HARMONIC CURVATURE IN DIMENSION FOUR

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Abstract. We provide a step towards classifying Riemannian four-manifolds in which the curvature tensor has zero divergence, or – equivalently – the Ricci tensor Ric satisfies the Codazzi equation. Every known compact manifold of this type belongs to one of five otherwise-familiar classes of examples. The main result consists in showing that, if such a manifold (not necessarily compact or even complete) lies outside of the five classes – a non-vacuous assumption – then, at all points of a dense open subset, Ric has four distinct eigenvalues, while suitable local coordinates simultaneously diagonalize Ric, the metric and, in a natural sense, also the curvature tensor. Furthermore, in a local orthonormal frame formed by Ricci eigenvectors, the connection form (or, curvature tensor) has just twelve (or, respectively, six) possibly-nonzero components, which together satisfy a specific system, not depending on the point, of homogeneous polynomial equations. A part of the classification problem is thus reduced to a question in real algebraic geometry.

Introduction

One says that a Riemannian manifold has *harmonic curvature* if its curvature tensor R satisfies the local-coordinate relation $R_{ijk}{}^{p}{}_{,p} = 0$, that is,

$$\operatorname{div} R \,=\, 0. \tag{0.1}$$

See [3, Sect. 16.33]. Let us now consider the condition

$$\begin{array}{l}(K+c)^3 + 3(K+c)\Delta K - 6|dK|^2 = r^3, \text{ where} \\ r, c \in \mathrm{I\!R} \;\text{ and }\; K+c \neq 0 \;\text{ at every point of }\; Q,\end{array} \tag{0.2}$$

imposed on the Gaussian curvature K of a Riemannian surface (Q,h), with $\Delta = h^{ij}\nabla_{\!i}\nabla_{\!j}$ denoting the h-Laplacian and $|\;|$ the h-norm.

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The following four-manifolds all have harmonic curvature (Remark 4.2).

(0.3)
a) Einstein manifolds of dimension four.
b) Conformally flat 4-manifolds of constant scalar curvature.
c) Riemannian products of a one-dimensional manifold and a conformally flat 3-manifold with constant scalar curvature.
d) Products of surfaces having constant Gaussian curvatures.
e) Warped products $(Q \times S, (h \times h^c)/(K + c)^2)$, where (Q, h) and (S, h^c) are Riemannian surfaces, (S, h^c) is of constant Gaussian curvature c, and (Q, h) satisfies (0.2).

In (0.3.e) we treat K as a function on $Q \times S$, constant along the S factor. Thus, $2(K + c)$ equals the scalar curvature of the product metric $h \times h^c$.

All known examples of *compact* four-manifolds (M, g) with $\operatorname{div} R = 0$ belong to the five (non-disjoint) local-isometry types (0.3) in the sense that

(0.4) each $x \in M$ has a neighborhood isometric to one of (0.3).

However, $\operatorname{div} R = 0$ in some *complete* Riemannian four-manifolds not containing open submanifolds of types (0.3). See [8] and Remark 7.1.

This paper is a step towards classifying Riemannian four-manifolds with harmonic curvature that lie outside of the five classes (0.3). The next section states in full detail our two main results, here summarized only briefly.

According to the first of them, Theorem 1.2, at generic points of such a manifold (M, g), its Ricci tensor Ric has four distinct eigenvalues, and suitable local coordinates simultaneously diagonalize g, Ric and R. Note that the local orthonormal frame e_i, $i = 1, \ldots, 4$, obtained by normalizing the coordinate vector fields, then gives rise to an orthogonal web of codimension-one foliations or, equivalently, satisfies, for all distinct $i, j \in \{1, 2, 3, 4\}$, the Lie-bracket relations

(0.5) $[e_i, e_j] = F_{ji} e_i - F_{ij} e_j$ (no summation), with some functions F_{ji}.

As shown by Tod [18], coordinate-diagonalizability of a metric, in dimension four, generically imposes restrictions on the third derivative of the Weyl tensor. Simplicity of the Ricci eigenvalues implies in turn that a harmonic-curvature manifold satisfying the above assumptions cannot be a nontrivial warped product with any fibre dimension p greater than one [8, Corollary 1.3], although the case $p = 1$ does occur [8].

The second result, Theorem 1.3, states that the twelve functions F_{ji} in (0.5) and the six sectional-curvature functions $R_{ijij} = R(e_i, e_j, e_i, e_j)$ together form, at every generic point x, a solution of a specific system, not depending on x, of homogeneous polynomial equations. Thus, when these F_{ji} and R_{ijij} are treated as the components of a mapping Φ from a neighborhood of a generic point into $\mathbb{R}^{18}$, the values of Φ lie in an explicitly defined real algebraic variety $\mathcal{V} \subseteq \mathbb{R}^{18}$. Consequently, Theorem 1.3 relates the classification of four-dimensional Riemannian manifolds with $\operatorname{div} R = 0$, different from the types (0.3), to a problem in real algebraic geometry.

The text is organized as follows. Sections 1 and 7 provide detailed statements of the main results and an outline of the proof of parts (a) – (b) in Theorem 1.2. Preliminary and expository material, presented in Sections 2 through 6, and 8, is followed by Section 9, describing the three *a priori* possible cases that arise under the hypotheses Theorem 1.2. After the exclusion of two of these cases (Sections 10 – 12), the conclusions about the third case lead, in Section 13, to proofs of Theorem 1.2(a)-(b) and Theorem 1.3. The four final sections are devoted to proving part (c) of Theorem 1.2.

1. Detailed statements of the main results

As shown by DeTurck and Goldschmidt [10], in suitable local coordinates,

$$\text{every metric with } \operatorname{div} R = 0 \text{ is real-analytic.} \tag{1.1}$$

For a fixed oriented Riemannian four-manifold (M, g) with $\operatorname{div} R = 0$, let us denote by $\boldsymbol{r} \in \{1, 2, 3, 4\}$ and $\boldsymbol{w} \in \{1, 2, 3\}$ the maximum number of distinct eigenvalues of the Ricci tensor Ric acting on the tangent bundle TM and, respectively, of the self-dual Weyl tensor W^+ acting on the bundle of self-dual bivectors (see Section 3). Due to (1.1), both maxima $\boldsymbol{r}$ and $\boldsymbol{w}$ are simultaneously attained at all points of a dense open subset of M.

Proofs of Lemma 1.1, (a), (b) in Theorem 1.2 along with Theorem 1.3, and Theorem 1.2(c) are given, respectively, in Sections 4, 13 and 14 – 17.

Lemma 1.1. *For any oriented Riemannian four-manifold* (M, g) *having* $\operatorname{div} R = 0$*, the following two conditions are equivalent*:

(i) (M, g) *belongs to one of the local-isometry types* (0.3)*, as in* (0.4)*,*
(ii) g *is locally reducible, or* $\boldsymbol{r} \in \{1, 2\}$*, or* $\boldsymbol{w} \in \{1, 2\}$*.*

Theorem 1.2. *Suppose that* $\operatorname{div} R = 0$ *for the curvature tensor* R *of an oriented Riemannian four-manifold* (M, g) *which does not satisfy* (0.4)*. The following conclusions then hold on some dense open set* $U \subseteq M$*.*

(a) *Locally in* U *there exist functions* F_{ji} *and real-analytic orthonormal vector fields* e_i *diagonalizing* Ric*, with the Lie brackets given by* $[e_i, e_j] = F_{ji} e_i - F_{ij} e_j$ *whenever* $i, j \in \{1, 2, 3, 4\}$ *are distinct.*
(b) e_i *also diagonalize* R*, in the sense of Section* 3*.*
(c) $\boldsymbol{r} = 4$ *and* Ric *has four distinct eigenvalues at every point of* U*.*

About the local-coordinate aspect of (a), mentioned in the Introduction, see Remark 13.3,

Theorem 1.2 is non-vacuous (Remark 7.1) and, according to Lemma 1.1, the assumptions made about (M, g) amount to its being locally irreducible, four-dimensional, oriented and having $\operatorname{div} R = 0$ along with

$$(\boldsymbol{r}, \boldsymbol{w}) \in \{3, 4\} \times \{3\} \text{ or, equivalently, } \boldsymbol{r} \notin \{1, 2\} \text{ and } \boldsymbol{w} \notin \{1, 2\}. \tag{1.2}$$

The next theorem and the remainder of the paper, except Section 4, use the convention that the indices i, j, k, l always range over $\{1, \ldots, 4\}$, repeated indices are *not summed over* and, unless stated otherwise,

(1.3) if some of i, j, k, l appear in an equality, they are assumed to be mutually distinct and preceded by a universal quantifier.

Thus, the presence of i, j, k will tacitly imply the preamble

$$\text{for all } i, j, k \in \{1, \dots, 4\} \text{ with } i \neq j \neq k \neq i. \tag{1.4}$$

Rather than using the sectional-curvature functions $R_{ijij} = R(e_i, e_j, e_i, e_j)$ (see the Introduction), it is more convenient to phrase our second main result in terms of the analogous components $\sigma_{ij} = W(e_i, e_j, e_i, e_j)$ of the Weyl tensor W, along with the scalar curvature s, and the eigenvalue functions $\lambda_i = b(e_i, e_i)$ of the traceless Ricci tensor $b = \mathrm{Ric} - \mathrm{s}g/4$. The latter are easily expressed through the former, cf. equality (2.2) below, and vice versa:

$$R_{ijij} = \sigma_{ij} + \frac{1}{2}(\lambda_i + \lambda_j) + \frac{\mathrm{s}}{12} \quad \text{if } i, j \in \{1, 2, 3, 4\} \text{ and } i \neq j. \tag{1.5}$$

See the line following formula (8.5) in Section 8.

Theorem 1.3. *For* $(M, g), U, e_i$ *and* F_{ji} *as in Theorem* 1.2*,* F_{ji} *and the functions* σ_{ij}, λ_i *defined above satisfy the polynomial equations*

$$\begin{aligned}
&\lambda_i + \lambda_j + \lambda_k + \lambda_l = \sigma_{ij} - \sigma_{ji} = \sigma_{ij} - \sigma_{kl} = \sigma_{ij} + \sigma_{ik} + \sigma_{il} = 0,\\
&[(\lambda_k - \lambda_l)\sigma_{kl} + (\lambda_l - \lambda_i)\sigma_{li} + (\lambda_i - \lambda_k)\sigma_{ik}](F_{kl}F_{li} + F_{lk}F_{ki} - F_{ki}F_{li})\\
&\quad = [(\lambda_k - \lambda_l)\sigma_{kl} + (\lambda_l - \lambda_j)\sigma_{lj} + (\lambda_j - \lambda_k)\sigma_{jk}](F_{kl}F_{lj} + F_{lk}F_{kj} - F_{kj}F_{lj}),
\end{aligned}$$

with the conventions (1.3) – (1.4)*. Choosing the frame* e_i *is to be positive oriented, we may rewrite the second displayed equation as*

$$H_{ji}Z_j = -H_{ij}Z_i \quad \text{whenever } i, j \in \{1, 2, 3, 4\} \text{ and } i \neq j, \tag{1.6}$$

where H_{ij} *and* Z_l *are uniquely characterized by* $H_{ij} = F_{kl}F_{lj} + F_{lk}F_{kj} - F_{kj}F_{lj}$ *when* $\{i, j, k, l\} = \{1, 2, 3, 4\}$ *and* $Z_l = (\lambda_i - \lambda_j)\sigma_{ij} + (\lambda_j - \lambda_k)\sigma_{jk} + (\lambda_k - \lambda_i)\sigma_{ki}$ *if* (i, j, k, l) *is an even permutation of* $(1, 2, 3, 4)$*. The twelve functions* H_{ij} *are subject to a further system of polynomial equations, namely*

$$\operatorname{rank}\begin{bmatrix}
H_{12} & H_{13} & H_{14} & 0 & 0 & 0 & 1\\
H_{21} & 0 & 0 & H_{23} & H_{24} & 0 & 1\\
0 & H_{31} & 0 & H_{32} & 0 & H_{34} & 1\\
0 & 0 & H_{41} & 0 & H_{42} & H_{43} & 1
\end{bmatrix} \leq 3. \tag{1.7}$$

2. Preliminaries

Manifolds, mappings and tensor fields are by definition C^∞-differentiable. Unless stated otherwise, a manifold is assumed connected. Our conventions about the exterior derivative of a 1-form ζ and the curvature tensor R of a Riemannian metric g are such that, for tangent vector fields u, v, w,

$$\begin{aligned}
(d\zeta)(u, v) &= d_u[\zeta(v)] - d_v[\zeta(u)] - \zeta([u, v]),\\
R(v, w)u &= \nabla_w\nabla_v u - \nabla_v\nabla_w u + \nabla_{[v,w]}u.
\end{aligned} \tag{2.1}$$

The Ricci tensor Ric and scalar curvature s of g give rise to the Schouten tensor $\mathrm{Sch} = \mathrm{Ric} - [2(n-1)]^{-1}\mathrm{s}g$ and the Weyl conformal tensor $W = R - (n-2)^{-1}g \wedge \mathrm{Sch}$, in dimensions $n \ge 3$, where $\wedge$ is a natural bilinear pairing of symmetric 2-tensors, valued in covariant 4-tensors [3, formula (1.116)]. In coordinates, with R_{ij} denoting the components of Ric,

$$\begin{aligned} W_{ijkl} \,=\, & R_{ijkl} \,-\, (n-2)^{-1}(g_{ik}R_{jl} \,+\, g_{jl}R_{ik} \,-\, g_{jk}R_{il} \,-\, g_{il}R_{jk}) \\ & +\, (n-1)^{-1}(n-2)^{-1}\mathrm{s}(g_{ik}g_{jl} \,-\, g_{jk}g_{il}). \end{aligned} \tag{2.2}$$

A *Codazzi tensor* [3, Sect, 16.5] on a Riemannian manifold is a twice-covariant symmetric tensor field b with $db = 0$ (in coordinates: $b_{ki,j} = b_{kj,i}$)

As usual, by the *warped product* of the Riemannian manifolds (Q,h) (the *base*) and (Σ,γ) (the *fibre*) with the *warping function* $\phi : Q \to (0,\infty)$ we mean the Riemannian manifold

$$(M,g) \,=\, (Q \times \Sigma,\, h + \phi^2\gamma), \tag{2.3}$$

where h,γ,ϕ also denote the pullbacks of the original h,γ,ϕ to $Q \times \Sigma$.

Remark 2.1. Since (2.3) amounts to $(M,g) = (Q \times \Sigma,\, \phi^2[\phi^{-2}g + h])$, a warped product is the same as a Riemannian manifold conformal to a Riemannian product in such a way that the conformal-factor function is constant along one of the constituent manifolds.

3. Algebraic Weyl tensors and diagonalizability

Given a Euclidean vector space $\mathcal{T}$ of any dimension n, by an *algebraic Weyl tensor* in $\mathcal{T}$ we mean a quadrilinear mapping $A : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathrm{I\!R}$ having the usual symmetries of the Weyl conformal tensor (skew-symmetry in the first and last pairs of arguments, the first Bianchi identity, and vanishing of the Ricci contraction). If the Ricci-contraction requirement is relaxed, one calls A an *algebraic curvature tensor*. Any such A also forms a linear endomorphism $A : \mathcal{T}^{\wedge 2} \to \mathcal{T}^{\wedge 2}$ of the space $\mathcal{T}^{\wedge 2}$ of bivectors, with $A(v \wedge w) = A(v, w, \cdot\,, \cdot\,) \in [\mathcal{T}^*]^{\wedge 2} = \mathcal{T}^{\wedge 2}$, where $[\mathcal{T}^*]^{\wedge 2} = \mathcal{T}^{\wedge 2}$ due to the identification $\mathcal{T}^* = \mathcal{T}$ provided by the inner product [3, Sect. 1.108].

Following [3, Sect. 16.18], one says that an orthogonal basis $e_1, \ldots, e_n$ of $\mathcal{T}$ *diagonalizes* an algebraic curvature tensor A if $A(e_i, e_j, e_k, e_l) = 0$ whenever $\{i,j\} \ne \{k,l\}$ or, equivalently, if all the nonzero exterior products $e_i \wedge e_j$ are eigenvectors of $A : \mathcal{T}^{\wedge 2} \to \mathcal{T}^{\wedge 2}$. Riemannian manifolds (M,g) whose curvature tensor is diagonalized, at every point x, by some orthogonal basis of T_xM, were first studied by Maillot [12] and referred to by him as having *pure curvature operator*.

An algebraic curvature tensor A and a symmetric bilinear form b on $\mathcal{T}$ will be called *simultaneously diagonalizable* if some orthogonal basis of $\mathcal{T}$ diagonalizes both b (in the usual sense) and A.

For a proof of the next fact, see [9, Theorem 1].

Lemma 3.1. *Let b be a Codazzi tensor on a Riemannian manifold (M,g), and let $v_i \in T_xM$ be eigenvectors of $b(x)$ at a point $x \in M$ correspoding to* *some eigenvalues* λ_i, $i=1,2,3$. *The curvature tensor* $R=R(x)$ *of* (M,g) *at* x *then satisfies the relation* $R(v_1,v_2)v_3=0$ *whenever* $\lambda_1\neq\lambda_3\neq\lambda_2$.

