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Chapter 4

Einstein metrics

Hermitian Einstein metrics†

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0. INTRODUCTION

Let (M, g) be an *oriented* four-dimensional Riemannian manifold. We shall say that (M, g) is *locally conformally Kählerian* if every point x of M has a neighbourhood U with a function $F > 0$ on U such that Fg is a Kähler metric for some complex structure in U, *compatible with the original orientation*.

The aim of the present paper is to discuss some properties and examples of *Einstein* manifolds which are locally conformally Kählerian. In the case of orientable Einstein 4-manifolds, this condition does not seem too unnatural. It holds, for example (for some orientation), if the manifold has sufficiently many isometries with fixed points (see §3.5), as well as for *all known examples of compact orientable Einstein* 4-*manifolds* ([7], §5). The list of these examples (up to finite isometric coverings) consists of: (1) products of surfaces, (2) locally irreducible locally symmetric spaces, (3) compact complex Kähler surfaces having $c_1 < 0$ or $c_1 = 0$ with Kähler–Einstein metrics (cf. the existence theorems of Yau and Aubin [2,15,18]), (4) the complex surface obtained by blowing up a point in CP^2 (homeomorphic to $CP^2 \# (-CP^2)$) with the $U(2)$-invariant Hermitian Einstein metric constructed by Page [4,14].

In Sections 1 and 2 we list some properties of locally conformally Kählerian Einstein 4-manifolds. Some examples of *complete* manifolds of this type are described in Section 3; these examples are either locally conformal to products of surfaces (§3.3; by §2.5, there is no chance of

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finding essentially new compact examples in this way) or self-dual (§3.4). An interesting feature of our examples is that, for many of them, the conformally related Kähler metrics are essentially non-complete.

For a given Riemannian metric g, we shall use the symbols ∇, Ric, Scal, $W = W(g)$ and $\Delta = -g^{ij}\nabla_i\nabla_j$ for its Riemannian connection, Ricci tensor, scalar curvature, Weyl conformal tensor and Laplace operator.

1. KÄHLER MANIFOLDS OF REAL DIMENSION 4 WHICH ARE LOCALLY CONFORMALLY EINSTEINIAN

Before discussing the Einstein 4-manifolds which are locally conformally Kählerian, let us consider the corresponding (locally conformally Einsteinian) Kähler 4-manifolds. The characterization of the latter is, generically, very simple (equation (1) below). Conditions, characterizing locally conformally Einsteinian manifolds among arbitrary Riemannian ones, were found by Brinkmann [5].

For a Riemannian manifold (M, g), the *Bach tensor* $B = B(g)$ (first studied by Bach [3]) is defined by $B_{ij} = \nabla^p\nabla^q W_{pijq} + \frac{1}{2}(\mathrm{Ric})^{pq}W_{pijq}$. Under a conformal change of metric in dimension 4, the Bach tensor transforms like $B(Fg) = B(g)/F$. If g is an Einstein metric, we have $\nabla^p\nabla^q W_{pijq} = 0$, $(\mathrm{Ric})^{pq}W_{pijq} = 0$ and hence $B(g) = 0$. Consequently, *condition $B(g) = 0$ is necessary in order that a Riemannian 4-manifold (M, g) be locally conformally Einsteinian.*

1.1

A Kähler 4-manifold (M, J, g) satisfies $B(g) = 0$ if and only if

$$(1) \qquad 2\nabla^2\mathrm{Scal} + \mathrm{Scal}\cdot\mathrm{Ric} = (\mathrm{Scal}^2/4 - \Delta\mathrm{Scal}/2)g$$

([7], Lemma 5). Conversely, for a Kähler 4-manifold (M, J, g) satisfying (1), the conformally related metric $\bar{g} = g/\mathrm{Scal}^2$ (defined wherever Scal $\neq 0$) is Einsteinian. The Einstein metric conformal to g is essentially unique and, if Scal is not constant, $\bar{g}$ is not locally symmetric (since $24\bar{g}(W^+(\bar{g}), W^+(\bar{g})) = \mathrm{Scal}^6$; cf. Section 2 and [7], Proposition 3). Therefore, *at points where* Scal $\neq 0$, *a Kähler 4-manifold is locally conformally Einsteinian if and only if it satisfies* (1).

