quad g^{n-1,n-1} = -e^{-2T} g_{22},
$$
$$
g^{n-1,n} = -e^{-2T} g_{12} \quad \text{and} \quad g^{nn} = -e^{-2T} g_{11}.
$$

Now we can compute the following components of the Riemannian connection:

$$\Gamma^1_{11} = \partial_1 T - e^{-2T}, \quad \Gamma^1_{12} = \partial_2 T, \quad \Gamma^1_{22} = e^{-2T} - \partial_1 T, \quad \Gamma^2_{11} = -\partial_2 T,$$

$$\Gamma^2_{12} = \partial_1 T + e^{-2T}, \quad \Gamma^2_{22} = \partial_2 T, \quad \Gamma^x_{11} = \Gamma^x_{22} = (n-2)^{-1}\varepsilon e^{2T} u^x,$$

$$\Gamma^x_{12} = 0, \quad \Gamma^1_{ij} = \Gamma^2_{ij} = \Gamma^x_{ij} = 0 \quad \text{if } i > 2 \text{ or } j > 2,$$

$$\Gamma^{n-1}_{11} = u^n[2\partial_1 T\cdot\partial_2 T - \partial_1\partial_2 T - 2e^{-2T}\partial_2 T] + $$
$$+ u^{n-1}[(n-2)^{-1}\varepsilon e^{2T} + 3e^{-2T}\partial_1 T - (\partial_2 T)^2] \quad \text{in view of (76)},$$

$$\Gamma^{n-1}_{12} = u^n[(\partial_2 T)^2 - (\partial_1 T)^2 + \partial_1\partial_1 T + 2e^{-4T} + e^{-2T}\partial_1 T] +$$
$$+ u^{n-1}[\partial_1 T\cdot\partial_2 T + 2e^{-2T}\partial_2 T] + (n-2)^{-1}\varepsilon e^{-T}\sum_x \varepsilon_x (u^x)^2 +$$
$$+ (n-2)\varepsilon F^{-1} e^{-T},$$

$$\Gamma^{n-1}_{1x} = 0, \quad \Gamma^{n-1}_{1,n-1} = -e^{-2T}, \quad \Gamma^{n-1}_{1n} = -\partial_2 T, \quad \Gamma^{n-1}_{2,n-1} = 0,$$

$$\Gamma^{n-1}_{22} = u^n[4e^{-2T}\partial_2 T - 2\partial_1 T\cdot\partial_2 T + \partial_1\partial_2 T] + u^{n-1}[2e^{-4T} - e^{-2T}\partial_1 T - (\partial_1 T)^2],$$

$$\Gamma^{n-1}_{2x} = -(n-2)^{-1}\varepsilon\varepsilon_x e^T u^x, \quad \Gamma^{n-1}_{2n} = \partial_1 T - e^{-2T},$$

$$\Gamma^n_{11} = u^n[2e^{-4T} - 3e^{-2T}\partial_1 T - (\partial_2 T)^2] + u^{n-1}[\partial_1\partial_2 T - 2\partial_1 T\cdot\partial_2 T] -$$
$$-(n-2)^{-1}\varepsilon e^{-T}\sum_x \varepsilon_x(u^x)^2 - (n-2)\varepsilon F^{-1} e^{-T},$$

$$\Gamma^n_{12} = u^n[\partial_1 T\cdot\partial_2 T - 2e^{-2T}\partial_2 T] +$$
$$+ u^{n-1}[(\partial_1 T)^2 - (\partial_2 T)^2 + \partial_2\partial_2 T + 3e^{-2T}\partial_1 T + 2e^{-4T}],$$

$$\Gamma^n_{1x} = -(n-2)^{-1}\varepsilon e^T \varepsilon_x u^x, \quad \Gamma^n_{1,n-1} = \partial_2 T, \quad \Gamma^n_{1n} = e^{-2T}, \quad \Gamma^n_{2x} = 0,$$

$$\Gamma^n_{2,n-1} = -\partial_1 T - e^{-2T}, \quad \Gamma^n_{2n} = 0, \quad \Gamma^{n-1}_{ij} = \Gamma^n_{ij} = 0 \quad \text{if } i, j > 2.$$

It is easy to verify that

$$R_{121}{}^1 = 0, \quad R_{121}{}^2 = \partial_1\partial_1 T + \partial_2\partial_2 T + 2e^{-4T},$$
$$R_{122}{}^1 = -\partial_1\partial_1 T - \partial_2\partial_2 T - 2e^{-4T},$$

and

$$R_{121}{}^{n-1} = -2u^n(n-2)^{-1}\varepsilon - 2u^{n-1}(n-2)^{-1}\varepsilon e^{2T}\partial_2 T + (n-2)^{-2}e^{3T}\sum_x \varepsilon_x(u^x)^2 +$$
$$+ [\partial_1\partial_1 T + \partial_2\partial_2 T + 2e^{-4T} + (n-2)^{-1}\varepsilon e^{2T}][2u^n e^{-2T} - 2u^n\partial_1 T + u^{n-1}\partial_2 T] +$$
$$+ u^n\partial_1[\partial_1\partial_1 T + \partial_2\partial_2 T + 2e^{-4T} + (n-2)^{-1}\varepsilon e^{2T}].$$