From Lemma 3.1 it is immediate that a Codazzi tensor b on a Riemannian n-manifold (M,g) and the curvature tensor R are simultaneously diagonalizable at each point $x\in M$ where b has n distinct eigenvalues. On the other hand, the condition of simultaneous diagonalizability of b and R at any given point x clearly implies the same condition for b and the Ricci tensor Ric (that is, the bundle endomorphisms of TM corresponding to b and Ric then commute at x) and, consequently, also for b and the Weyl tensor $W=W(x)$, cf. (2.2). However, in dimension 4, even a weaker assumption on b yields the same conclusion [7, proof of Lemma 2]:

Lemma 3.2. *Let* b *be a Codazzi tensor on an oriented Riemannian four-manifold* (M,g). *Then* b *and the Weyl tensor* W *are simultaneously diagonalizable at every point at which* b *is not a multiple of* g.

In an oriented Euclidean 4-space $\mathcal{T}$, the Hodge star $*:\mathcal{T}^{\wedge 2}\to\mathcal{T}^{\wedge 2}$ acting on bivectors may be characterized by $*(e_1\wedge e_2)=e_3\wedge e_4$ whenever $e_1,\dots,e_4$ is a positive-oriented orthonormal basis of $\mathcal{T}$. This makes $*$ an involution, with $\mathcal{T}^{\wedge 2}=\mathcal{L}^+\oplus\mathcal{L}^-$, where $\mathcal{L}^+=\mathrm{Ker}\,(*-\mathrm{Id})$ and $\mathcal{L}^-=\mathrm{Ker}\,(*+\mathrm{Id})$, the spaces of *self-dual* and *anti-self-dual* bivectors, are the eigenspaces of $*$. Any $e_1,\dots,e_4$ as above clearly lead to a basis of $\mathcal{L}^\pm$ formed by the bivectors

$$e_1\wedge e_2\pm e_3\wedge e_4,\quad e_1\wedge e_3\pm e_4\wedge e_2,\quad e_1\wedge e_4\pm e_2\wedge e_3\,\in\,\mathcal{L}^\pm. \tag{3.1}$$

Any algebraic Weyl tensor $A:\mathcal{T}^{\wedge 2}\to\mathcal{T}^{\wedge 2}$ leaves the subspaces $\mathcal{L}^\pm$ invariant, since [16, Theorem 1.3] it commutes with $*$, which results in

(3.2) the restrictions $A^\pm:\mathcal{L}^\pm\to\mathcal{L}^\pm$, both self-adjoint and traceless.

If $\mathcal{T}=T_xM$, where (M,g) is an oriented Riemannian four-manifold and $x\in M$, we denote $\mathcal{L}^\pm$ by $\Lambda^\pm_xM$, which leads to the vector subbundles $\Lambda^\pm M$ of $[TM]^{\wedge 2}=\Lambda^+M\oplus\Lambda^-M$. The restrictions $W^\pm:\Lambda^\pm M\to\Lambda^\pm M$ of the Weyl tensor W of (M,g) satisfy in view of (3.2) the conditions

$$\mathrm{tr}\,W^+\,=\,\mathrm{tr}\,W^-\,=\,0. \tag{3.3}$$

Lemma 3.3. *Given an orthonormal basis of a Euclidean* 4*-space* $\mathcal{T}$ *which diagonalizes an algebraic Weyl tensor* A, *let* $\sigma_{ij}=\sigma_{ji}$ *be its eigenvalues, with* $\sigma_{ij}=A(e_i,e_j,e_i,e_j)$, *where* $i,j\in\{1,2,3,4\}$ *and* $i\neq j$. *If* $\{i,j,k,l\}=\{1,2,3,4\}$, *then, for either fixed orientation,* A^+ *has the same eigenvalues* $\sigma_{ij},\sigma_{ik},\sigma_{il}$ *as* A^-, *while* $\sigma_{ij}=\sigma_{kl}$ *and* $\sigma_{ij}+\sigma_{ik}+\sigma_{il}=0$.

Proof. With standard normalizations, $A(e_i\wedge e_j)=\sigma_{ij}e_i\wedge e_j$ (no summation). If $\{i,j,k,l\}=\{1,2,3,4\}$ and we use the orientation determined by e_i,e_j,e_k,e_l then, by (3.1), $e_i\wedge e_j+e_k\wedge e_l$ lies in $\mathcal{L}^+$, and so does its A-image $\sigma_{ij}e_i\wedge e_j+\sigma_{kl}e_k\wedge e_l$, equal to its own $*$-image $\sigma_{kl}e_i\wedge e_j+\sigma_{ij}e_k\wedge e_l$. This last equality gives $\sigma_{ij}=\sigma_{kl}$, while $\sigma_{ij}+\sigma_{ik}+\sigma_{il}$ vanishes, being the trace of $A(e_i,\,\cdot\,,e_i,\,\cdot\,)$, that is, $a(e_i,e_i)$ for the Ricci contraction a of A.

Now $e_i \wedge e_j \pm e_k \wedge e_l \in \mathcal{L}^{\pm}$ is an eigenvector of both $A^{\pm}$ with the eigenvalue $\sigma_{ij} = \sigma_{kl}$, due to (3.1) with $A(e_i \wedge e_j) = \sigma_{ij} e_i \wedge e_j$. □

Remark 3.4. The mapping that assigns to a positive-oriented orthonormal basis $e_1, \dots, e_4$ of $\mathcal{T}$ the pair (3.1) of positive-oriented orthogonal bases of $\mathcal{L}^+$ and $\mathcal{L}^-$, with all vectors of length $\sqrt{2}$, is a two-fold covering, equivariant relative to the two-fold covering homomorphism $\mathrm{SO}(4) \to \mathrm{SO}(3) \times \mathrm{SO}(3)$, while $\mathcal{L}^{\pm}$ are both canonically oriented [3, Sect. 16.58].

Remark 3.5. Let (M, g) be a Kähler manifold of real dimension four, with the canonical orientation. Its self-dual Weyl tensor W^+ acting on the bundle $\Lambda^+ M$ of self-dual bivectors then has fewer than three distinct eigenvalues at every point [3, formula (16.64)].

Remark 3.6. In an oriented Riemannian four-manifold (M, g) with g conformal to a product $\hat{g}$ of surface metrics, the conclusion of Remark 3.5 applies to both W^+ and W^-. (As a special case, (M, g) might be here a warped product of two orientable Riemannian surfaces, cf. Remark 2.1.) This follows since $\hat{g}$ then is a Kähler metric for two local complex structures compatible with the two mutually opposite orientations.

4. Harmonic curvature

For any Riemannian manifold, the second Bianchi identity implies the equality $\operatorname{div} R = -\,d\,\mathrm{Ric}$, where the Ricci tensor Ric treated as a 1-form valued in 1-forms. (Its coordinate version reads $R_{ijk}{}^p{}_{,p} = R_{ki,j} - R_{kj,i}$.) This leads to equivalence between (0.1) and the Codazzi equation

$$d\,\mathrm{Ric} = 0, \qquad \text{that is,} \quad R_{ki,j} = R_{kj,i}. \tag{4.1}$$

Consequently, $\operatorname{div} R = 0$ if and only if Ric is a Codazzi tensor (Section 2). Contracting the identity $R_{ijk}{}^p{}_{,p} = R_{ki,j} - R_{kj,i}$ with g^{ik} one gets

$$2 \operatorname{div} \mathrm{Ric} = d\mathrm{s}, \quad \text{that is,} \quad 2g^{jk} R_{ij,k} = \mathrm{s}_{,i} \tag{4.2}$$

for any Riemannian metric g. Therefore, from (4.1),

(4.3) whenever $\operatorname{div} R = 0$ the scalar curvature s must be constant.

Since $2(n-1)(n-2) \operatorname{div} W = -(n-3)\, d[2(n-1)\,\mathrm{Ric} - \mathrm{s}g]$ for Riemannian metrics g in dimensions $n \geq 4$, cf. [3, Sect. 16.3], (4.1) and (4.3) imply that

$$\operatorname{div} R = 0 \quad \text{if and only if} \quad \operatorname{div} W = 0 \text{ and } d\mathrm{s} = 0. \tag{4.4}$$

As an obvious consequence of (0.1), a Riemannian product has harmonic curvature if and only if so do both factor manifolds. For a surface metric, harmonic curvature means constant Gaussian curvature, which follows from (4.3). In dimension $n = 3$ one always has $W = 0$, and conformal flatness amounts to the condition $d[2(n-1)\,\mathrm{Ric} - \mathrm{s}g] = 0$, cf. [3, Sect. 1.170], so that, by (4.1), having harmonic curvature is the same as being conformally flat and of constant scalar curvature.

Remark 4.1. A Riemannian product is conformally flat if and only if both factors have constant sectional curvatures K, K' and $K' = -K$ or one factor is of dimension 1. See [19, Section 5], [3, Sect. 1.167].

Remark 4.2. Condition (0.1) for the manifolds (0.3.a) – (0.3.b), or (0.3.c) – (0.3.d), or (0.3.e), follows from (4.1) and (4.4), or from the paragraph following (4.4) or, respectively, from [7, Lemma 3(ii)].

Lemma 4.3. *Let* $\operatorname{div} R = 0$ *for a warped product* $(M, g) = (I \times N,\, dt^2 + Fh)$ *of an open interval* $I \subset \mathrm{I\!R}$ *carrying the standard metric* dt^2 *and a Riemannian* 3*-manifold* (N, h)*, where* $F : I \to (0, \infty)$ *and* dt^2, F, h *are identified with their pullbacks to* M*. Then* (M, g) *is of type* (0.3.c) *or* (0.3.b)*, depending on whether* F *is constant or not.*

Proof. If F is nonconstant, h, being an Einstein metric [6, Lemma 4], has constant sectional curvature. We can now use Remarks 2.1 and 4.1. □

For an oriented Riemannian four-manifold (M, g) having $\operatorname{div} R = 0$, we denote by $\boldsymbol{w}^- \in \{1, 2, 3\}$ the maximum number of distinct eigenvalues of the anti-self-dual Weyl tensor W^- acting on the bundle $\Lambda^- M$ of anti-self-dual bivectors (Section 3). This amounts to the analog of $\boldsymbol{w}$, defined in Section 1, for (M, g) with the opposite orientation, and

(4.5) if $\operatorname{div} R = 0$, one has $\boldsymbol{w}^- = \boldsymbol{w}$ unless g is an Einstein metric.

Namely, (4.1) and Lemma 3.2 then imply simultaneous diagonalizability of Ric and the Weyl tensor W at every point of an open dense subset of M, cf. (1.1), and we can apply Lemma 3.3 to $A = W(x)$ at any point $x \in M$.

Lemma 4.4. *A non-Einstein oriented* 4*-manifold* (M, g) *with* $\operatorname{div} R = 0$ *has* $\boldsymbol{w} = 1$*, or* $\boldsymbol{w} = 2$*, if and only if* g *is conformally flat or, respectively, every point of* M *lies in an open submanifold of type* (0.3.d) – (0.3.e)*.*

Proof. The claim about $\boldsymbol{w} = 1$ trivially follows from (3.3) and (4.5). The warped-product case of Remark 3.6 yields the 'if' part for $\boldsymbol{w} = 2$ by showing that $\boldsymbol{w} \le 2$ (and $\boldsymbol{w} \ne 1$ since the relation $K \ne -c$ in (0.2) precludes conformal flatness via Remark 4.1). Now let $\boldsymbol{w} = 2$. By [7, Prop. 1], $W^+ \ne 0$ everywhere, and we may consider the metric $\hat{g} = |W|^{2/3} g$ on M, conformal to g. Since (4.4) gives $\operatorname{div} W = 0$ whenever $\operatorname{div} R = 0$, (4.5) and [7, Theorem 2] imply that $\hat{g}$, restricted to some neighborhood of any point at which $\mathrm{Ric} \ne \mathrm{s}g/4$, is a product of surface metrics. Due to real-analyticity – see (1.1) – the same must also be the case for points x having $\mathrm{Ric} = \mathrm{s}g/4$, as a simply connected neighborhood U of x contains a nonempty open connected subset carrying a $\hat{g}$-parallel two-dimensional distribution, and the distribution clearly admits an extension to U. Some constant multiple of $\hat{g}$ then has, locally, the form $h \times h^c$ of (0.3.e), with $g = (h \times h^c)/(K + c)^2$. Namely, as our hypotheses give $\nabla \mathrm{Ric} \ne 0$ somewhere, this last claim follows from [7, Theorem 2 and Lemma 3(i)] for points with $\mathrm{Ric} \ne \mathrm{s}g/4$, while real-analyticity of the metrics involved (including the surface metrics h, h^c

defined, locally, at all points of M) allows us to relax the requirement that $\mathrm{Ric} \neq \mathrm{s}g/4$, completing the proof. ☐

Proof of Lemma 1.1. First, (i) leads to (ii). Namely, for (0.3.a) (or (0.3.b), or (0.3.c)) one has $\boldsymbol{r} = 1$ (or $\boldsymbol{w} = 1$, or local reducibility). In (0.3.d) or (0.3.e), the warped-product claim in Remark 3.6 and (1.1) give $\boldsymbol{w} \leq 2$.

Conversely, let (ii) hold. If g is locally reducible, we have (0.3.c) or (0.3.d), cf. the paragraph following (4.4), while the case $\boldsymbol{r} = 1$ yields (0.3.a). Suppose now that g is not locally reducible and $\boldsymbol{r} > 1$. Thus, $\nabla \mathrm{Ric} \neq 0$ somewhere. If $\boldsymbol{r} = 2$, it follows from [6, Theorem 1(i)], via Remark 2.1, that (M, g) has an open submanifold conformal to the Riemannian product of an interval and a three-manifold of constant sectional curvature, making g conformally flat due to Remark 4.1 combined with (1.1), and so (4.3) then yields (0.3.b). This leaves the cases $\boldsymbol{w} = 1$ and $\boldsymbol{w} = 2$, in which, since $\boldsymbol{r} > 1$, Lemma 4.4 gives (0.3.b), (0.3.d) or (0.3.e). ☐

5. The local types (0.3.a) – (0.3.d)

The focus of our discussion does *not* include the local types (0.3.a) – (0.3.d), since each of them is of independent interest and has been studied extensively. We list here some known facts about them, in the compact case.

The simplest examples of compact Einstein four-manifolds are, arguably, spaces of constant curvature. Their complex counterparts ($\mathbb{C}\mathrm{P}^2$, complex 2-tori, and compact quotients of $\mathbb{C}\mathrm{H}^2$) carry well-known Kähler-Einstein metrics, as does any Riemannian product of two oriented surfaces having the same constant Gaussian curvature.

Generally, for a compact complex manifold M to admit a Kähler-Einstein metric, its Lie algebra $\mathfrak{h}(M)$ of holomorphic vector fields must be reductive, as shown by Matsushima [13], while $c_1(M)$ has to be negative, zero or positive. Conversely, when $c_1(M) < 0$, the Calabi conjecture, proved by Aubin [2] and Yau [20], guarantees that M carries a Kähler-Einstein metric, unique up to a factor. Also, Yau's proof [20] of another conjecture made by Calabi implies in particular the existence of Ricci-flat Kähler metrics on K3 surfaces (which, besides the complex 2-tori, are the only Kähler-type compact complex surfaces having $c_1(M) = 0$).

For del Pezzo surfaces (compact complex surfaces M with $c_1(M) > 0$) the analog of the Calabi conjecture is false. In fact, defining M to be the one-point or two-point blow-up of $\mathbb{C}\mathrm{P}^2$, one has $c_1(M) > 0$, yet no Kähler-Einstein metric exists on M, since $\mathfrak{h}(M)$ is not reductive. However, these two surfaces carry conformally-Kähler Einstein metrics: the former was constructed by Page [14], the latter discovered, much more recently, by Chen, LeBrun and Weber [5].

On the other hand, Tian [17] showed that all the remaining del Pezzo surfaces do admit Kähler-Einstein metrics. Besides $\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, these surfaces arise as k-point blow-ups of $\mathbb{C}\mathrm{P}^2$, for $k \in \{3, 4, \ldots, 8\}$.

The class of conformally flat manifolds includes spaces of constant curvature, as well as the Riemannian products listed in Remark 4.1, and is closed under a family of connected-sum operations, cf. [15, p. 479]. As shown by Kuiper [11, Theorem 6], a compact simply connected conformally flat manifold must be conformally diffeomorphic to a standard sphere.

For any compact conformally flat manifold, the additional requirement that the scalar curvature be constant can always be realized, according to Aubin's and Schoen's solutions of the Yamabe problem [1, 15], by a suitable conformal change of the metric. In dimensions $n \in \{3,4\}$ relevant to us, this result is due to Schoen [15].

6. Compact manifolds of the local type (0.3.e)

Following [7, Example 4], we now describe how (0.3.e) leads to compact Riemannian four-manifolds (M,g) having $\operatorname{div} R = 0$, with M obtained as total spaces of flat $\mathrm{SO}(3)$ bundles of 2-spheres over closed surfaces.