1.2

For a Kähler 4-manifold (M, J, g) satisfying (1), $J(\nabla\mathrm{Scal})$ *is a holomorphic Killing vector field* ([7], Proposition 4). Thus, g is then an *extremal Kähler metric* in the sense of Calabi [6].

2. LOCALLY CONFORMALLY KÄHLERIAN FOUR-DIMENSIONAL EINSTEIN MANIFOLDS

Let M be an oriented smooth 4-manifold. If a Riemannian metric g (or a conformal structure) is chosen in M, the bundle $\Lambda^2 M$ of 2-forms on M can be written as the Whitney sum $\Lambda^+ M + \Lambda^- M$, where the 3-plane bundles $\Lambda^\pm M$ consist of all 2-forms ζ with $*\zeta = \pm\zeta$. The Weyl tensor W of (M, g), viewed as an endomorphism of $\Lambda^2 M$, always commutes with $*$ ([16]) and hence it leaves $\Lambda^\pm M$ invariant; we shall denote by $W^\pm = W^\pm(g)\colon \Lambda^\pm M \to \Lambda^\pm M$ the resulting restrictions of W.

2.1

In a Kähler 4-manifold (M, J, g) endowed with natural orientation (so that the Kähler form ω is a section of $\Lambda^+ M$), the eigenvalues of W^+ at any point are $\{\mathrm{Scal}/6, -\mathrm{Scal}/12, -\mathrm{Scal}/12\}$ and ω is an eigenvector of W^+ for the 'simple' eigenvalue Scal/6 (cf. [11]). By the conformal invariance of W, this implies that the *condition* $\#\,\mathrm{Spec}\, W^+ \leqslant 2$ ('W^+ has less than three distinct eigenvalues at any point') *is necessary for an oriented Riemannian 4-manifold to be locally conformally Kählerian* (in the sense of Section 0).

2.2 Uniqueness of a Kähler metric within a given conformal class

Let (M, J, g) be a Kähler 4-manifold, U an open subset of M with $W^+ \neq 0$ everywhere in U and $F > 0$ a function on U such that $\tilde{g} = Fg$ is a Kähler metric for some complex structure $\tilde{J}$ compatible with the orientation determined by J. Then F is *constant* and $\tilde{J} = \pm J$. In fact, in view of (2.1), the Kähler form ω of g (resp. $\tilde{\omega}$ of $\tilde{g}$) is determined (up to a sign) by $W^+(g)$ (resp. by $W^+(\tilde{g}) = W^+(g)/F$), at any point of U. Thus, $\tilde{\omega} = \pm F\omega$ and hence $\tilde{J} = \pm J$; since $d\tilde{\omega} = d\omega = 0$, F must be constant.

2.3

Let (M, g) be an oriented Einstein 4-manifold with $\#\,\mathrm{Spec}\, W^+ \leqslant 2$.

(a) If $W^+ = 0$ identically, then (M, g) is called an *anti-self-dual* Einstein manifold ([1]). In the case where M is compact and Scal $\geqslant 0$, these manifolds have been classified by N. Hitchin [12,13] (see also [10]).

(b) If W^+ does not vanish identically, then $W^+ \neq 0$ *everywhere*. The global conformal change $\tilde{g} = |W^+|^{2/3} g$ gives rise to a Kähler metric $\tilde{g}$ on M or on some twofold cover of M (the Kähler form for $\tilde{g}$ is determined by g only up to a sign) and the scalar curvature of $\tilde{g}$ is non-zero everywhere ([7], Proposition 5).