Simplifying these formulae with the aid of (76), it is easy to see that the only non-zero components of R_{hijk}, R_{ij}, $R_{ij,k}$ and C_{hijk} are those related to

$$R_{1212} = -2u^n(n-2)^{-1}\varepsilon e^{3T}\partial_1 T - 2u^{n-1}(n-2)^{-1}\varepsilon e^{3T}\partial_2 T + \\ + 2(n-2)^{-2}e^{4T}\sum_x \varepsilon_x(u^x)^2 + F^{-1}e^{4T},$$

$$R_{121,n-1} = -(n-2)^{-1}\varepsilon e^{3T}, \quad R_{122n} = (n-2)^{-1}\varepsilon e^{3T},$$

$$R_{1x1x} = R_{2x2x} = -(n-2)^{-1}\varepsilon\varepsilon_x e^{2T},$$

and

(79) $$R_{11} = R_{22} = \varepsilon e^{2T},$$

and

(80) $$R_{11,1} = 2\varepsilon, \quad R_{12,2} = R_{22,1} = -2\varepsilon,$$

and

(81) $$C_{1212} = -F^{-1}e^{4T}.$$

It is now easy to verify that our metric is e.c.s. By (81) and (79), R_{ij} is ε-semidefinite and the fundamental function is equal to F. Defining vector fields a, b by $a = \partial/\partial u^n$, $b = \partial/\partial u^{n-1}$, so that their covariant components are given by $a_1 = e^T$, $a_2 = \ldots = a_n = 0$ and $b_2 = e^T$, $b_1 = b_3 = \ldots = b_n = 0$, and using (79) and (80), we obtain (38) and (39) with $S = e^{-3T}$. In view of Remark 4, this completes the proof.

A Riemannian manifold M is called *locally homogeneous* if for any two points p, q of M there exists an isometry of a neighbourhood of p onto a neighbourhood of q, which sends p onto q.

THEOREM 5. *Let M be an elliptic e.c.s. manifold with constant fundamental function. Then the following two conditions are equivalent*:

(i) *M is locally homogeneous.*

(ii) *The fundamental invariant S of M (defined by (42)) is constant.*

Proof. By (42), (i) implies (ii). Now suppose that S is constant (hence $R_{ij,k} \neq 0$ everywhere) and let $p, q \in M$. In the notation of Theorem 4, the equality $S = e^{-3T}$ together with (76) implies

(82) $$e^{-3T} = (2n-4)^{-1/2}, \quad \varepsilon = -1.$$

Choosing now coordinate systems centered at p and q according to (i) of Theorem 4, we note that the expressions for the metric thus obtained are identical (the number of minus signs among the ε_x is an algebraic invariant of the metric). Thus, p and q are congruent by a local isometry, which completes the proof.

As an immediate consequence of (82), we have

COROLLARY 3. *The Ricci tensor of any locally homogeneous elliptic e.c.s. manifold is negative semidefinite.*

Remark 5. Using Theorems 4 and 5, we can describe the local structure of locally homogeneous elliptic e.c.s. manifolds as follows. For any point p of such a manifold M there exists a local coordinate system $u^1, \ldots, u^n$ in a neighbourhood of p such that $u^i(p) = 0$, $i = 1, \ldots, n$ ($n = \dim M \geqslant 4$) and the only non-zero components of the metric are

$$g_{11} = 2u^n e^{-T} + (n-2)^{-1} e^{2T} \sum_x \varepsilon_x (u^x)^2 + (n-2) F^{-1} e^{2T},$$

$$g_{22} = g_{11} - 4u^n e^{-T}, \quad g_{12} = g_{21} = -2u^{n-1} e^{-T},$$

$$g_{xx} = \varepsilon_x, \quad g_{1n} = g_{n1} = g_{2,n-1} = g_{n-1,2} = e^T,$$

where $T = \frac{1}{8}\log(2n-4)$, $|\varepsilon_x| = 1$, and $F \neq 0$ is the (constant) fundamental function of M. Conversely, the metric defined by these data is locally homogeneous, e.c.s., elliptic and its fundamental function equals F.

The argument used in the proof of Theorem 5 shows also that two locally homogeneous elliptic e.c.s. manifolds of equal dimensions, indices and fundamental functions are locally isometric. Moreover, from (3) it is clear how multiplying the metric by a constant affects the fundamental function. Thus, we obtain

COROLLARY 4. *If M_1 and M_2 are two locally homogeneous elliptic e.c.s. manifolds of equal dimensions and equal indices, then they are locally homothetic. If their (constant) fundamental functions coincide, then they are locally isometric.*

By the *index* of (the metric of) a Riemannian manifold (M, g) we mean here the number of negative entries in a diagonal form of g at any point of M. For the metric g given by (75), we have clearly (index of g) $= 2 +$ (the number of x with $\varepsilon_x = -1$).

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