Example 6.1. Given $c, r \in (0,\infty)$, a metric h^c of constant Gaussian curvature c on S^2, a closed Riemannian surface (Q,h) with the Gaussian curvature K satisfying (0.2) as well as having $K + c > 0$ everywhere in Q, and a group homomorphism $\varphi : \pi \to \mathrm{SO}(3)$, for $\pi = \pi_1 Q$, we define $(\tilde{M}, \tilde{g})$ to be the manifold obtained when, in (0.3.e), one sets $S = S^2$ and, instead of (Q,h), uses its Riemannian universal covering space $(\tilde{Q}, \tilde{h})$. Then $\tilde{g}$ descends to a metric g on the quotient manifold $M = \tilde{M}/\pi$, the free properly discontinuous action of π on $\tilde{M} = \tilde{Q} \times S^2$ by $\tilde{g}$-isometries [7, Example 4] being given by $\gamma(x,y) = (\gamma(x), [\varphi(\gamma)](y))$ whenever $(\gamma, x, y) \in \pi \times \tilde{Q} \times S^2$, with $\gamma(x)$ corresponding to the action of π on $\tilde{Q}$ via deck transformations.

An obvious question that arises is whether Example 6.1 really gives rise to anything interesting, which here means manifolds not belonging to the local types (0.3.a), (0.3.b), (0.3.c) or (0.3.d). As explained below, the answer is known to be 'yes' for Q homeomorphic to S^2, or $\mathrm{I\!RP}^2$, or a closed orientable surface of genus greater than 1.

According to [7, Proposition 4] (or, [7, Proposition 2]), on the closed orientable surface of any genus $p > 1$ (or, respectively, on $\mathrm{I\!RP}^2$) there exists an uncountable set $\mathcal{E}$ of pairwise nonhomothetic metrics h having the properties required in Example 6.1 (and, in the case of $\mathrm{I\!RP}^2$, rotationally invariant). The set $\mathcal{E}$ is homeomorphic to $\mathrm{I\!R}^{6p-5}$ and contains a codimension-one subset formed by metrics of constant Gaussian curvature; or, respectively, $\mathcal{E}$ is the union of a countably infinite family of subsets homeomorphic to $\mathrm{I\!R}$ which all contain a fixed constant-curvature metric, and are otherwise mutually disjoint.

Any h as above on $\mathrm{I\!RP}^2$ can obviously be pulled back to S^2.

The metrics h just mentioned all give rise, as in Example 6.1, to compact manifolds of the local type (0.3.e) which do not simultaneously represent any of the local types (0.3.a), (0.3.b) or (0.3.c) [7, Theorems 4 and 5]. However, all such nonflat metrics known to exist on the torus or Klein bottle lead to four-manifolds that also belong to type (0.3.c). See [7, Example 5].

7. Outline of proof of Theorem 1.2(a)–(b)

For a fixed oriented Riemannian four-manifold (M,g) with $\operatorname{div} R = 0$, let $\boldsymbol{r} \in \{1,2,3,4\}$ and $\boldsymbol{w}^{\pm} \in \{1,2,3\}$ denote the maximum number of distinct eigenvalues of the Ricci tensor Ric (acting on the tangent bundle TM) and, respectively, of the (anti)self-dual Weyl tensor $W^{\pm}$, acting on the bundle $\Lambda^{\pm}M$ of (anti)self-dual bivectors. See Section 3. For simplicity we write $\boldsymbol{w}$ instead of $\boldsymbol{w}^{+}$. Note that, by (4.5), $\boldsymbol{w}^{-} = \boldsymbol{w}$ unless (M,g) is an Einstein manifold. In view of DeTurck and Goldschmidt's result (1.1), there is a dense open subset of M consisting of *generic* points, meaning

(7.1) points at which the maxima $\boldsymbol{r}$, $\boldsymbol{w}$ are simultaneously attained.

Four possible cases may occur:

(A) $\boldsymbol{r} = 1$: an Einstein manifold – type (0.3.a).
(B) $\boldsymbol{r} > 1$ and $\boldsymbol{w} = 1$: type (0.3.b), as a consequence of Lemma 4.4.
(C) $\boldsymbol{r} > 1$ and $\boldsymbol{w} = 2$: locally, type (0.3.d) or (0.3.e) – see Lemma 4.4.
(D) $\boldsymbol{r} > 1$ and $\boldsymbol{w} = 3$. By Lemma 8.1(i), some neighborhood U of any generic point x admits orthonormal analytic vector fields $e_1, \dots, e_4$ which diagonalize both W (in the sense of Section 3), and Ric.

In case (D), let $\boldsymbol{d} \in \{0,1,2,3,4\}$ denote the maximal number of integers $l \in \{1,2,3,4\}$ for which there exist i,j,k with $\{i,j,k,l\} = \{1,2,3,4\}$ and $g(\nabla_{e_i} e_j, e_k) \neq 0$ somewhere in U. As shown in Lemma 9.2 and Section 11, $\boldsymbol{d} \notin \{2,3,4\}$, so that there are just two possible subcases:

(D1) $\boldsymbol{d} = 1$: according to Theorem 12.3, (M,g) is, locally, of type (0.3.c).
(D0) $\boldsymbol{d} = 0$, that is, $g(\nabla_{e_i} e_j, e_k) = 0$ on U whenever $i \neq j \neq k \neq i$.

Subcase (D0) clearly yields assertions (a) – (b) in Theorem 1.2, under the assumption (equivalent, by Lemma 1.1, to the hypotheses of Theorem 1.2) that (M,g) contains no open submanifolds of types (0.3.a) – (0.3.e). With Lemma 4.4 already established, (a) – (b) *in Theorem* 1.2 *will thus follow from Lemmas* 8.1(i), 9.2*, the claims made in Section* 11*, and Theorem* 12.3.

Remark 7.1. According to [8, the lines following formula (0.3)], there exist complete, locally irreducible, non-Ricci-parallel Riemannian four-manifolds (M,g), which are not conformally flat, having – in addition to some further properties – harmonic curvature and $\boldsymbol{r} = 4$. (Their local-isometry types form a five-dimensional moduli space.) All those manifolds satisfy the hypotheses of our Theorem 1.2. In fact, $\boldsymbol{w}$ must equal 3, as the case $\boldsymbol{w} \leq 2$ would, by Lemma 4.4, lead to the local type (0.3.d) or (0.3.e), making (M,g), locally, a warped product with a two-dimensional fibre and harmonic curvature. Consequently [8, Corollary 1.3], its Ricci tensor would have a multiple eigenvalue at every point, contrary to the relation $\boldsymbol{r} = 4$.

It is not known whether the above class contains any compact manifolds.

8. The Codazzi-Weyl simultaneous diagonalizability

Unlike in Section 2, from now on repeated indices are *not* summed over.

Lemma 8.1. *Let there be given an oriented Riemannian four-manifold* (M,g) *with a Codazzi tensor field* b *on* (M,g) *having* $4b \neq (\mathrm{tr}_g b)g$ *everywhere and an algebraic Weyl tensor field* A *such that* $\mathrm{div}\,A = 0$*, while* A, b *are simultaneously diagonalizable at each point in the sense of Section* 3*, and the bundle morphism* $A^+ : \Lambda^+M \to \Lambda^+M$ *arising as the restriction of* $A : [TM]^{\wedge 2} \to [TM]^{\wedge 2}$ *to self-dual bivectors, cf.* (3.2)*, has three distinct eigenvalues at every point of* M*.*

The above hypotheses imply the following conclusions.

(i) *An orthonormal frame* $e_1, \dots, e_4$ *diagonalizing both* A *and* b *at any* $x \in M$ *is unique up to permuting and/or changing signs of* e_i *and, passing to a finite covering of* M *if necessary, we may assume that* e_i *are* C^∞ *vector fields on* M*.*

(ii) *The directional derivative* D_i *in the direction of* e_i *and the functions* $\Gamma_{ij}^k, \lambda_i, \sigma_{ij}$ *given, with* i,j,k,l *ranging over* $\{1,2,3,4\}$*, by*

$$\Gamma_{ij}^k = g(\nabla_{e_i} e_j, e_k), \quad \lambda_i = b(e_i, e_i) \quad \sigma_{ij} = A(e_i, e_j, e_i, e_j), \tag{8.1}$$

satisfy, whenever $\{i,j,k,l\} = \{1,2,3,4\}$*, the conditions*

$$\begin{array}{rl} \text{a)} & \Gamma_{ij}^k + \Gamma_{ik}^j = 0 \quad \text{and} \quad \sigma_{ij} \neq \sigma_{ik} \neq \sigma_{il} \neq \sigma_{ij}, \\ \text{b)} & \sigma_{ij} = \sigma_{ji} = \sigma_{kl} \quad \text{and} \quad \sigma_{ij} + \sigma_{ik} + \sigma_{il} = 0, \\ \text{c)} & (\lambda_j - \lambda_k)\Gamma_{ij}^k = (\lambda_k - \lambda_i)\Gamma_{jk}^i = (\lambda_i - \lambda_j)\Gamma_{ki}^j, \\ \text{d)} & (\sigma_{ij} - \sigma_{ik})\Gamma_{ij}^k = (\sigma_{jk} - \sigma_{ji})\Gamma_{jk}^i = (\sigma_{ki} - \sigma_{kj})\,\Gamma_{ki}^j, \\ \text{e)} & D_i\lambda_j = (\lambda_j - \lambda_i)\Gamma_{jj}^i, \\ \text{f)} & D_j\sigma_{ij} = (\sigma_{ij} - \sigma_{ik})\Gamma_{kk}^j + (\sigma_{ij} - \sigma_{il})\Gamma_{ll}^j. \end{array} \tag{8.2}$$

(iii) *At any point of* M*, the morphism* $A^+ : \Lambda^+M \to \Lambda^+M$ *has the same eigenvalues, including multiplicities, as* $A^- : \Lambda^-M \to \Lambda^-M$*.*

Proof. Both (iii) and (8.2.a) – (8.2.b) trivially follow from Lemma 3.3 and the fact that e_i are orthonormal, while (i) is immediate from Remark 3.4 along with the essential uniqueness of eigenvectors of $A^\pm$, which itself is due to the assumption about eigenvalues. Equalities (8.2.c) – (8.2.f) amount in turn to the Codazzi equation for b and the relation $\mathrm{div}\,A = 0$. □

Suppose now that an oriented Riemannian four-manifold (M,g) has

$$\mathrm{div}\,R = 0 \ \text{ with } \ \boldsymbol{r} > 1 \ \text{ and } \ \boldsymbol{w} = 3, \tag{8.3}$$

$\boldsymbol{r}$ and $\boldsymbol{w}$ being defined as at the beginning of Section 7 (or Section 1). The hypotheses of Lemma 8.1 are then satisfied by A equal to the Weyl conformal tensor W and the traceless Ricci tensor $b = \mathrm{Ric} - \mathrm{s}g/4$ of g, on any fixed connected component of the dense open set of generic points, defined as in (7.1). This follows from (4.1), (4.4), Lemma 3.2 with $\boldsymbol{r} > 1$, and the equality $\boldsymbol{w} = 3$. (The same would be true if we set $b = \mathrm{Ric}$ instead.) The conclusions of Lemma 8.1 thus hold as well, which makes (4.5) a consequence of Lemma 8.1(iii). Furthermore, (8.3) also implies that, whenever $i \neq j$,

$$\sigma_{ij} \neq 0 \quad \text{everywhere in some dense open subset of } M. \tag{8.4}$$

Otherwise, let $\sigma_{ij} = 0$ on a nonempty open set; since σ_{ij} is an eigenvalue function of both W^+ and W^-, using [3, Proposition 16.72] one then gets $W = 0$ on that set, even though $\boldsymbol{w} = 3$. Finally, setting

$$R_{ijkl} = g(R(e_i, e_j)e_k, e_l) \qquad \text{for} \quad i, j, k, l \in \{1, 2, 3, 4\}, \tag{8.5}$$

we obtain (1.5) from (2.2) and (8.1) for $A = W$ and $b = \mathrm{Ric} - \mathrm{s}g/4$. Also,

$$R_{ijkl} = 0 \text{ unless } \{i, j\} = \{k, l\} \subseteq \{1, 2, 3, 4\} \text{ is a 2-element set,} \tag{8.6}$$

as $R_{ijkl} = 0$ in (2.2) when $\{i, j, k, l\}$ has more than two elements, it being clearly the case for all the other terms of (2.2), where the components refer this time to the frame in Lemma 8.1(i), for $(A, b) = (W, \mathrm{Ric} - \mathrm{s}g/4)$.

Next, we may define the functions S_l and y_l, $l = 1, 2, 3, 4$, by

$$S_l = (\sigma_{ij} - \sigma_{ik})\Gamma_{ij}^k\,, \quad y_l = (\lambda_j - \lambda_k)\Gamma_{ij}^k \text{ with } \{i, j, k, l\} = \{1, 2, 3, 4\}, \tag{8.7}$$

since, due to (8.2.a) – (8.2.d), the definition is correct, namely, S_l and y_l do not depend on the choice of i, j and k.

Remark 8.2. If $\{i, j, k, l\} = \{1, 2, 3, 4\}$, then $\lambda_i, \lambda_j, \lambda_k$ are all equal (or, all distinct) wherever $S_l \neq 0 = y_l$ (or, respectively, $S_l \neq 0 \neq y_l$). In fact, (8.7) with $S_l \neq 0$ gives $\Gamma_{ij}^k \neq 0$, and so, again by (8.7), the relation $y_l = 0$ (or, $y_l \neq 0$) yields $\lambda_i = \lambda_j = \lambda_k$ (or, $\lambda_i \neq \lambda_j \neq \lambda_k \neq \lambda_i$).

Lemma 8.3. *Under the hypotheses of Lemma* 8.1, *let* $l \in \{1, 2, 3, 4\}$ *and* $x \in M$ *be such that* $S_l(x) \neq 0 \neq y_l(x)$ *in* (8.7). *Then the function* α *defined, on a neighborhood of* x, *by*

$$\alpha = S_l/y_l \neq 0, \tag{8.8}$$

satisfies, for any i, j, k *with* $\{i, j, k, l\} = \{1, 2, 3, 4\}$, *the relations*

(i) $\sigma_{ij} - \sigma_{ik} = (\lambda_j - \lambda_k)\alpha$,
(ii) $3\sigma_{ij} = (\lambda_i + \lambda_j - 2\lambda_k)\alpha$,
(iii) $D_k\sigma_{ik} = D_k\sigma_{lj} = [(\lambda_k - \lambda_j)\Gamma_{jj}^k + (\lambda_i - \lambda_j)\Gamma_{ll}^k]\alpha$,
(iv) $D_i\sigma_{jk} = D_i\sigma_{li} = [(\lambda_j - \lambda_i)\Gamma_{jj}^i + (\lambda_k - \lambda_i)\Gamma_{kk}^i]\alpha$,
(v) $D_k\sigma_{jk} = D_k\sigma_{li} = [(\lambda_k - \lambda_i)\Gamma_{ii}^k + (\lambda_j - \lambda_i)\Gamma_{ll}^k]\alpha$,
(vi) $D_i\sigma_{ik} = D_i\sigma_{jl} = [(\lambda_i - \lambda_j)\Gamma_{jj}^i + (\lambda_k - \lambda_j)\Gamma_{ll}^i]\alpha$,
(vii) $D_l\sigma_{ik} = D_l\sigma_{jl} = [(\lambda_k - \lambda_j)\Gamma_{ii}^l + (\lambda_i - \lambda_j)\Gamma_{kk}^l]\alpha$,
(viii) $D_l\sigma_{jk} = D_l\sigma_{il} = [(\lambda_k - \lambda_i)\Gamma_{jj}^l + (\lambda_j - \lambda_i)\Gamma_{kk}^l]\alpha$,
(ix) $D_i\alpha = 2\alpha\Gamma_{ll}^i$,
(x) $D_i\lambda_i = (\lambda_i - \lambda_j)\Gamma_{jj}^i + (\lambda_i - \lambda_k)\Gamma_{kk}^i + (\lambda_j + \lambda_k - 2\lambda_i)\Gamma_{ll}^i$,
(xi) $(\lambda_i - \lambda_j)(D_l\alpha - 2\alpha\Gamma_{kk}^l) = \alpha(2\lambda_i + 2\lambda_j - \mathrm{tr}_g b)(\Gamma_{jj}^l - \Gamma_{ii}^l)$,
(xii) $D_i(\mathrm{tr}_g b) = (\mathrm{tr}_g b - 4\lambda_i)\Gamma_{ll}^i$.

(xiii) $\sigma_{ki} - \sigma_{kj} = (\lambda_i - \lambda_j)\alpha$, *which is the* i, j *version of* (i).