2.4

Let (M, g) be a *compact* oriented Einstein 4-manifold such that # Spec $W^+ \leqslant 2$ and W^+ is *not parallel*. Then the universal covering manifold $\tilde{M}$ of M is diffeomorphic to $S^2 \times S^2$ or to a connected sum $CP^2 \# (-kCP^2)$, $0 \leqslant k \leqslant 8$, while the pull-back of the metric $|W^+|^{2/3}g$ to $\tilde{M}$ is a Kähler metric with positive non-constant scalar curvature ([7], Theorem 2).

From (2.3) it follows that, for oriented Einstein 4-manifolds with $W^+ \neq 0$, local conformal equivalence to a Kähler manifold implies global one (at least for a twofold cover). The only known example of a *compact* manifold of this type which is not (locally) Kählerian (i.e. satisfies the hypothesis of §2.4) is $CP^2 \# (-CP^2)$ with the Page metric (Section 0). The following argument shows that no new compact Einstein 4-manifolds can be obtained by conformal deformations of products of surfaces, even without insisting that the underlying manifold be globally diffeomorphic to a product.

2.5

Proposition *Let (M, g) be a compact Einstein 4-manifold. If, for some non-void connected open subset U of M and a function $F > 0$ on U, the Riemannian manifold (U, Fg) is isometric to a product of surfaces, then either g is a metric of constant curvature, or F is constant and (M, g) is isometrically covered by a product of surfaces.*

Proof By a result of DeTurck and Kazdan [9], (M, g) is analytic. If g is not of constant curvature, we may assume that M is oriented so that $W^+ \neq 0$ somewhere. However, a manifold conformal to a product of surfaces satisfies # Spec $W^+ \leqslant 2$ for *both orientations*, so that $W^+ \neq 0$ everywhere by §2.3(b). Combining §2.3(b) with §2.2 we see that $|W^+|^{2/3}/F$ is constant, so that analyticity together with the de Rham decomposition theorem implies that the universal covering space $(\tilde{M}, G)$ of $(M, |W^+|^{2/3}g)$ is isometric to a product of surfaces. If $|W^+|$ were not constant, §2.4 would imply that $(\tilde{M}, G)$ is isometric to $(S^2, g_1) \times (S^2, g_2)$ for some metrics g_1, g_2 on S^2, one of which has non-constant Gauss curvature. On the other hand, G is an extremal Kähler metric (the gradient of its scalar curvature is holomorphic, cf. §1.1, §1.2) and so the same would hold for g_1 and g_2. By a theorem of Calabi ([6], p. 276), both g_1 and g_2 would have constant curvatures. This contradiction completes the proof. □

3. EXAMPLES

By §2.3 and §1.1, the only way of obtaining locally conformally Kählerian Einstein 4-manifolds with $W^+ \neq 0$ is, essentially, to take a Kähler 4-manifold (M, J, g) satisfying (1) and define the Einstein metric by $\bar{g} =$

g/Scal^2. Although this procedure is possible only in the set U where Scal $\neq 0$, it always gives rise to *complete* Einstein metrics in the components of U, provided that g is complete and Scal is bounded. Actually, even weaker conditions are sufficient for completeness of $\bar{g}$. For convenience, we shall now consider Riemannian manifolds with boundary (empty or not); they are, naturally, metric spaces and their completeness is equivalent to the existence of end-points for any curve of finite length.

3.1

Lemma *Let f be a bounded C^∞ function on a complete Riemannian manifold (N, g) (with boundary). Set $\bar{g} = g/f^2$ wherever $f \neq 0$. Then, for any component Q of the set $N \backslash f^{-1}(\{0\})$, $(Q, \bar{g})$ is a complete Riemannian manifold (with boundary).*