Proof. Fix i, j, k, l with $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and $S_l y_l \neq 0$ at x. Then (i) is obvious from (8.8) and (8.7), while adding (i) to its version obtained by interchanging i, j and using (8.2.b) we get (ii). Next, (8.2.f), with j, k switched, (8.2.b), (i) and (xiii) yield (iii). Similarly, (8.2.f) for l, i rather than i, j, (8.2.b), (i) and (xiii) imply (iv). Invariance of our assumptions under permutations of i, j, k gives (v) and (vi) from (iii) by switching i with j, or i with k, and using (8.2.b). Next, (vii) follows since $D_l\sigma_{ik} = D_l\sigma_{jl} = (\sigma_{jl}-\sigma_{ji})\Gamma_{ii}^l + (\sigma_{jl}-\sigma_{jk})\Gamma_{kk}^l$ due to (8.2.b) and (8.2.f), while $\sigma_{jl}-\sigma_{ji} = \sigma_{ik}-\sigma_{ij} = (\lambda_k-\lambda_j)\alpha$ and $\sigma_{jl}-\sigma_{jk} = \sigma_{ki}-\sigma_{kj} = (\lambda_i-\lambda_j)\alpha$ by (8.2.b), (i) and (xiii). Switching i with j in (vii), we obtain (viii). Applying D_k or D_i to (xiii), we now see that $D_k\sigma_{ik} - D_k\sigma_{jk} - (\lambda_i - \lambda_j)D_k\alpha - \alpha(D_k\lambda_i - D_k\lambda_j) = 0$ as well as $\alpha D_i\lambda_i = D_i\sigma_{ik} - D_i\sigma_{jk} - (\lambda_i - \lambda_j)D_i\alpha + \alpha D_i\lambda_j$. The first of these two equalities becomes $(\lambda_j - \lambda_i)[D_k\alpha - 2\alpha\Gamma_{ll}^k] = 0$ if one replaces the directional derivatives with the corresponding right-hand sides in (iii), (v) and (8.2.e). As $y_l \neq 0$ in (8.7), Remark 8.2 now yields (ix). Analogously, (v), (iv), (ix) and (8.2.e) combined with the second equality give (x) multiplied by α, which implies (x), since $\alpha \neq 0$ by (i) and (8.2.a). Also, D_l applied to (xiii) gives $(\lambda_i - \lambda_j)D_l\alpha = D_l\sigma_{ki} - D_l\sigma_{kj} + [(\lambda_j-\lambda_l)\Gamma_{jj}^l + (\lambda_i - \lambda_l)\Gamma_{ii}^l]\alpha$, where $D_l\lambda_j$ and $D_l\lambda_i$ have been replaced with the expressions provided by (8.2.e). Using (vii) and (viii), we now easily get (xi). Finally, as $\mathrm{tr}_g b = \lambda_i + \lambda_j + \lambda_k + \lambda_l$, (x) and (8.2.e) yield (xii), completing the proof. □

9. Three a priori possible cases

Throughout this section we assume the hypotheses of Lemma 8.1 and use the notations of its conclusions. Let $\boldsymbol{d} \in \{0, 1, 2, 3, 4\}$ be the maximum value in M of the function ζ assigning to $x \in M$ the number $\zeta(x)$ of indices $i \in \{1, 2, 3, 4\}$ for which $S_i(x) \neq 0$ in (8.7). Obviously, $\zeta = \boldsymbol{d}$ on some nonempty open subset of M.

Remark 9.1. The invariant $\boldsymbol{d}$ is still well-defined if, instead of the hypotheses of Lemma 8.1, one assumes (8.3): we then just take the maximum of ζ over the (possibly disconnected) dense open set of generic points, cf. (7.1).

Lemma 9.2. *Under the hypotheses of Lemma* 8.1, $\boldsymbol{d} \in \{0, 1, 2\}$. *In other words, for the functions* S_i *given by* (8.7) *we have* $S_iS_jS_k = 0$ *whenever* i, j, k *are all distinct, that is, at any point of* M *at least two of* S_i *must be zero. Furthermore, with* λ_i *and* y_i *as in* (8.1) *and* (8.7),

$$(9.1) \qquad \begin{aligned} &\text{a)} \quad S_kS_l(\lambda_i + \lambda_j - \lambda_k - \lambda_l) = 0 \quad \text{if} \quad \{i, j, k, l\} = \{1, 2, 3, 4\}, \\ &\text{b)} \quad y_k/S_k = -y_l/S_l \quad \text{whenever} \quad k \neq l \quad \text{and} \quad S_kS_l \neq 0. \end{aligned}$$

Proof. Fix i,j,k,l with $\{i,j,k,l\} = \{1,2,3,4\}$. To prove (9.1), suppose that $S_kS_l \neq 0$ at some $x \in M$. Then, from (8.7) and (8.2.b), at x,

$$\frac{\lambda_i - \lambda_k}{\sigma_{ji}-\sigma_{jk}} = \frac{y_l}{S_l} = \frac{\lambda_i - \lambda_j}{\sigma_{ki}-\sigma_{kj}} = -\frac{\lambda_i - \lambda_j}{\sigma_{li}-\sigma_{lj}} = -\frac{y_k}{S_k} = -\frac{\lambda_j - \lambda_l}{\sigma_{ij}-\sigma_{il}} = \frac{\lambda_l - \lambda_j}{\sigma_{ji}-\sigma_{jk}},$$

the denominators being nonzero by (8.2.a). Now (9.1) follows.

As for the first claim, suppose on the contrary that $\{i,j,k,l\} = \{1,2,3,4\}$ and $S_iS_jS_k \neq 0$ in an open subset U' of M. Then, by (9.1.a), $\lambda_i + \lambda_j = \lambda_k + \lambda_l$, $\lambda_j + \lambda_k = \lambda_i + \lambda_l$, and $\lambda_i + \lambda_k = \lambda_j + \lambda_l$ everywhere in U'. This gives $\lambda_i = \lambda_j = \lambda_k = \lambda_l$, even though Lemma 8.1 assumes that $4b \neq (\mathrm{tr}_g b)g$. $\square$

We now restrict our discussion to any fixed connected component U of the nonempty open subset of M in which $\zeta = \boldsymbol{d}$. Also, let us rearrange S_i (that is, the orthonormal vector fields e_i) so as to make those among S_1, S_2, S_3, S_4 which vanish on U precede those which do not. In view of Lemma 9.2, one of three cases must occur:

$$\text{(9.2)}\qquad \begin{array}{rllll} \text{a)} & \boldsymbol{d} = 2 & \text{and} & S_1 = S_2 = 0 \neq S_3S_4 & \text{everywhere in } U, \\ \text{b)} & \boldsymbol{d} = 1 & \text{and} & S_1 = S_2 = S_3 = 0 \neq S_4 & \text{at each point of } U, \\ \text{c)} & \boldsymbol{d} = 0 & \text{and} & S_1 = S_2 = S_3 = S_4 = 0 & \text{identically in } U. \end{array}$$

In view of (8.2.a) and (8.7), there are the following implications.

$$\text{(9.3)}\qquad \begin{array}{llll} \text{Case (9.2.a):} & \Gamma_{ij}^k = 0 & \text{if} \quad i \neq j \neq k \neq i & \text{and} \quad 3,4 \in \{i,j,k\}. \\ \text{Case (9.2.b):} & \Gamma_{ij}^k = 0 & \text{if} \quad i \neq j \neq k \neq i & \text{and} \quad 4 \in \{i,j,k\}. \\ \text{Case (9.2.c):} & \Gamma_{ij}^k = 0 & \text{whenever} \quad i,j,k & \text{are distinct}. \end{array}$$

It is immediate from (9.1.a) with $S_3S_4 \neq 0$ that

$$\text{(9.4)}\qquad \text{in case (9.2.a),} \quad \mathrm{tr}_g b = 2(\lambda_1 + \lambda_2) = 2(\lambda_3 + \lambda_4).$$

Here are some more consequences of (9.2.a). First, as $S_3S_4 \neq 0$, we have

$$\text{(9.5)}\qquad y_3y_4 \neq 0 \text{ everywhere in } U,$$

or else Remark 8.2 would make three of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ equal to one another and, by (9.4), they would all be equal, contrary to the assumption that $4b \neq (\mathrm{tr}_g b)g$ in Lemma 8.1. We may thus apply Lemma 8.3(i) to both $l \in \{3,4\}$, and then, in view of (8.2.a),

$$\text{(9.6)}\qquad \lambda_k \neq \lambda_l \text{ for } k,l \in \{1,2,3,4\}, \text{ unless } k = l \text{ or } \{k,l\} = \{3,4\}.$$

Also, by (9.1.b) and (8.8), $\alpha_3 = -\alpha_4 = -\alpha$ if $\alpha = \alpha_4$ as in Lemma 8.3, for $l = 4$. Thus, (i) – (x) in Lemma 8.3 remain valid for $l = 3$, provided that α is everywhere replaced with $-\alpha$ and so, from Lemma 8.3(vii),

$$\text{(9.7)}\qquad D_i\alpha = 2\alpha\Gamma_{kk}^i \text{ if } k \in \{3,4\} \text{ and } i \neq k.$$

Finally, due to (9.4), assertion (x) in Lemma 8.3 with $\{i,l\} = \{3,4\}$ becomes $D_i(\lambda_i + \lambda_l) = (\lambda_l - \lambda_i)\Gamma_{ll}^i$ which, by (8.2.e), equals $D_i\lambda_l$, giving

$$\text{(9.8)}\qquad D_3\lambda_3 = D_4\lambda_4 = 0 \qquad \text{everywhere in } U,$$

10. Further consequences of (9.2.a)

The following theorem is used in Section 11 to show that, in the case of harmonic curvature, $\boldsymbol{d} \neq 2$ (cf. Section 7); it only describes the curvature components needed for this purpose. The theorem is valid in a more general situation, which is why, for the sake of completeness and a possible reference, the remaining components are listed in the Appendix.

Theorem 10.1. *Let tensor fields A and b on an oriented Riemannian four-manifold (M,g) satisfy the hypotheses of Lemma 8.1 and (9.2.a). If $\mathrm{tr}_g b = 0$ and $\sigma_{12} \neq 0$ everywhere in M, then the following holds at each point for some function λ, some constant μ, and $\alpha = S_4/y_4$ as in (8.8).*

(a) *The eigenvalue functions λ_i of b and σ_{ij} of $A^{\pm}$ are given by $\lambda_1 = \lambda = -\lambda_2$, $\lambda_3 = \mu = -\lambda_4$ and $3\sigma_{12} = 3\sigma_{34} = -2\mu\alpha$, $3\sigma_{13} = 3\sigma_{24} = (\mu + 3\lambda)\alpha$, $3\sigma_{14} = 3\sigma_{23} = (\mu - 3\lambda)\alpha$.*

(b) *σ_{12}, μ and α are nonzero constants, while $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4 \neq \lambda_1$.*

(c) *$\nabla_{e_3} e_3 = \nabla_{e_3} e_4 = \nabla_{e_4} e_3 = \nabla_{e_4} e_4 = 0$ and $R(e_3,e_4)e_3 = R(e_3,e_4)e_4 = 0$.*

(d) *Whenever $\{i,j\} = \{1,2\}$ and $\{k,l\} = \{3,4\}$, one has*

$$
\begin{aligned}
&R_{ikik} \,=\, \frac{D_k D_k \lambda_i}{\lambda_i - \lambda_k} \,-\, \frac{2(D_k\lambda_i)^2}{(\lambda_i - \lambda_k)^2} \,+\, \frac{y_l^2}{\lambda_i(\lambda_i + \lambda_k)},\\
&(\lambda_i + \lambda_k)R_{ikil} = D_k D_l \lambda_i - \frac{2\lambda_i (D_k\lambda_i) D_l \lambda_i}{\lambda_i^2 - \lambda_k^2} + \frac{(\lambda_i^2 + \lambda_k^2) y_k y_l}{\lambda_i(\lambda_i^2 - \lambda_k^2)},\\
&[e_i, e_j] = -\frac{D_j\lambda_i}{2\lambda_i}\, e_i + \frac{D_i\lambda_i}{2\lambda_i}\, e_j + \frac{2\lambda_i}{\lambda_k^2 - \lambda_i^2}(y_l e_k + y_k e_l),\\
&(\lambda_i - \lambda_k)R_{ijik} = D_j D_k \lambda_i - D_i y_l + (\lambda_i^2 + 2\lambda_i\lambda_k - \lambda_k^2)\frac{y_l D_i\lambda_i - (D_k\lambda_i)D_j\lambda_i}{\lambda_i(\lambda_i^2 - \lambda_k^2)}\\
&(\lambda_k^2 - \lambda_i^2)R_{ijkl} \,=\, 2(y_k D_k \lambda_i \,-\, y_l D_l \lambda_i),
\end{aligned}
$$

with y_k, y_l, R_{ijkl} as in (8.7), (8.5) and (2.1), and $\lambda_i(\lambda_i^2 - \lambda_k^2) \neq 0$.

Proof. As $\mathrm{tr}_g b = 0$, (9.4) means nothing else than

$$\lambda_1 + \lambda_2 \,=\, \lambda_3 + \lambda_4 \,=\, 0. \tag{10.1}$$

Our assumption (9.2.a) leads to (9.5), and so $S_3 S_4 y_3 y_4 \neq 0$ everywhere. We may thus use Lemma 8.3(xii) which, with $\mathrm{tr}_g b = 0$, yields $\lambda_1 \Gamma_{33}^1 = \lambda_2 \Gamma_{33}^2 = \lambda_1 \Gamma_{44}^1 = \lambda_2 \Gamma_{44}^2 = 0$. Since $\lambda_1\lambda_2 \neq 0$ (for otherwise (10.1) would give $\lambda_1 = \lambda_2 = 0$, contradicting (9.6)), we thus get $\Gamma_{33}^i = \Gamma_{44}^i = 0$ for $i = 1,2$. On the other hand, by (9.8), (10.1) and (8.2.e), $0 = -D_k\lambda_k = D_k\lambda_l = (\lambda_l - \lambda_k)\Gamma_{ll}^k$ whenever $\{k,l\} = \{3,4\}$. Therefore, $\Gamma_{44}^3 = \Gamma_{33}^4 = 0$, or else we would have $\lambda_3 = \lambda_4$ and, by (10.1), $\lambda_3 = \lambda_4 = 0$ which, due to Lemma 8.3(ii) and (10.1), would yield $\sigma_{12} = 0$, even though we assumed that $\sigma_{12} \neq 0$. Finally, (9.3) implies that $\Gamma_{34}^i = \Gamma_{43}^i = 0$ for $i = 1,2$. Consequently, by (8.2.a),

$$
\begin{array}{ll}
\text{i)} & \Gamma_{33}^i \,=\, \Gamma_{44}^i \,=\, \Gamma_{34}^i \,=\, \Gamma_{43}^i \,=\, 0 \quad \text{for all} \quad i = 1,2,3,4,\\
\text{ii)} & \Gamma_{ik}^l \,=\, \Gamma_{kl}^i \,=\, \Gamma_{ki}^l \,=\, 0 \quad \text{if} \quad k,l \in \{3,4\}.
\end{array}
\tag{10.2}
$$

In view of (8.1) and (2.1), relations (10.2.i) prove (c). Also, using (8.2.e), (10.1), (10.2) and (9.8), we easily conclude that λ_3 and λ_4 are constant. By (10.1), this yields (a), with the function $\lambda = \lambda_1$ and constant $\mu = \lambda_3$, as one sees evaluating $3\sigma_{12}, 3\sigma_{13}, 3\sigma_{14}$ from Lemma 8.3(ii) applied to i, j, k, l such that $\{i, j, k\} = \{1, 2, 3\}$, while $l = 4$, and then invoking (8.2.b). Now (b) follows from (9.6), (9.7), (10.2) and (a), as $\sigma_{12} \neq 0$.

Next, let us fix i, j, k, l with $\{i, j\} = \{1, 2\}$ and $\{k, l\} = \{3, 4\}$.

Due to (8.1), (10.2.ii) and (8.7), with $\lambda_1 \neq \lambda_2$ by (9.6),

$$\begin{aligned}
&\nabla_{\!e_k} e_i = \Gamma_{ki}^j e_j = (\lambda_i - \lambda_j)^{-1} y_l e_j, \qquad \nabla_{\!e_i} e_k = -\Gamma_{ii}^k e_i + \Gamma_{ik}^j e_j,\\
&g(\nabla_{\!e_i}\nabla_{\!e_k} e_i,\, e_k) = (\lambda_i - \lambda_j)^{-1} y_l \Gamma_{ij}^k = (\lambda_i - \lambda_j)^{-1}(\lambda_j - \lambda_k)^{-1} y_l^2,\\
&g(\nabla_{\!e_i}\nabla_{\!e_k} e_i,\, e_l) = (\lambda_i - \lambda_j)^{-1} y_l \Gamma_{ij}^l = (\lambda_i - \lambda_j)^{-1}(\lambda_j - \lambda_l)^{-1} y_k y_l.
\end{aligned} \tag{10.3}$$

From (8.1), with $\nabla_{\!e_k} e_k = \nabla_{\!e_k} e_l = 0$ in (c), $g(\nabla_{\!e_k}\nabla_{\!e_i} e_i,\, e_k) = D_k[g(\nabla_{\!e_i} e_i,\, e_k)] = D_k \Gamma_{ii}^k$ and $g(\nabla_{\!e_k}\nabla_{\!e_i} e_i,\, e_l) = D_k[g(\nabla_{\!e_i} e_i,\, e_l)] = D_k \Gamma_{ii}^l$. Since (8.2.e) and (9.6) give $\Gamma_{ii}^k = (\lambda_i - \lambda_k)^{-1} D_k \lambda_i$ and $\Gamma_{ii}^l = (\lambda_i - \lambda_l)^{-1} D_l \lambda_i$, we get $g(\nabla_{\!e_k}\nabla_{\!e_i} e_i,\, e_k) = D_k[(\lambda_i - \lambda_k)^{-1} D_k \lambda_i]$ and $g(\nabla_{\!e_k}\nabla_{\!e_i} e_i,\, e_l) = D_k[(\lambda_i - \lambda_l)^{-1} D_l \lambda_i]$. Thus, with λ_k and λ_l both constant, cf. (b),

$$\begin{aligned}
&g(\nabla_{\!e_k}\nabla_{\!e_i} e_i,\, e_k) = (\lambda_i - \lambda_k)^{-1} D_k D_k \lambda_i - (\lambda_i - \lambda_k)^{-2} (D_k \lambda_i)^2,\\
&g(\nabla_{\!e_k}\nabla_{\!e_i} e_i,\, e_l) = (\lambda_i - \lambda_l)^{-1} D_k D_l \lambda_i - (\lambda_i - \lambda_l)^{-2} (D_k \lambda_i) D_l \lambda_i.
\end{aligned} \tag{10.4}$$

By the first line of (10.3), $[e_i, e_k] = \nabla_{\!e_i} e_k - \nabla_{\!e_k} e_i = -\Gamma_{ii}^k e_i + (\Gamma_{ik}^j - \Gamma_{ki}^j) e_j$. Now (8.1) yields $g(\nabla_{[e_i,e_k]} e_i, e_k) = -(\Gamma_{ii}^k)^2 + (\Gamma_{ik}^j - \Gamma_{ki}^j)\Gamma_{ji}^k$ and $g(\nabla_{[e_i,e_k]} e_i, e_l) = -\Gamma_{ii}^k \Gamma_{ii}^l + (\Gamma_{ik}^j - \Gamma_{ki}^j)\Gamma_{ji}^l$. Consequently, from (8.2.e) and (8.7),

$$\begin{aligned}
g(\nabla_{[e_i,e_k]} e_i, e_k) &= -(\lambda_i - \lambda_k)^{-2} (D_k \lambda_i)^2\\
&\quad + (\lambda_k - \lambda_i)^{-1} [(\lambda_i - \lambda_j)^{-1} - (\lambda_k - \lambda_j)^{-1}] y_l^2,\\
g(\nabla_{[e_i,e_k]} e_i, e_l) &= -(\lambda_i - \lambda_k)(\lambda_i - \lambda_l)(D_k \lambda_i) D_l \lambda_i\\
&\quad + (\lambda_l - \lambda_i)^{-1} [(\lambda_i - \lambda_j)^{-1} - (\lambda_k - \lambda_j)^{-1}] y_k y_l.
\end{aligned} \tag{10.5}$$

Relations (10.3) – (10.5), (2.1), (8.5) and (10.1) prove the first two lines of (d). Also, $\nabla_{\!e_j} e_i = -\Gamma_{jj}^i e_j + \Gamma_{ji}^k e_k + \Gamma_{ji}^l e_l$, from (8.1) and (8.2.a), so that

$$\nabla_{\!e_j} e_i = (\lambda_i - \lambda_j)^{-1} (D_i \lambda_j) e_j + (\lambda_i - \lambda_k)^{-1} y_l e_k + (\lambda_i - \lambda_l)^{-1} y_k e_l \tag{10.6}$$

by (8.2.e) and (8.7). The third line of (d) now follows if one switches i, j in (10.6), subtracts, and uses (b) to replace λ_j, λ_l with $-\lambda_i, -\lambda_k$. On the other hand, (8.1) yields $\nabla_{\!e_i} e_i = \Gamma_{ii}^j e_j + \Gamma_{ii}^k e_k + \Gamma_{ii}^l e_l$, and so, from (10.2.ii), $g(\nabla_{\!e_j}\nabla_{\!e_i} e_i,\, e_k) = \Gamma_{ii}^j \Gamma_{jj}^k + D_j \Gamma_{ii}^k$. Similarly, $\nabla_{\!e_j} e_i = -\Gamma_{jj}^i e_j + \Gamma_{ji}^k e_k + \Gamma_{ji}^l e_l$, in the line preceding (10.6), and hence $g(\nabla_{\!e_i}\nabla_{\!e_j} e_i,\, e_k) = -\Gamma_{jj}^i \Gamma_{ij}^k + D_i \Gamma_{ji}^k$, while the third line of (d) and (10.2.ii) give $2\lambda_i g(\nabla_{[e_i,e_j]} e_i, e_k) = -\Gamma_{ii}^k D_j \lambda_i + \Gamma_{ji}^k D_i \lambda_i$. Replacing Γ_{ii}^j, Γ_{jj}^k, Γ_{ii}^k, Γ_{jj}^i, Γ_{ij}^k, Γ_{ji}^k, in the right-hand sides of the three just-derived relations of the form $g(\nabla\ldots, e_k) = \ldots$, with the expressions provided by (8.2.e) and the second equality of (8.7), and noting that, in (a), $\lambda_j = -\lambda_i$ and $\lambda_l = -\lambda_k$ is constatnt, we obtain the fourth line of (d). Finally, $\nabla_{\!e_i}e_k = -\Gamma_{ii}^k e_i + \Gamma_{ik}^j e_j$ in (10.3). Thus, $g(\nabla_{\!e_j}\nabla_{\!e_i}e_k,\, e_l) = -\Gamma_{ii}^k\Gamma_{ji}^l + \Gamma_{ik}^j\Gamma_{jj}^i$ and, with i,j switched, $g(\nabla_{\!e_i}\nabla_{\!e_j}e_k,\, e_l) = -\Gamma_{jj}^k\Gamma_{ij}^l + \Gamma_{jk}^i\Gamma_{ii}^j$, while $g(\nabla_{[e_i,e_j]}e_k, e_l) = 0$ by (10.2.ii). The last line of (d) now follows if one replaces $\Gamma_{ii}^k, \Gamma_{ji}^l, \Gamma_{ik}^j, \Gamma_{jj}^i, \Gamma_{jj}^k, \Gamma_{ij}^l, \Gamma_{jk}^i, \Gamma_{ii}^j$ as before, using (8.2.e) and (8.7). □

11. Exclusion of case (9.2.a) when $\operatorname{div} R = 0$

We now proceed to derive a contradiction from the assumption that (9.2.a) holds and $(A, b) = (W, \mathrm{Ric} - \mathrm{s}g/4)$. We are allowed to invoke Theorem 10.1, since $\sigma_{12} \neq 0$ as a consequence of (8.4).

First, $\mathrm{s} = 8\mu\alpha$, from Theorem 10.1(a) and (1.5) for $(i,j) = (3,4)$, where $R_{ijij} = 0$ in (1.5) by Theorem 10.1(c). Thus, by (1.5) and Theorem 10.1(a),

$$\begin{aligned} &R_{1212} = 0,\ \ R_{1313} = (\alpha + 1/2)(\mu + \lambda), R_{1414} = (\alpha - 1/2)(\mu - \lambda),\\ &R_{2323} = (\alpha + 1/2)(\mu - \lambda), R_{2424} = (\alpha - 1/2)(\mu + \lambda),\ \ R_{3434} = 0. \end{aligned} \tag{11.1}$$

Choosing (i,k) in the first equality of Theorem 10.1(d) to be $(1,3)$, $(2,3)$, $(2,4)$ or, respectively, $(1,4)$, we get

$$\begin{aligned} \frac{D_3D_3\lambda}{\lambda - \mu} - \frac{2(D_3\lambda)^2}{(\lambda - \mu)^2} + \frac{y_4^2}{\lambda(\lambda + \mu)} &= (\alpha + 1/2)(\mu + \lambda),\\ \frac{D_3D_3\lambda}{\lambda + \mu} - \frac{2(D_3\lambda)^2}{(\lambda + \mu)^2} + \frac{y_4^2}{\lambda(\lambda - \mu)} &= (\alpha + 1/2)(\mu - \lambda),\\ \frac{D_4D_4\lambda}{\lambda - \mu} - \frac{2(D_4\lambda)^2}{(\lambda - \mu)^2} + \frac{y_3^2}{\lambda(\lambda + \mu)} &= (\alpha - 1/2)(\mu + \lambda),\\ \frac{D_4D_4\lambda}{\lambda + \mu} - \frac{2(D_4\lambda)^2}{(\lambda + \mu)^2} + \frac{y_3^2}{\lambda(\lambda - \mu)} &= (\alpha - 1/2)(\mu - \lambda). \end{aligned} \tag{11.2}$$

The linear combinations of the first (or, last) two lines of (11.2) with the coefficients $\mu - \lambda$ and $\mu + \lambda$ yield

$$\begin{aligned} 4\mu[(D_3\lambda)^2 + y_4^2] &= -(2\alpha + 1)(\lambda^2 - \mu^2)^2,\\ 4\mu[(D_4\lambda)^2 + y_3^2] &= -(2\alpha - 1)(\lambda^2 - \mu^2)^2. \end{aligned} \tag{11.3}$$

On the other hand, Theorem 10.1(a)-(b) and (9.5) imply that

$$\lambda\mu(\lambda^2 - \mu^2)y_3y_4 \neq 0 \quad \text{everywhere.} \tag{11.4}$$

The last equality of Theorem 10.1(d), for $(i,j,k,l) = (1,2,3,4)$, combined with (8.6) and (11.4), gives $y_4D_4\lambda = y_3D_3\lambda$. Thus, at every point, the vectors $(D_4\lambda, y_3)$ and $(D_3\lambda, y_4)$, both nonzero due to (11.4), are linearly dependent in $\mathbb{R}^2$, and so $(D_4\lambda, y_3) = (qD_3\lambda, qy_4)$ for some function q without zeros. Now, by (11.3), $(2\alpha - 1) = (2\alpha + 1)q^2$ and so, as both sides of both equalities in (11.3) are nonzero, cf. (11.4),

$$q \text{ is a nonzero constant} \tag{11.5}$$

in view of Theorem 10.1(b). However, according to Theorem 10.1(a),

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \,=\, (\lambda, -\lambda, \mu, -\mu). \tag{11.6}$$

Therefore, the fourth equality of Theorem 10.1(d), with the left-hand side equal to 0 by (8.6), applied to fixed i, j with $\{i, j\} = \{1, 2\}$ and (k, l) replaced by $(3, 4)$ or, respectively, $(4, 3)$, reads

$$\begin{aligned} D_j D_3 \lambda_i - D_i y_4 + (\lambda^2 + 2\lambda_i \mu - \mu^2) \frac{y_4 D_i \lambda_i - (D_3 \lambda_i) D_j \lambda_i}{\lambda_i (\lambda^2 - \mu^2)} &= 0, \\ D_j D_4 \lambda_i - D_i y_3 + (\lambda^2 - 2\lambda_i \mu - \mu^2) \frac{y_3 D_i \lambda_i - (D_4 \lambda_i) D_j \lambda_i}{\lambda_i (\lambda^2 - \mu^2)} &= 0. \end{aligned}$$

Replacing the pair $(D_4 \lambda_i, y_3)$ in the second equality above by $(q D_3 \lambda_i, q y_4)$ and subtracting the result from the first equality multiplied by q we obtain

$$4q\lambda_i \mu [y_4 D_i \lambda_i - (D_3 \lambda_i) D_j \lambda_i] \,=\, 0 \quad \text{whenever } \{i, j\} = \{1, 2\}, \tag{11.7}$$

due to constancy of q established in (11.5). We have two matrix equalities

$$\text{a)} \begin{bmatrix} D_3 \lambda & y_4 \\ -y_4 & D_3 \lambda \end{bmatrix} \begin{bmatrix} D_1 \lambda \\ D_2 \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{b)} \begin{bmatrix} y_3 & -y_4 \\ y_4 & y_3 \end{bmatrix} \begin{bmatrix} D_3 \lambda \\ D_4 \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{11.8}$$

with *both determinants nonzero in view of* (11.4). Namely, (11.8.a) follows if one lets (i, j) be $(1, 2)$ or $(2, 1)$ in (11.7), and uses (11.4) – (11.6). By (11.8.a), $D_1 \lambda = D_2 \lambda = 0$. Thus, the third equality of Theorem 10.1(d) and (11.4) give $y_4 D_3 \lambda + y_3 D_4 \lambda = 0$. Now (11.8.b) follows: $y_4 D_4 \lambda = y_3 D_3 \lambda$, as we saw in the second line after (11.4).

Both determinants in (11.8.b) being nonzero, we conclude that λ is constant, which in turn contradicts the second equality of Theorem 10.1(d), since $y_3 y_4 \neq 0 = R_{ikil}$ by (11.4) and (8.6).

12. Case (9.2.b)

In this section we show that four-manifolds with harmonic curvature, having $\boldsymbol{d} = 1$ (see Section 7) are locally of type (0.3.c).

For the Levi-Civita connections ∇ and $\hat{\nabla}$ of conformally related metrics g and $\hat{g} = |\beta|^{1/2} g$ on a manifold, with a nowhere-zero function β, one has

$$\hat{\nabla}_v w \,=\, \nabla_v w \,+\, \frac{d_v \beta}{4\beta}\, w \,+\, \frac{d_w \beta}{4\beta}\, v \,-\, g(v, w) \frac{\nabla \beta}{4\beta}, \tag{12.1}$$

cf. [3, p. 58], v, w, d_v and $\nabla \beta$ being, respectively, any two vector fields, the v-directional derivative, and the g-gradient of β.

Lemma 12.1. *Let* $A, b, (M, g)$ *satisfy the assumptions of Lemma* 8.1 *and* (9.2.b). *If* $\mathrm{tr}_g b = 0$, *then there exists a dense open set* $U \subseteq M$ *such that, on every connected component* U' *of* U, *with some function* $\beta : U' \to \mathrm{I\!R} \smallsetminus \{0\}$, *and* e_i *as in Lemma* 8.1(i),

$$\nabla_{e_4} e_4 = 0, \quad 4\beta \nabla_{e_i} e_4 = -(D_4 \beta) e_i, \quad D_i \beta = 0 \ \text{ for } \ i = 1, 2, 3. \tag{12.2}$$

Proof. It suffices to exhibit $\beta : U' \to \mathbb{R} \smallsetminus \{0\}$ having

$$\text{(12.3)} \qquad \text{i) } \Gamma_{44}^i = D_i\beta = 0, \quad \text{ii) } 4\beta\Gamma_{i4}^i = -D_4\beta \quad \text{whenever } \ i \in \{1,2,3\}.$$

Namely, the last equality in (12.2) follows from (12.3.i), the first two – from (12.3) and (8.1), as $\Gamma_{44}^4 = 0$ by (8.2.a), while $\Gamma_{i4}^j = 0$ whenever $\{i,j,k\} = \{1,2,3\}$ due to (9.2.b) and (9.3).

Let $U \subseteq M$ be the open dense set of points x such that $y_4(x) \neq 0$ or $y_4 = 0$ on a neighborhood of x. For any connected component U' of U, one of the following two conditions holds throughout U' (see Remark 8.2):

(a) $\lambda_1 = \lambda_2 = \lambda_3$ and $y_4 = 0$, for the eigenvalue functions λ_i of b,
(b) $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$ and $y_4 \neq 0$.

We thus need to show that either of (a) – (b) implies (12.3) on U'.

First, in case (a), since $\mathrm{tr}_g b = 0$, setting $\beta = \lambda_i$, $i = 1,2,3$, we get $\lambda_4 = -3\beta$. Also, $\beta \neq 0$ everywhere, for otherwise we would have $b = 0$, even though Lemma 8.1 assumes that $4b \neq (\mathrm{tr}_g b)g$. Thus, from (8.2.e), $D_i\beta = 0$ and $D_i\lambda_4 = 0$. However, (8.2.e) with $D_i\lambda_4 = 0$ gives $-4\beta\Gamma_{44}^i = 0$, $i = 1,2,3$, proving (12.3.i). Finally, (8.2.e) with $i = 4$, any $j \in \{1,2,3\}$, and $\lambda_4 = -3\beta$ reads $D_4\beta = 4\beta\Gamma_{jj}^4$, and (8.2.a) yields (12.3.ii).

Suppose now that condition (b) holds. Note that

$$\text{(12.4)} \qquad \text{i) } R_{1424} = 0, \quad \text{ii) } R_{ij4k} = 0 \quad \text{whenever } \ \{i,j,k\} = \{1,2,3\}$$

with the notation of (8.5). Namely, Lemma 3.1 and (b) give $R(e_i, e_j)e_k = 0$, as well as $R(e_1, e_4)e_2 = 0$ (or, $R(e_2, e_4)e_1 = 0$) at points where $\lambda_2 \neq \lambda_4$ (or, respectively, $\lambda_1 \neq \lambda_4$), so that the usual symmetries of R imply (12.4).