Proof We claim that for any C^1 curve γ: $[a, b) \to Q$ with $a < b \leqslant \infty$, $g(\dot{\gamma}, \dot{\gamma}) = 1$ and of finite $\bar{g}$-length

$$L = \int_a^b (\mathrm{d}t/|f(\gamma(t))|),$$

there exists a limit $\gamma(b) \in Q$. If $b = \infty$, then $L = \infty$, since f is bounded. For $b < \infty$, completeness of (N, g) implies the existence of $\gamma(b) \in N$. Note that $\mathrm{d}[f(\gamma(t))]/\mathrm{d}t = g(\nabla f, \dot{\gamma})$ is bounded for $t < b$, since $\gamma([a, b])$ is compact. If we had $\gamma(b) \in f^{-1}(\{0\})$, so that $f(\gamma(b)) = 0$, then this would give $|f(\gamma(t))| \leqslant A(b - t)$ for some $A > 0$ and all $t \in [a, b]$ and hence $L \geqslant \infty$, contradicting our assumption. □

3.2

Proposition *Let a Kähler 4-manifold (M, J, g) with non-constant scalar curvature satisfy* (1).

(i) *Every (non-empty) component N of the set* $\mathrm{Scal}^{-1}([0, \infty))$ *(or* $\mathrm{Scal}^{-1}((-\infty, 0])$ *which is not a single point, is a four-dimensional submanifold of M (possibly with boundary). (N, g) is complete if (M, g) is also.*

(ii) *For N as in* (i), *if (N, g) is complete,* Scal *is bounded on N and Q is the subset of N given by* Scal $\neq 0$, *then $(Q, g/\mathrm{Scal}^2)$ is a complete Einstein 4-manifold (without boundary), which is not locally symmetric.*

In fact, by §1.2 and equation (1), $\mathrm{Scal}^{-1}(\{0\})$ is a union of disjoint hypersurfaces and isolated points (note that $\mathrm{Scal}(x) = 0$ and $\nabla \mathrm{Scal}(x) = 0$ implies $\mathrm{Hess\,Scal}(x) = -\Delta \mathrm{Scal}(x) \cdot g(x)/4 \neq 0$, since the Killing field $J(\nabla \mathrm{Scal})$ is determined by its 1-jet at x). Applying §1.1 and §3.1 with N as in (i) and $f = \mathrm{Scal}$, we obtain (ii).

Using §3.2, we shall now describe various explicit examples of complete conformally Kählerian Einstein 4-manifolds.

3.3

Example Let our Kähler 4-manifold be a product of surfaces. (This case was studied by Tashiro [17].) Relation (1) holds if and only if both surfaces satisfy

(2) $$2\nabla^2\kappa = (\varepsilon - \kappa^2)g,$$

g, ∇ and κ being now the metric, the connection and the Gauss curvature of the surface, while ε is a common constant; by rescaling g, we shall assume that $\varepsilon \in \{-1, 0, 1\}$. For a surface (S, g) satisfying (2), we have $|\nabla\kappa|^2 = -\kappa^3/3 + \varepsilon\kappa - r = P_{\varepsilon, r}(\kappa) \geqslant 0$ for some real r and so the integral curves of $\nabla\kappa$ are geodesics. The length of such a geodesic γ: $(t_1, t_2) \to S$, containing no critical points of κ, is given by

(3) $$L(\gamma) = \int_{\kappa_1}^{\kappa_2} [P_{\varepsilon, r}(\kappa)]^{-1/2} d\kappa, \qquad \kappa_i = \lim \kappa(\gamma(t)).$$

If S is oriented, the complex structure tensor J gives rise to the Killing field $J(\nabla\kappa)$. It is now easy to verify that a complete local description of such surfaces at points with $\nabla\kappa \neq 0$ is

(4) $$g = dt^2 + a(d\kappa/dt)^2 dx^2$$

in suitable local coordinates (t, x), where $a > 0$ and $\kappa = \kappa(t)$ is any solution of the equation $2d^2\kappa/dt^2 = \varepsilon - \kappa^2$, i.e. of