Let us now fix $x \in U'$. By (b), at least two of $\lambda_1(x), \lambda_2(x), \lambda_3(x)$ are nonzero. Rearranging e_1, e_2, e_3, we may assume that $\lambda_1\lambda_2 \neq 0$ at x. Then, from Lemma 8.3(xii) with $l = 4$ and $\mathrm{tr}_g b = 0$, on a neighborhood of x,

$$\text{(12.5)} \qquad \Gamma_{44}^1 \,=\, \Gamma_{44}^2 \,=\, 0.$$

In view of (9.2.b) and (9.3), relations (8.1), (8.2.a) and (12.5) yield

$$\text{(12.6)} \qquad \begin{aligned} &\nabla_{\!e_i} e_j \,=\, \Gamma_{ij}^i e_i \,+\, \Gamma_{ij}^k e_k \quad \text{if } \ \{i,j,k\} = \{1,2,3\}, \\ &\nabla_{\!e_4} e_1 \,=\, \nabla_{\!e_4} e_2 \,=\, 0, \qquad \nabla_{\!e_4} e_3 \,=\, -\Gamma_{44}^3 e_4 \end{aligned}$$

as $g(\nabla_{\!e_4} e_i, e_4) = 0$ for $i = 1,2$ by (12.5). From (8.1), (8.2.a) and (9.3),

$$\text{(12.7)} \qquad \text{i) } \nabla_{\!e_i} e_4 = -\Gamma_i e_i \ \text{ for } \ i = 1,2,3, \quad \text{where ii) } \Gamma_i = \Gamma_{ii}^4 = -\Gamma_{i4}^i.$$

By (12.6), we thus have $[e_4, e_1] = \Gamma_1 e_1$. and so $g(\nabla_{[e_4,e_1]} e_2, e_4) = 0$. Also, from (12.6), $g(\nabla_{\!e_4}\nabla_{\!e_1} e_2, e_4) = -\Gamma_{12}^3\Gamma_{44}^3$ and $\nabla_{\!e_1}\nabla_{\!e_4} e_2 = 0$. Combining the last three equalities with (2.1) and noting that $y_4 = (\lambda_2 - \lambda_3)\,\Gamma_{12}^3$ in (8.7), we now get $(\lambda_3 - \lambda_2) g(R(e_1, e_4)e_2, e_4) = y_4\Gamma_{44}^3$. Thus, by (12.4) and (b), $\Gamma_{44}^3 = 0$, which, along with (12.5), proves the first part of (12.3.i), while the second part then follows from Lemma 8.3(ix) for $l = 4$ if we set, this time, $\beta = \alpha^2$, where α is given by (8.8) with $l = 4$. In view of (12.7.ii), since $\mathrm{tr}_g b = 0$, Lemma 8.3(xi) can be rewritten as

$$(\lambda_i - \lambda_j)(D_4\alpha - 2\alpha\Gamma_k) = 2\alpha(\lambda_i + \lambda_j)(\Gamma_j - \Gamma_i) \tag{12.8}$$

for i, j, k with $\{i, j, k\} = \{1, 2, 3\}$. Furthermore, R_{ij4k} evaluated from (2.1), (12.6), (12.7) and the resulting relation $[e_i, e_j] = \Gamma_{ij}^i e_i - \Gamma_{ji}^j e_j + (\Gamma_{ij}^k - \Gamma_{ji}^k)e_k$, valid if $\{i, j, k\} = \{1, 2, 3\}$, equals $(\Gamma_j - \Gamma_k)\Gamma_{ij}^k - (\Gamma_i - \Gamma_k)\Gamma_{ji}^k$. As $R_{ij4k} = 0$ in (12.4), this means that $y_4[(\lambda_i - \lambda_k)^{-1}(\Gamma_i - \Gamma_k) - (\lambda_j - \lambda_k)^{-1}(\Gamma_j - \Gamma_k)] = 0$ whenever $\{i, j, k\} = \{1, 2, 3\}$, cf. (8.7). Thus, as $y_4 \neq 0$ in (b). for some function ψ not depending on the choice of $i, j \in \{1, 2, 3\}$, one has

$$\Gamma_i - \Gamma_j = (\lambda_i - \lambda_j)\psi. \tag{12.9}$$

In view of (b) and (12.9), relation (12.8) now becomes

$$\Gamma_k = \frac{D_4\alpha}{2\alpha} + (\lambda_i + \lambda_j)\psi \quad \text{if} \quad \{i, j, k\} = \{1, 2, 3\}.$$

Hence $\Gamma_k - \Gamma_j = (\lambda_j - \lambda_k)\psi$ is the *opposite* of the value in (12.9), implying, via (b), that $\psi = 0$. The last displayed equality now yields $D_4\alpha = 2\alpha\Gamma_i$ for $i = 1, 2, 3$ and, as $\beta = \alpha^2$, (12.3.ii) follows from (12.7.ii). □

Lemma 12.2. *Under the hypotheses of Lemma* 12.1*, for the function* β *in* (12.2)*,* (M, g) *is locally isometric to a warped product* $(I\times N,\, dt^2 + |\beta|^{-1/2}h)$*, with the notation of Lemma* 4.3*, where* β *is constant in the* N *direction.*

In fact, (12.1) and (12.2) give $\hat\nabla\hat e_4 = 0$, where $\hat e_4 = |\beta|^{-1/4}e_4$ and $\hat\nabla$ is the Levi-Civita connection of the metric $\hat g = |\beta|^{1/2}g$, conformal to g, Thus, $(M, \hat g)$ is locally isometric to a Riemannian product of an interval I and a Riemannian 3-manifold (N, h). Since $D_i\beta = 0$ for $i = 1, 2, 3$ by (12.2), our claim follows.

Theorem 12.3. *Given an oriented non-Einstein Riemannian four-manifold* (M, g) *with* $\operatorname{div} R = 0$ *such that* $W^+ : \Lambda^+M \to \Lambda^+M$ *has three distinct eigenvalues at some point of* M*, let* $\boldsymbol{d}$ *be the invariant mentioned in Remark* 9.1*. If* $\boldsymbol{d} = 1$*, then* (M, g) *is locally of type* (0.3.c)*.*

This is immediate from Lemmas 12.2 and 4.3, as the assumption on W^+ precludes conformal flatness of (M, g).

13. Case (9.2.c).

This section discusses, in some detail, four-manifolds with harmonic curvature such that $\boldsymbol{d} = 0$ (notation of Section 7).

Lemma 13.1. *Let* A, b *and* (M, g) *satisfy the hypotheses of Lemma* 8.1 *and* (9.2.c)*, with* $e_i, \Gamma_{ij}^i, \lambda_i, \sigma_{ij}$ *as in Lemma* 8.1(i) *and* (8.1)*. Setting*

$$F_{ji} = \Gamma_{ij}^i \text{ and } H_{ji} = F_{kl}F_{li} + F_{lk}F_{ki} - F_{ki}F_{li} \quad \text{if} \quad \{i, j, k, l\} = \{1, 2, 3, 4\}, \tag{13.1}$$

one has the following relations, valid as long as $\{i,j,k,l\}=\{1,2,3,4\}$.

$$(13.2)\quad \begin{array}{rl} \text{a)} & \nabla_{\!e_i}e_i\,=\,-\,F_{ji}e_j\,-\,F_{ki}e_k\,-\,F_{li}e_l\,, \qquad \nabla_{\!e_i}e_j\,=\,F_{ji}e_i\,,\\ \text{b)} & D_i\lambda_j\,=\,(\lambda_i\,-\,\lambda_j)F_{ij}\,,\\ \text{c)} & D_k\sigma_{ij}\,=\,(\sigma_{kj}-\sigma_{ij})F_{ki}\,+\,(\sigma_{ki}-\sigma_{ij})F_{kj}\,,\\ \text{d)} & [e_i,e_j]\,=\,F_{ji}e_i\,-\,F_{ij}e_j\,,\\ \text{e)} & D_iF_{ij}\,+\,D_jF_{ji}\,+\,F_{ij}^2\,+\,F_{ji}^2\,+\,F_{ki}F_{kj}\,+\,F_{li}F_{lj}\,=\,-R_{ijij}\,,\\ \text{f)} & D_iF_{jk}\,-\,(F_{ji}-F_{jk})F_{ik}\,=\,R_{kijk}\,. \end{array}$$

In the case where $\mathrm{tr}_g b$ *is constant and* $\{i,j,k,l\}=\{1,2,3,4\}$,

$$(13.3)\qquad D_k\lambda_k\,=\,(\lambda_l\,-\,\lambda_k)F_{kl}\,+\,(\lambda_i\,-\,\lambda_k)F_{ki}\,+\,(\lambda_j\,-\,\lambda_k)F_{kj}\,.$$

If $A=W$ *and* $b=\mathrm{Ric}\,-\,\mathrm{s}g/4$ *then, whenever* $\{i,j,k,l\}=\{1,2,3,4\}$,

$$(13.4)\quad \begin{array}{rl} \text{i)} & D_iF_{jk}\,=\,(F_{ji}-F_{jk})F_{ik}\,,\\ \text{ii)} & (\sigma_{ki}\,-\,\sigma_{li})(D_kF_{lk}\,-\,D_lF_{kl})\,=\,3(H_{ji}\,-\,H_{ij})\sigma_{ij}\,,\\ \text{iii)} & (\lambda_k\,-\,\lambda_l)(D_kF_{lk}\,-\,D_lF_{kl})\\ & \qquad=\,(\lambda_k\,+\,\lambda_l\,-\,2\lambda_i)H_{ji}\,+\,(\lambda_k\,+\,\lambda_l\,-\,2\lambda_j)H_{ij}\,,\\ \text{iv)} & D_iD_kF_{lk}\,+\,2F_{ik}D_kF_{lk}\,=\,(F_{li}-F_{lk})D_kF_{ik}\,,\\ \text{v)} & D_iD_kF_{kl}+(F_{ik}+F_{il})D_kF_{kl}=F_{il}D_kF_{ki}+(F_{ki}-F_{kl})H_{jl}\,. \end{array}$$

Proof. Assertions (13.2.a) – (13.2.c) are obvious from (8.1) and (8.2.a) with (9.2.c) and (9.3) or, respectively, (8.2.e) – (8.2.f), while (13.2.d) is immediate from (13.2.a). Evaluating R_{ijij} and R_{kijk} from (8.5), (2.1) and (13.2.d), we easily obtain (13.2.e) – (13.2.f).

If $\mathrm{tr}_g b=\lambda_i+\lambda_j+\lambda_k+\lambda_l$ is constant, (13.2.b) implies (13.3).

The assumptions made in the line preceding (13.4) yield (13.4.i) as a consequence of (13.2.f), since Lemma 3.2 applied to $b=\mathrm{Ric}\,-\,\mathrm{s}g/4$ gives $R_{kijk}=0$. Relations (13.4.ii) – (13.4.iii), or (13.4.iv) – (13.4.v), then follow in turn from the equalities $D_iD_j\sigma_{il}-D_jD_i\sigma_{il}=D_w\sigma_{il}$ and $D_kD_l\lambda_k-D_lD_k\lambda_k=D_w\lambda_k$, where $w=[e_k,e_l]$ (or, respectively, $D_iD_kF_{lk}-D_kD_iF_{lk}=D_wF_{lk}$ and $D_iD_kF_{kl}-D_kD_iF_{kl}=D_wF_{kl}$, where $w=[e_i,e_k]$), combined with (13.2.d), (13.2.b), (13.4.i) and (13.3). □

Proof of Theorem 1.2(a)-(b). Due to (1.2), $\boldsymbol{w}=3$ and $\boldsymbol{r}\in\{3,4\}$. The assumptions of Lemma 8.1 thus hold for $(A,b)=(W,\mathrm{Ric}-\mathrm{s}g/4)$. Lemma 9.2 now gives $\boldsymbol{d}\in\{0,1,2\}$, and so $\boldsymbol{d}=0$, the cases $\boldsymbol{d}=2$ and $\boldsymbol{d}=1$ being excluded by the argument in Section 11 and, respectively, Theorem 12.3. Assertions (a) and (b) of Theorem 1.2 are now immediate from (13.2.d) and the choice of e_i in Lemma 8.1(i). □

Proof of Theorem 1.3. The first displayed equation is obvious from (8.2.b) and the definition of λ_i. Next, adding $\lambda_l-\lambda_k$ times (13.4.ii) to (13.4.iii) multiplied by $\sigma_{ki}-\sigma_{li}$, we obtain $0=2(H_{ij}Z_{klj}-H_{ji}Z_{kli})$, where $2Z_{klj}=$

$3(\lambda_k - \lambda_l)\sigma_{ij} + (\lambda_k + \lambda_l - 2\lambda_j)(\sigma_{ki} - \sigma_{li})$ and

$$\begin{aligned} 2Z_{kli} &= 3(\lambda_k - \lambda_l)\sigma_{ji} + (\lambda_k + \lambda_l - 2\lambda_i)(\sigma_{kj} - \sigma_{lj}) \\ &= 3(\lambda_k - \lambda_l)\sigma_{ij} + (\lambda_k + \lambda_l - 2\lambda_i)(\sigma_{li} - \sigma_{ki}). \end{aligned}$$

(By (8.2.b), $(\sigma_{ji}, \sigma_{kj}, \sigma_{lj}) = (\sigma_{ij}, \sigma_{li}, \sigma_{ki})$.) Thus, as $\sigma_{ij} = \sigma_{kl}$ and $\sigma_{ki} = \sigma_{ik}$ in (8.2.b), the last displayed line gives $2Z_{kli} = 3(\lambda_k - \lambda_l)\sigma_{kl} + (\lambda_k + \lambda_l - 2\lambda_i)\sigma_{li} + (2\lambda_i - \lambda_k - \lambda_l)\sigma_{ik}$, that is, $2Z_{kli} = 2(\lambda_k - \lambda_l)\sigma_{kl} + (\lambda_k - \lambda_l)\sigma_{kl} + 2(\lambda_l - \lambda_i)\sigma_{li} + (\lambda_k - \lambda_l)\sigma_{li} + 2(\lambda_i - \lambda_k)\sigma_{ik} + (\lambda_k - \lambda_l)\sigma_{ik}$. Due to (8.2.b), or the definition of Z_j in Theorem 1.3, the second, fourth and sixth (or, first, third and fifth) terms in the last six-term sum add up to 0 or, respectively, to $2Z_j$, if (i, j, k, l), and hence (k, l, i, j), is an even permutation of $(1, 2, 3, 4)$. Thus, $Z_{kli} = Z_j$ while, the permutation (k, l, j, i) being odd, $Z_{klj} = -Z_i$, and so the equality $H_{ij}Z_{klj} = H_{ji}Z_{kli}$ obtained above amounts to (1.6). Furthermore, $Z_i + Z_j + Z_k + Z_l = 0$ if (i, j, k, l) is an even permutation of $(1, 2, 3, 4)$, since the term $(\lambda_i - \lambda_j)\sigma_{ij}$ in $Z_l = (\lambda_i - \lambda_j)\sigma_{ij} + (\lambda_j - \lambda_k)\sigma_{jk} + (\lambda_k - \lambda_i)\sigma_{ki}$ gets cancelled by $(\lambda_j - \lambda_i)\sigma_{ji}$ in Z_k, as a consequence of (8.2.b) and evenness of the permutation (j, i, l, k). Therefore,

$$Z_1 + Z_2 + Z_3 + Z_4 = 0 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \tag{13.5}$$

the second relation in (13.5) being a part of the already-established first displayed equation in Theorem 1.3. From (1.6) and (13.5) we get

$$\begin{bmatrix} Z_1 & Z_2 & Z_3 & Z_4 \end{bmatrix} \boldsymbol{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{13.6}$$

where $\boldsymbol{H}$ denotes the 4×7 matrix in (1.7). With λ_4 replaced by $-\lambda_1 - \lambda_2 - \lambda_3$, cf. (13.5), the definition of Z_j gives

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} \sigma_{24} - \sigma_{43} & 2\sigma_{24} - \sigma_{32} - \sigma_{43} & \sigma_{24} + \sigma_{32} - 2\sigma_{43} \\ \sigma_{13} + \sigma_{34} - 2\sigma_{41} & \sigma_{34} - \sigma_{41} & 2\sigma_{34} - \sigma_{13} - \sigma_{41} \\ 2\sigma_{14} - \sigma_{21} - \sigma_{42} & \sigma_{21} + \sigma_{14} - 2\sigma_{42} & \sigma_{14} - \sigma_{42} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}. \tag{13.7}$$

Due to (8.2.b), the 3×3 matrix appearing in (13.7) equals

$$\begin{bmatrix} \sigma_{13} - \sigma_{12} & 3\sigma_{13} & -3\sigma_{12} \\ -3\sigma_{14} & \sigma_{12} - \sigma_{14} & 3\sigma_{12} \\ 3\sigma_{14} & -3\sigma_{13} & \sigma_{14} - \sigma_{13} \end{bmatrix},$$

and so it has the determinant $-8(\sigma_{13} - \sigma_{12})(\sigma_{12} - \sigma_{14})(\sigma_{14} - \sigma_{13})$, nonzero according to (8.2.a). while (1.2) and (13.5) give $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$. Thus, $(Z_1, Z_2, Z_3) \neq (0, 0, 0)$ in (13.7), and (1.7) follows from (13.6). □

Remark 13.2. Let the tangent bundle TM of an n-dimensional manifold M be trivialized by vector fields $e_1, \dots, e_n$ satisfying the Lie-bracket relations (0.5). Then, for the Levi-Civita connection ∇ of the metric g on M such that $e_1, \dots, e_n$ are g-orthonormal,

$$\nabla_{e_i} e_j = F_{ji} e_i \quad \text{if } i \neq j, \qquad \nabla_{e_i} e_i = -\sum_{k \neq i} F_{ki} e_k. \tag{13.8}$$

In fact, the connection *defined* by (13.8) is torsion-free and makes g parallel.

Remark 13.3. The assumptions of Remark 13.2 will still hold if $e_1, \dots, e_n$ are replaced by $\phi_1 e_1, \dots, \phi_n e_n$, for any functions ϕ_i without zeros.

Locally, such ϕ_i may then be chosen so that $\phi_i e_i$ commute, as one sees setting $\phi_i e_i = \partial_i$, where ∂_i are the coordinate vector fields for local coordinates x^i, with each x^i constant along the integrable distribution $\mathrm{Span}\{e_j : j \neq i\}$.

Remark 13.4. Let the objects appearing in Remark 13.2 also have the property that $[e_i, e_k] = 0$ for some fixed $m \in \{1, \dots, n-1\}$ and all i, k with $i \le m < k$. Then the distributions $\mathcal{V} = \mathrm{Span}\{e_1, \dots, e_m\}$ and $\mathcal{H} = \mathrm{Span}\{e_{m+1}, \dots, e_n\}$ are g-parallel. Namely, whenever $i \le m < k$, Remark 13.2 with $F_{ik} = F_{ki} = 0$ implies that e_k is g-parallel along e_i, and vice versa. Thus, $\mathcal{V}$ and $\mathcal{H}$ are g-parallel along each other. Hence $\mathcal{V} = \mathcal{H}^\perp$ is g-parallel along $\mathcal{V}$ as well.

14. Proof of Theorem 1.2(c): part one

We will prove Theorem 1.2(c) by assuming its negation which, by (1.2), means that $\boldsymbol{r} = 3$, and – at the end of Section 17 – obtaining a contradiction.

Throughout this and the next three sections (M, g) is a fixed oriented Riemannian four-manifold with $\operatorname{div} R = 0$, belonging to class (D0) of Section 7 and having $\boldsymbol{r} = 3$, while U denotes the set of all generic points (Section 7) at which $\sigma_{ij}\sigma_{ik}\sigma_{il} \neq 0$, that is, the eigenvalues of W^+ and W^- are all nonzero. The indices i, j, k, l are always assumed to satisfy the condition

$$\{i, j\} = \{1, 2\}, \qquad \{k, l\} = \{3, 4\}. \tag{14.1}$$

By (8.4) and (1.1), U is an open dense subset of M. We use the notation of Lemma 13.1, for $A = W$ and $b = \mathrm{Ric} - \mathrm{s}g/4$, cf. the lines following (8.3), so that, without loss of generality, on every connected component U' of U, for some function λ,

$$\begin{aligned} &\text{a)} \quad \lambda_1 = \lambda_2 = \lambda \neq \lambda_3 \neq \lambda_4 \neq \lambda, \\ &\text{b)} \quad \lambda_3 + \lambda_4 = -2\lambda \end{aligned} \tag{14.2}$$

everywhere in U'. Setting $\sigma = \sigma_{12}$, we also define the function

$$G_k = (\lambda_k - \lambda)^{-1} D_k \lambda \quad \text{for } k \in \{3, 4\}, \tag{14.3}$$

and a metric $\hat{g}$ on U' by requiring $\hat{g}$-orthonormality of $\hat{e}_1, \dots, \hat{e}_4$, where

$$\hat{e}_i = \sigma^{-1/3} e_i \ \text{ if } \ i = 1, 2 \ \text{ and } \ \hat{e}_k = (\lambda_k - \lambda)^{-1} e_k \ \text{ if } \ k = 3, 4. \tag{14.4}$$

Assuming (14.1), we now obtain from (13.2.d), as in Remark 13.3,

$$
(14.5)\qquad
\begin{array}{rl}
\text{i)} & [\hat{e}_i,\hat{e}_j]=\hat{F}_{ji}\hat{e}_i-\hat{F}_{ij}\hat{e}_j, \qquad [\hat{e}_k,\hat{e}_l]=\hat{F}_{lk}\hat{e}_k-\hat{F}_{kl}\hat{e}_l,\\
\text{ii)} & [\hat{e}_i,\hat{e}_k]=\hat{F}_{ki}\hat{e}_i-\hat{F}_{ik}\hat{e}_k, \qquad \text{where}\\
\text{iii)} & \hat{F}_{ik}=\sigma^{-1/3}(F_{ik}-D_i\log|\lambda_k-\lambda|^{-1}),\\
\text{iv)} & \hat{F}_{ki}=(\lambda_k-\lambda)^{-1}(F_{ki}-D_k\log|\sigma|^{-1/3}),\\
\text{v)} & \hat{F}_{ij}=\sigma^{-1/3}(F_{ij}-D_i\log|\sigma|^{-1/3}),\\
\text{vi)} & \hat{F}_{kl}=(\lambda_k-\lambda)^{-1}(F_{kl}-D_k\log|\lambda_l-\lambda|^{-1}).
\end{array}
$$

Lemma 14.1. *Under the hypotheses* (14.1) – (14.5)*, for* $G_k,\hat{g}$ *as above,*

(a) $D_i\lambda = D_iG_k = 0$ *and* $H_{ji}=H_{ij}=F_{lk}G_k+F_{kl}G_l-G_kG_l$,
(b) $F_{ki} = G_k = D_k\log|\sigma|^{-1/3}$ *and* $-2F_{kl} = D_k\log|\sigma^{-1/3}(\sigma_{il}-\sigma_{jl})|$,
(c) $D_kF_{lk}=D_lF_{kl}$, $(F_{lk}G_k+F_{kl}G_l-G_kG_l)\lambda=0$*, and* $H_{lj}=F_{ik}G_k$,
(d) $F_{ik} = D_i\log|\lambda_k-\lambda|^{-1}$ *and* $H_{lj}-H_{kl}=(F_{ik}-F_{il})(G_k-F_{kl})$,
(e) $\hat{e}_k$ *is* $\hat{g}$*-parallel along* $\hat{e}_i$*, and vice versa, so that* $[\hat{e}_i,\hat{e}_k]=0$,
(f) $\hat{g}$ *is, locally, a Riemannian product of two surface metrics, with the factor distributions* $\mathcal{V}=\mathrm{Span}\{e_1,e_2\}$ *and* $\mathcal{H}=\mathrm{Span}\{e_3,e_4\}$,
(g) $\hat{F}_{kl}=(\lambda_l-\lambda)^{-1}(F_{kl}-G_k)$ *and* $D_i\hat{F}_{kl}=D_k\hat{F}_{ij}=0$,
(h) $D_i\lambda_k$ *is nonzero on a dense open set, and* $(D_k\lambda+4\lambda F_{kl})G_l=0$.

Proof. By (13.2.b) and (14.2), $D_i\lambda = D_i\lambda_j = (\lambda-\lambda)F_{ij} = 0$ and $D_k\lambda = D_k\lambda_i = (\lambda_k-\lambda)F_{ki}$. Therefore $D_i\lambda=0$, while $F_{ki}=G_k$ does not depend on $i\in\{1,2\}$, so that (13.4.i) yields $D_iG_k = D_iF_{kj} = (G_k-G_k)F_{ij} = 0$. Combined with (13.1), this proves (a) and the first equality in (b). Since, by (8.2.b), $\sigma_{kj}+\sigma_{ki} = -\sigma_{kl} = -\sigma_{ij} = -\sigma$, (13.2.c) similarly gives $D_k\sigma = D_k\sigma_{ij} = (\sigma_{kj}-\sigma_{ij})G_k + (\sigma_{ki}-\sigma_{ij})G_k = -3\sigma G_k$ as well as $D_k(\sigma_{il}-\sigma_{jl}) = (\sigma_{jl}-\sigma_{il})(G_k+2F_{kl})$, and the remainder of (b) follows. Now (13.4.iii) reads $(\lambda_k-\lambda_l)(D_kF_{lk}-D_lF_{kl}) = 8\lambda(G_kG_l-F_{lk}G_k-F_{kl}G_l)$, cf. (14.2). Thus, (c) is clear from (13.4.ii), (a) and (13.1). By (a), (14.2) and (13.2.b), $D_i(\lambda_k-\lambda) = D_i\lambda_k = (\lambda-\lambda_k)F_{ik}$. This, along with (13.1), (b) and (c), implies (d). Next, (14.5.i) – (14.5.ii) and Remark 13.4 lead to (e) – (f), since $\hat{F}_{ik} = \hat{F}_{ki} = 0$ from (14.5.iii) – (14.5.iv), (b) and (d).

Also, by (e), $\hat{e}_i$ commutes with $[\hat{e}_k,\hat{e}_l]$ and $\hat{e}_k$ with $[\hat{e}_i,\hat{e}_j]$, which in view of (14.5.i) yields $D_i\hat{F}_{lk} = D_k\hat{F}_{ij} = 0$, and consequently (g): from (14.2), (13.2.b) and (14.3), we get $(\lambda_l-\lambda)D_k\log|\lambda_l-\lambda|^{-1} = D_k(\lambda-\lambda_l) = (\lambda_k-\lambda)G_k-(\lambda_k-\lambda_l)F_{kl}$, and thus $(\lambda_k-\lambda)(\lambda_l-\lambda)\hat{F}_{kl}$ equals $(\lambda_k-\lambda)(F_{kl}-G_k)$ according to (14.5.vi), while $\lambda_k\neq\lambda$ (see (14.2.a)).

To prove the first claim in (h), we may suppose that, on the contrary, $D_i\lambda_k=0$ for both $k=3,4$ and both $i=1,2$ since, according to (14.2.b) and (a), $D_i\lambda_3=0$ if and only if $D_i\lambda_4=0$. In view of (a) and (d), $F_{ik}=0$, so that (8.2.b) and (13.2.c) give $D_i\sigma = D_i\sigma_{ij} = D_i\sigma_{kl} = 0$. Thus, from (a), $D_i[(\lambda_k-\lambda)\sigma^{-1/3}]=0$ and, as $\sigma^{-1/3}e_k = (\lambda_k-\lambda)\sigma^{-1/3}\hat{e}_k$, the relation $[\hat{e}_i,\hat{e}_k]=0$ in (e) yields $[\hat{e}_i,\sigma^{-1/3}e_k]=0$, that is, $[\sigma^{-1/3}e_i,\sigma^{-1/3}e_k]=0$.

Applying Remarks 13.3 and 13.4 to the vector fields $\sigma^{-1/3}e_i, \sigma^{-1/3}e_k$ and to the metric $\sigma^{2/3}g$ making them orthonormal, we see that g is, locally, conformal to a Riemannian product of two surface metrics. Therefore, by Remark 3.6, $\boldsymbol{w} \in \{1,2\}$, contradicting the definition of class (D0) in Section 7, which includes the requirement that $\boldsymbol{w} = 3$.

By (14.2.b), $\lambda_l - \lambda = -(\lambda_k + 3\lambda)$, and so (g) implies that

$$\text{(14.6)} \qquad \begin{array}{rl} \text{i)} & F_{lk} = G_l + (\lambda_k - \lambda)\hat{F}_{lk}, \qquad F_{kl} = G_k - (\lambda_k + 3\lambda)\hat{F}_{kl}, \\ \text{ii)} & (\hat{F}_{lk}G_k - \hat{F}_{kl}G_l)\lambda\lambda_k = (\hat{F}_{lk}G_k + 3\hat{F}_{kl}G_l)\lambda^2 - \lambda G_k G_l, \end{array}$$

(14.6.ii) being the result of using (14.6.i) to rewrite the second equality in (c). Thus, $\lambda\hat{F}_{lk}G_k = \lambda\hat{F}_{kl}G_l$, or else, in an open set on which $\lambda\hat{F}_{lk}G_k \neq \lambda\hat{F}_{kl}G_l$, the formula for λ_k arising from (14.6.ii) would, by (a) and (g), show that $D_i\lambda_k = 0$ for both $i = 1,2$, contradicting the second part of (h). Since $\lambda\hat{F}_{lk}G_k = \lambda\hat{F}_{kl}G_l$, (14.6.ii) reads

$$\text{(14.7)} \qquad 4\lambda^2\hat{F}_{lk}G_k = 4\lambda^2\hat{F}_{kl}G_l = \lambda G_k G_l.$$

By (14.6), (14.7) and (14.3), $4\lambda^2 F_{kl}G_l = 4\lambda^2 G_k G_l - 4(\lambda_k + 3\lambda)\lambda^2\hat{F}_{kl}G_l = (\lambda - \lambda_k)\lambda G_k G_l = -\lambda G_l D_k\lambda$. This proves the second equality in (h), as it obviously holds on any open set on which $\lambda = 0$. $\square$

Remark 14.2. Assuming (14.1), we have $F_{ki} = G_k$ and $\lambda = \lambda_i$ in view of Lemma 14.1(b) and (14.2.a), while $\sigma = \sigma_{ij} = \sigma_{kl}$ by (8.2.b), so that combining (13.2.e) with (1.5) we obtain $D_iF_{ij} + D_jF_{ji} + F_{ij}^2 + F_{ji}^2 + G_k^2 + G_l^2 = -\sigma - \lambda - \mathrm{s}/12$ along with $D_iF_{ik} + D_kG_k + F_{ik}^2 + G_k^2 + F_{ji}F_{jk} + G_lF_{lk} = -\sigma_{ik} - (\lambda + \lambda_k + \mathrm{s}/6)/2$ and, from (14.2.b), $D_kF_{kl} + D_lF_{lk} + F_{kl}^2 + F_{lk}^2 + F_{ik}F_{il} + F_{jk}F_{jl} = -\sigma + \lambda - \mathrm{s}/12$.

15. Proof of Theorem 1.2(c): part two

Throughout this section, again, $\{i,j\} = \{1,2\}$ and $\{k,l\} = \{3,4\}$. According to (14.2.b) and (8.2.b), there exist functions μ, τ, not depending on the choice of $i \in \{1,2\}$ and $k \in \{3,4\}$, such that, with $\sigma = \sigma_{ij}$,

$$\text{(15.1)} \qquad \begin{array}{l} \text{a)}\ \lambda_k = (-1)^k\mu - \lambda, \quad \text{b)}\ \sigma_{ik} = (-1)^{i+k}\tau - \sigma/2, \\ \text{c)}\ \sigma_{ij} = \sigma_{kl} = \sigma, \quad \text{d)}\ -2F_{kl} = D_k\log|\mu| + [1 - 2(-1)^k\lambda/\mu]G_k, \end{array}$$

where (15.1.d) follows from (15.1.a) and (14.3), since $D_k\lambda_l = (\lambda_k - \lambda_l)F_{kl}$, cf. (13.2.b). Note that $\mu \neq 0$ (and (15.1.d) holds) on an open set which is nonempty (and therefore dense, by (1.1)), or else (15.1.a) would give $\lambda_3 = \lambda_4$, contradicting (14.2.a). In view of Lemma 14.1(g) and (15.1),

$$\text{(15.2)} \qquad 2(-1)^k\hat{F}_{kl} = [D_k\log|\mu| + \{3 - 2(-1)^k\lambda/\mu\}G_k]/[\mu + 2(-1)^k\lambda].$$

Using (8.2.b), Lemma 14.1(b) and equations (13.2.b) – (13.2.c), we see that

$$\text{(15.3)} \qquad \begin{array}{l} \text{a)}\ 2D_i\sigma = -3(F_{ik} + F_{il})\sigma - 2(-1)^{i+k}(F_{ik} - F_{il})\tau, \\ \text{b)}\ 4D_i\tau = -3(-1)^{i+k}(F_{ik} - F_{il})\sigma - 2[4F_{ij} + F_{ik} + F_{il}]\tau, \\ \text{c)}\ D_k\sigma = -3G_k\sigma, \quad D_k\tau = -(2F_{kl} + G_k)\tau, \quad D_k\lambda = [(-1)^k\mu - 2\lambda]G_k, \end{array}$$

the last equality in (15.3.c) being due to (14.3) and (15.1.a). As $D_i\lambda = 0$, cf. Lemma 14.1(a), from Lemma 14.1(d) and (15.1.a) we get

$$\text{(15.4)}\qquad \begin{aligned} F_{ik} + F_{il} \,&=\, D_i \log|\mu^2 - 4\lambda^2| \,=\, 2(\mu^2 - 4\lambda^2)^{-1}\mu D_i\mu,\\ (-1)^{i+k}(F_{ik} - F_{il}) \,&=\, D_i \log|[\mu + 2(-1)^i\lambda]/[\mu - 2(-1)^i\lambda]|\\ &=\, -4(-1)^i(\mu^2 - 4\lambda^2)^{-1}\lambda D_i\mu. \end{aligned}$$