(5) $$(d\kappa/dt)^2 = -\kappa^3/3 + \varepsilon\kappa - r = P_{\varepsilon, r}(\kappa)$$

for some r. The essentially distinct local types of surfaces satisfying (2) with $\nabla\kappa \neq 0$ are, thus, parametrized by $\varepsilon \in \{-1, 0, 1\}$ and $r \in R$; the parameter a in (4), locally irrelevant ($a dx^2 = d(a^{1/2}x)^2$), will later have global meaning. We can now describe some examples of such surfaces (S, g) with 'as much completeness as possible'. Our $g = g_{\varepsilon, r, a}$ will be the S^1-invariant metric, defined by (4) and (5) on the product $(\inf \kappa, \sup \kappa) \times S^1$ (sometimes 'completed' by adding a point), where the second coordinate $x \in S^1 = R/2\pi Z$, while the first one is parametrized by κ instead of t. For brevity, we shall say that (S, g) is *half-complete from above* (resp. *from below*) if, for all $q \in (\inf \kappa, \sup \kappa)$, $\kappa^{-1}([q, \infty))$ (resp. $\kappa^{-1}((-\infty, q])$) is a complete surface with boundary.

Let $\kappa_0 = \kappa_0(\varepsilon, r)$ be the lowest real root of $P = P_{\varepsilon, r}$. Note that P may have three distinct real roots (if $\varepsilon = 1$ and $|r| < 2/3$), a simple and a double one (if $\varepsilon = 1$, $|r| = 2/3$), a triple one ($\varepsilon = r = 0$), or only one, simple real root (otherwise). Let $\kappa = \kappa_{\varepsilon, r}$ be a solution of (5) with range $(-\infty, \kappa_0)$.

(A) If (ε, r) is none of $(0, 0)$, $(1, 2/3)$, $(1, -2/3)$, then $\kappa_0^2 \neq \varepsilon$ and, for $a = 4/(\varepsilon - \kappa_0^2)^2$, our S^1-invariant metric $g = g_{\varepsilon, r, a}$ on $(-\infty, \kappa_0) \times S^1 \approx R^2 \setminus \{0\}$, defined by (4), can be extended to a smooth S^1-invariant metric g' on R^2 with $\kappa(0) = \kappa_0$. The resulting surface $A_{\varepsilon, r} = (R^2, g')$ is half-complete from above (since κ is a proper function on R^2).

(B) If $\varepsilon = 1$ and $r = -2/3$, our metric $g = g_{\varepsilon, r, a}$ on $S = (-\infty, \kappa_0) \times S^1$ is half-complete from above for each a (which one can easily verify using (3)) and its Gauss curvature $\kappa \leqslant \kappa_0 = -1$. Notation: $(S, g) = B_a$.

In the case where all roots of $P = P_{\varepsilon, r}$ are real and the highest one, κ_1, is simple (i.e. $\varepsilon = 1$ and $-2/3 \leqslant r < 2/3$), denote by κ_2 the largest root with $\kappa_2 < \kappa_1$. Clearly, $-1 \leqslant \kappa_2 < 1 < \kappa_1 \leqslant 2$. Take a solution $\kappa = \kappa_{\varepsilon, r}$ of (5) with range (κ_2, κ_1).

(C) If $P'(\kappa_2) \neq 0$ (i.e. $(\varepsilon, r) \neq (1, -2/3)$), the choice of $a = 4/(1 - \kappa_1^2)^2$ (resp. $a = 4/(1 - \kappa_2^2)^2$) allows us to attach a point x_0 to $(\kappa_2, \kappa_1) \times S^1$ so that our metric $g_{1, r, a}$ can be extended to an S^1-invariant metric g' on a disc D^2 with $\kappa(x_0) = \kappa_1$ (resp. $\kappa(x_0) = \kappa_2$). The surface $(D^2, g') = C_r^+$ (resp. $(D^2, g') = C_r^-$) is half-complete from above (resp. from below).

(D) If $P'(\kappa_2)) = 0$ $((\varepsilon, r, \kappa_1, \kappa_2) = (1, -2/3, 2, -1))$, our surface $D_a = ((\kappa_2, \kappa_1) \times S^1, g)$ is half-complete from below (by (3)) and, for $a = 4/9$, one can attach a point x_0 to D_a so that g has an extension to a metric g' on a disc D^2 with $\kappa(x_0) = \kappa_1$; the surface $(D^2, g') = D$ is *complete*.

Finally, we have the standard examples:

(E) Any complete surface (S, g) of constant curvature $c \in \{1, 0, -1\}$ satisfies (2) with $\varepsilon = |c|$. By an abuse of notation, we shall write here $(S, g) = E_c$ and associate with E_c the parameter $r = 2c/3$.

Among the above examples, complete ones occur only in (D) and (E). However, even for the non-complete surfaces of (A)–(C), certain additional conditions imply, for some of their four-dimensional products, that the subset of the product defined by Scal $\geqslant 0$ (or by Scal $\leqslant 0$) is complete (which follows from half-completeness of our surfaces) and Scal is bounded on it. Therefore, by §3.2(ii) they give rise to complete, conformally Kählerian, open Einstein 4-manifolds, which have negative scalar curvature (equal to $48(r + r')$, cf. notations in Table 1) and are not locally symmetric (unless both surfaces are of type (E)). The details are presented in Table 1 (with notational conventions like $\kappa_0 = \kappa_0(\varepsilon, r)$, $\kappa_0' = \kappa_0(\varepsilon, r')$, etc.). Each of the Einstein 4-manifolds described here has an isometry group of positive dimension.

Table 1

Product manifold	Additional condition	Relation defining a complete submanifold boundary	Topology of the 4-associated complete with Einstein 4-manifold
$A_{\varepsilon,r} \times A_{\varepsilon,r'}$	$\kappa_0 + \kappa_0' > 0$	Scal $\geqslant 0$	D^4
$A_{1,r} \times B_a$	$\kappa_0 > 2$	Scal $\geqslant 0$	$S^1 \times D^3$
$A_{1,r} \times C_{r'}^+$	$\kappa_2' < -\kappa_0 < \kappa_1'$	Scal $\geqslant 0$	D^4
$A_{1,r} \times D$	$\kappa_0 + 2 > 0$	Scal $\geqslant 0$	D^4
$A_{\varepsilon,r} \times E_c$	$\varepsilon = \lvert c\rvert$, $\kappa_0 > -c$	Scal $\geqslant 0$	$D^2 \times E_c$
$B_a \times C_r^+$	None	Scal $\geqslant 0$	$S^1 \times D^3$
$B_a \times D$	None	Scal $\geqslant 0$	$S^1 \times D^3$
$C_r^{\pm} \times E_{-1}$	None	Scal $\geqslant 0$ (resp. Scal $\leqslant 0$)	$D^2 \times E_{-1}$
$C_r^- \times C_{r'}^-$	$\kappa_2 + \kappa_2' < 0$	Scal $\leqslant 0$	D^4
$C_r^- \times D_a$	None	Scal $\leqslant 0$	$S^1 \times D^3$
$C_r^- \times D$	None	Scal $\leqslant 0$	$S^1 \times D^3$
$D_a \times D_a$	None	Scal $\leqslant 0$	$T^2 \times D^2$
$D_a \times D$	None	Scal $\leqslant 0$	$T^2 \times D^2$
$D_a \times E_{-1}$	None	Scal $\leqslant 0$	$S^1 \times D^1 \times E_{-1}$
$D \times D$	None	Scal $\geqslant 0$	D^4
		Scal $\leqslant 0$	$S^3 \times D^1$
$D \times E_c$	$\lvert c\rvert = 1$	Scal $\geqslant 0$	$D^2 \times E_c$
		Scal $\leqslant 0$	$S^1 \times D^1 \times E_c$

3.4

Example Consider a four-dimensional Lie algebra with basis $e_1, \ldots, e_4$ such that $[e_1, e_\alpha] = 0$ $(\alpha \geqslant 2)$, $[e_2, e_3] = 2(p - q^2)e_4$, $[e_2, e_4] = 2\varepsilon e_3$, $[e_3, e_4] = 2(p - q^2)e_2$, where $p, q \in R, p \neq q^2$ and $\varepsilon = \pm 1$. Let σ be any solution of the equation

(6) $\quad \mathrm{d}\sigma/\mathrm{d}t = 2(q - \sigma)(\sigma^2 - p)$

defined on an interval (a, b) and such that $\varepsilon(\sigma^2 - p) > 0$, $\varepsilon(q - \sigma) > 0$. The open subset $M = (a, b) \times H$ of a Lie group $R \times H$ associated to our Lie algebra (H locally isomorphic to $SU(2)$ or to $SL(2, R)$) can now be endowed with the Riemannian metric g given by $g(e_1, e_1) = g(e_3, e_3) = (q - \sigma)(\sigma^2 - p)$, $g(e_2, e_2) = g(e_4, e_4) = \varepsilon(q - \sigma)$, $g(e_\alpha, e_\beta) = 0$ $(\alpha \neq \beta)$, where σ depends on $t \in (a, b)$ and $e_1 = \partial/\partial t$, e_2, e_3, e_4 are viewed as right-invariant vector fields on $R \times H$. This metric is preserved by the right action of H on M and it is Kählerian for the complex structure J given by $Je_1 = e_3$, $Je_2 = e_4$. Moreover, (M, J, g) satisfies (1) and Scal $= 48\sigma$ is not constant. By

§1.2, $J(\nabla \mathrm{Scal})$ is a non-trivial H-invariant Killing field on M, so that the isometry group of (M, g) is four-dimensional (locally isomorphic to $S^1 \times H$). Another property of g is that it is *self-dual* ($W^- = 0$ for the natural orientation); for details, see [8]. Combining this construction with §3.2(ii), we shall now describe some examples of complete Einstein 4-manifolds. Take $\varepsilon = -1$, $p > 0$, $-p^{1/2} < q < 0$, $H = SU(2)$ and a solution σ of (6) defined on an interval (a, b) and having range $(q, p^{1/2})$. The $U(2)$-invariant metric g constructed as above on the manifold $(a, b) \times S^3$ (which we identify with a pointed ball $B^4 \setminus \{0\}$ in C^2) can be extended to a Kähler metric g' on B^4 with $\sigma(0) = \mathrm{Scal}(0)/48 = q$ ([8], §2). In (B^4, g'), relation $\mathrm{Scal} < 0$ defines a ball Q with compact closure. Thus, by §3.2(ii), $(Q, g'/\mathrm{Scal}^2) = M_{p,q}$ is a complete Hermitian Einstein 4-manifold, of negative scalar curvature $2^{12} \cdot 3^3 pq$, which is conformally Kählerian, not locally symmetric and self-dual for the natural orientation. The four-dimensional isometry group of $M_{p,q}$ has three-dimensional principal orbits; various examples of Einstein manifolds with the similar property of 'principal cohomogeneity one' have been constructed by Bérard Bergery [4].

3.5

Remark Let (M, g) be an orientable Einstein 4-manifold such that for each $x \in M$ there exists a non-trivial Killing field defined near x and vanishing at x (this happens, for example, if all orbits of the isometry group $I(M, g)$ are of dimensions less than $\dim I(M, g)$). By Lemma 9 of [7], we have $\# \operatorname{Spec} W^+ \leqslant 2$ for both orientations and so, by §2.3, (M, g) is locally conformally Kählerian (for an appropriate orientation). However, if (M, g) is not locally symmetric, one can easily prove, using the Killing field mentioned in §1.2, that the Lie algebra of germs of Killing fields at any point is four-dimensional and has principal orbits of dimension three; a general existence theorem for (incomplete) Einstein metrics of cohomogeneity one is due to Koiso and Bérard Bergery ([4], Proposition 3).

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