The functions P_i, Q_i given by $2P_i = (-1)^k(F_{ik} - F_{il})$ and $2Q_i = -(F_{ik} + F_{il})$ are clearly independent of the choice of k, l with $\{k, l\} = \{3, 4\}$. Also,

$$\text{(15.5)}\qquad \begin{aligned} &\text{i)}\quad F_{ik} \,=\, (-1)^k P_i - Q_i, \quad -F_{il} \,=\, (-1)^k P_i + Q_i,\\ &\text{ii)}\quad D_i\sigma = 3Q_i\sigma - 2(-1)^i P_i\tau,\\ &\text{iii)}\quad D_i\tau = (Q_i - 2F_{ij})\tau - 3(-1)^i P_i\sigma/2,\\ &\text{iv)}\quad D_i\mu = 2\lambda P_i + \mu Q_i, \quad \text{v)}\ \ \mu P_i + 2\lambda Q_i = 0. \end{aligned}$$

Namely, (15.5.i) is obvious, and (15.5.ii) – (15.5.iii) follow from (15.3). On the other hand, $(-1)^k D_i\mu = D_i\lambda_k = (\lambda - \lambda_k)F_{ik}$ by (15.1.a), Lemma 14.1(a), (13.2.b) and (14.2.a); simultaneously, (15.1.a) and (15.5.i) give $(\lambda - \lambda_k)F_{ik} = [2\lambda - (-1)^k\mu][(-1)^k P_i - Q_i]$. Hence $D_i\mu = [2\lambda - (-1)^k\mu][P_i - (-1)^k Q_i]$, that is, $D_i\mu = 2\lambda P_i + \mu Q_i - (-1)^k(\mu P_i + 2\lambda Q_i)$, which holds for both $k \in \{3, 4\}$, thus implying (15.5.iv) and (15.5.v). Next,

$$\text{(15.6)}\qquad \begin{aligned} D_i P_i &= 2P_i Q_i - F_{ji}P_j - (-1)^i\tau - \mu/2\\ &\quad -(-1)^k(D_k G_k - D_l G_l + G_k^2 - G_l^2 + G_l F_{lk} - G_k F_{kl})/2,\\ D_i Q_i &= P_i^2 + Q_i^2 - F_{ji}Q_j - (\sigma - \mathrm{s}/6)/2\\ &\quad +(D_k G_k + D_l G_l + G_k^2 + G_l^2 + G_l F_{lk} + G_k F_{kl})/2. \end{aligned}$$

This is immediate from the definitions of P_i and Q_i, (15.5.i), and the second conclusion of Remark 14.2, the right-hand side of which is equal, by (15.1.a) and (15.1.b), to $-(-1)^{i+k}\tau + [\sigma - (-1)^k\mu - \mathrm{s}/6]/2$. Furthermore,

(15.7) λ is nonzero everywhere in some dense open set and, whenever it is constant, $F_{ki} = G_k = D_k\sigma = 0$ for all $(i, k) \in \{1, 2\} \times \{3, 4\}$.

In fact, the second claim is obvious from (14.3) and Lemma 14.1(b). As for the first, let $\lambda = 0$ on a nonempty open set. On such a set, with (14.1), the last equality in (15.4) gives $F_{ik} = F_{il}$ or, in the notation of (15.5), $P_i = P_j = 0$. Since $G_k = G_l = 0$ by (15.7), the first formula in (15.6) now reads $(-1)^i\tau = -\mu/2$. This being the case for both $i \in \{1, 2\}$, it follows that $\tau = \mu = 0$. Consequently, as $\lambda = 0$, (15.1.a) yields $\lambda_3 = \lambda_4 = 0$, contradicting (14.2.a).

Subtracting from (15.6) its version obtained by switching i, j and noting that $(-1)^j = -(-1)^i$, we get

$$\text{(15.8)}\qquad \begin{aligned} D_i P_i - 2P_i Q_i - F_{ij}P_i - (D_j P_j - 2P_j Q_j - F_{ji}P_j) \,&=\, -2(-1)^i\tau,\\ D_i Q_i - P_i^2 - Q_i^2 - F_{ij}Q_i \,&=\, D_j Q_j - P_j^2 - Q_j^2 - F_{ji}Q_j. \end{aligned}$$

Since $\lambda_i = \lambda_j = \lambda$ and $F_{ki} = F_{kj} = G_k$, cf. (14.2.a) and Lemma 14.1(b), the second displayed equation in Theorem 1.3 with i and l switched takes, by

(15.5.i) and (15.1.a) – (15.1.c), the form

$$\tag{15.9} \begin{aligned}[(2F_{kl} - G_k)P_i - (-1)^k Q_i G_k][4\lambda\tau + 3(-1)^i\mu\sigma] \\ = [(-1)^k P_i G_k - Q_i G_k][4(-1)^k\lambda\tau - 2\mu\tau].\end{aligned}$$

16. Proof of Theorem 1.2(c): part three

We continue making the same assumptions as in Lemma 14.1.

Lemma 16.1. *One has* $G_3 G_4 = 0$ *everywhere.*

Proof. Suppose that, on the contrary, neither G_3 nor G_4 is identically 0. As $\lambda \neq 0$ somewhere by (15.7), using (1.1) we may restrict our discussion to a dense open set, at every point of which

$$\tag{16.1} \lambda G_3 G_4 \neq 0.$$

With (14.1), for some functions C_3, C_4, E_3, E_4, Π and $\varepsilon = \mathrm{sgn}\lambda \in \{1, -1\}$,

$$\tag{16.2} \begin{array}{rlrl} \mathrm{i)} & F_{kl} = D_k \log|\lambda|^{-1/4}, & \mathrm{ii)} & D_k C_l = 0, \\ \mathrm{iii)} & \lambda = \varepsilon(C_3 + C_4)^2, & \mathrm{iv)} & \varepsilon(C_3 + C_4) > 0, \\ \mathrm{v)} & \lambda - \lambda_k = 4\varepsilon(C_3 + C_4)C_k, & \mathrm{vi)} & \lambda_k = \varepsilon(C_3 + C_4)(C_l - 3C_k), \\ \mathrm{vii)} & D_i(C_3 + C_4) = 0, & \mathrm{viii)} & F_{ik} = -D_i \log|C_k|, \\ \mathrm{ix)} & C_k = E_k + (-1)^k\Pi, & \mathrm{x)} & D_k E_l = D_i E_k = D_k \Pi = 0, \\ \mathrm{xi)} & D_i C_k = (-1)^k D_i \Pi, & \mathrm{xii)} & \Pi \text{ is nonconstant.} \end{array}$$

In fact, (16.1) and the second claim in Lemma 14.1(h) imply (16.2.i). Setting $C_k = |\lambda|^{-1/2}(\lambda - \lambda_k)/4$ and using (14.2.b) we obtain (16.2.ii) (since (13.2.b), (16.2.i) and (14.2.b) give $4\lambda D_k \lambda_l = (\lambda_l - \lambda_k) D_k \lambda = 2(\lambda + \lambda_l) D_k \lambda$), as well as (16.2.iii) – (16.2.v), while subtraction of (16.2.v) from (16.2.iii) yields (16.2.vi). Equality (16.2.vii) is a consequence of (16.2.iii) and Lemma 14.1(a). Similarly, (16.2.v), (16.2.vii) and Lemma 14.1(d) lead to (16.2.viii). For ∂_i, ∂_k chosen as in Remark 13.3, the relations $\partial_k C_l = \partial_i(C_k + C_l) = 0$, due to (16.2.ii) and (16.2.vii), show that the function $\partial_i C_k = -\partial_i C_l$ is constant along $\mathcal{H} = \mathrm{Span}\{e_3, e_4\}$, and so C_k, for $k = 3, 4$, equals a function constant along $\mathcal{H}$ plus a function constant along $\mathcal{V} = \mathrm{Span}\{e_1, e_2\}$. Combined with (16.2.vii), this proves the existence of E_3, E_4 and Π satisfying (16.2.ix) – (16.2.xi). Finally, if Π were constant, (16.2.xi) and (16.2.vi) would give $D_i C_k = D_i \lambda_k = 0$ for both $i = 1, 2$ and both $k = 3, 4$, contradicting the first claim in Lemma 14.1(h).

Due to (14.3) and (16.2.ii) – (16.2.v), Lemma 14.1(b) gives $D_k \log|\sigma|^{-1/3} = D_k \log|C_k|^{-1/2}$, where $C_k \neq 0$ in view of (16.2.v) and (14.2.a). Thus, by (16.2.ii), $D_k(|C_3 C_4|^{-3/2}\sigma) = 0$, and so, for the function $L = |C_3 C_4|^{-3/2}\sigma$,

$$\tag{16.3} \sigma = |C_3 C_4|^{3/2} L \quad\text{and}\quad D_k L = 0 \quad\text{whenever}\quad k \in \{3, 4\}.$$

By (16.2.i), $2F_{kl} = D_k \log|\lambda|^{-1/2}$. Adding this to the second equality in Lemma 14.1(b), one gets $D_k S = 0$ for $S = (-1)^{i+k}|\lambda|^{-1/2}\sigma^{-1/3}(\sigma_{ik} - \sigma_{jk})$, with (14.1), and $k = 3, 4$, where, in view of (8.2.b), S does not depend on the choice of i, j, k, l satisfying (14.1). As (8.2.b) with $\sigma = \sigma_{ij}$ also yields $\sigma_{ik} - \sigma_{jk} = 2\sigma_{ik} + \sigma$, the definition of S amounts to

$$(16.4) \qquad 2\sigma_{ik} = -\sigma + (-1)^{i+k}|\lambda|^{1/2}\sigma^{1/3}S \text{ with (14.1) and } D_k S = 0.$$

Next, (13.2.c) and (8.2.b) give $D_i\sigma = D_i\sigma_{kl} = (\sigma_{il} - \sigma)F_{ik} + (\sigma_{ik} - \sigma)F_{il}$. Replacing σ_{ik}, σ_{il} and F_{ik}, F_{il} with the expressions provided by (16.4) and (16.2.viii), we easily verify that $2D_i\sigma = (-1)^{i+k}|\lambda|^{1/2}\sigma^{1/3}SD_i \log|C_k/C_l| + 3\sigma D_i \log|C_kC_l|$. At the same time, from (16.3), $2D_i\sigma = 2|C_kC_l|^{3/2}D_iL + 3\sigma D_i \log|C_kC_l|$. As $|\lambda|^{1/2} = \varepsilon(C_k + C_l)$ by (16.2.iii) – (16.2.iv), while $D_i\log|C_k/C_l| = (-1)^k(C_k^{-1} + C_l^{-1})D_i\Pi$, cf. (16.2.xi), and $\sigma^{1/3} = |C_3C_4|^{1/2}L^{1/3}$ from (16.3), equating the two expressions for $2D_i\sigma$ one gets, from (16.2.iv),

$$(16.5) \qquad (C_k^{-1} + C_l^{-1})^2 = 2|D_iL|/|SL^{1/3}D_i\Pi|$$

on a nonempty open set U'', where (16.2.xii) allows us to choose $i \in \{1, 2\}$ so that $|D_i\Pi| > 0$ on U''. Since, for ∂_i, ∂_k as in Remark 13.3, $|D_iL|/|D_i\Pi| = |\partial_iL|/|\partial_i\Pi|$, and so ∂_k applied to the right-hand side of (16.5) yields 0 in view of (16.4), (16.3) and (16.2.x), we have, from (16.2.ii), $0 = D_k(C_k^{-1} + C_l^{-1}) = -C_k^{-2}D_kC_k$. Thus, by (16.2.ii) – (16.2.iii), $D_k\lambda = 0$. Combined with (14.3) and (16.1), this leads to a contradiction, proving that $G_3G_4 = 0$. $\square$

Assume (14.1). In view of (13.2.a), for any function θ we have

$$(16.6) \qquad (\nabla d\theta)(e_i, e_j) = D_iD_j\theta - F_{ji}D_i\theta.$$

By Lemma 14.1(d), $F_{ik} = D_i\theta_k$, where $\theta_k = \log|\lambda_k - \lambda|^{-1}$. Now (13.4.i) reads $D_iD_j\theta_k = (F_{ji} - D_j\theta_k)D_i\theta_k$, and so, with $\theta = \theta_k$, from (16.6), $(\nabla d\theta)(e_i, e_j) + (D_j\theta)D_i\theta = 0$, that is, $(\nabla de^\theta)(e_i, e_j) = 0$. In other words, for $\psi_k = (\lambda_k - \lambda)^{-1}$,

$$(16.7) \qquad (\nabla d\psi_k)(e_i, e_j) = (\nabla d\psi_l)(e_i, e_j) = 0 \quad \text{if } \{k, l\} = \{3, 4\}.$$

When k, l with $\{k, l\} = \{3, 4\}$ are fixed, setting $\psi = \psi_k$ we get $\psi_l = -(4\lambda + 1/\psi)^{-1}$ from (14.2.b).Now (16.6) applied to $\theta = \psi_l$ and $\theta = \psi_k$ yields $-(4\lambda\psi + 1)^3(\nabla d\psi_l)(e_i, e_j) = (4\lambda\psi + 1)(\nabla d\psi)(e_i, e_j) - 8\lambda(D_j\psi)D_i\psi$, with $\psi = \psi_k$, as $D_i\lambda = 0$ (see Lemma 14.1(a)). In view of (16.7), this gives $(D_j\psi_k)D_i\psi_k = 0$, since $\lambda \neq 0$ by (15.7). Using (15.1.a) and, again, the relation $D_i\lambda = 0$ in Lemma 14.1(a), we see that

$$(16.8) \qquad (D_j\mu)D_i\mu = 0 \quad \text{if } \{i, j\} = \{1, 2\}.$$

From now on the symbols $\bullet$ stand for any indices for which the expression makes sense. For instance, $j, \bullet, \bullet$ in $H_{j\bullet\bullet}$ are mutually distinct.

Lemma 16.2. *One can fix $i, j \in \{1, 2\}$ and a nonempty open connected set U'', so that, whenever $\{k, l\} = \{3, 4\}$,*

(a) $D_j\mu = 0 \neq D_i\mu$ *everywhere,*
(b) $D_i\lambda = D_j\lambda = D_iG_k = D_jG_k = G_lG_k = 0$,
(c) $F_{j\bullet} = 0$ *and* $H_{j\bullet\bullet} = H_{\bullet j\bullet} = 0$,

(d) $D_j\lambda_k = D_j\sigma = D_j\sigma_{\bullet\bullet} = D_jF_{ij} = 0$.

Proof. Lemmas 14.1(a) and 16.1 imply (b) for $i, j \in \{1,2\}$. Thus, by real-analyticity, cf. (1.1), and (16.8), $D_j\mu = 0$ everywhere for one choice of j (but not for both, since that would, in view of (b) and (15.1.a), yield $D_i\lambda_k = 0$ whenever $i \le 2 < k$, contradicting Lemma 14.1(h)). Now (a) follows. From (a), (b), (15.1.a) and Lemma 14.1(d) we in turn obtain $D_j\lambda_k = F_{jk} = 0$, although, as we just saw, $D_i\lambda_k \ne 0$. (Here and below, $i, j \in \{1,2\}$ are fixed, so as to satisfy (a), and $\{k,l\} = \{3,4\}$.) Next, (16.7) and (16.6), for $\theta = \psi_k = (\lambda_k - \lambda)^{-1}$, give $D_iD_j\theta = F_{ji}D_i\theta$, while $D_j\theta = 0 \ne D_i\theta$ by (b) with $D_j\lambda_k = 0 \ne D_i\lambda_k$. Hence $F_{ji} = 0$, that is, $F_{j\bullet} = 0$, as required in (c), and the rest of (c) is obvious from (13.1). Also, combining (c) with the equality $(\sigma_{jk} - \sigma_{ik})(D_jF_{ij} - D_iF_{ji}) = 3(H_{lk} - H_{kl})\sigma_{kl}$, which is a special case of (13.4.ii) or, respectively, with the expression for $D_j\sigma_{\bullet\bullet}$ resulting from (13.2.c), we obtain (d), since $\sigma_{jk} \ne \sigma_{ik}$ and $\sigma = \sigma_{ij} = \sigma_{kl}$ according to (8.2.a) and (8.2.b), while we already saw that $D_j\lambda_k = 0$. $\square$

17. Proof of Theorem 1.2(c): conclusion

Fixing i, j with $\{i,j\} = \{1,2\}$ and U'' as in Lemma 16.2, for μ, σ, τ, P_i, Q_i appearing in (15.1) – (15.5), let us set $P = P_i$ and $Q = Q_i$. We have

$$
(17.1)\qquad
\begin{array}{rl}
\text{i)} & \mu P + 2\lambda Q = 0,\\
\text{ii)} & P_j = Q_j = F_{ji} = D_i\lambda = D_iG_k = 0,\\
\text{iii)} & D_iP = 2PQ + PF_{ij} - 2(-1)^i\tau,\\
\text{iv)} & D_iQ = P^2 + Q^2 + QF_{ij}.\\
\text{v)} & 2\lambda(P^2 - Q^2) = (-1)^i\mu\tau,\\
\text{vi)} & 2(-1)^i(2F_{ij}\mu - 5\lambda P)\tau = 3\lambda Q\sigma.
\end{array}
$$

In fact, (15.5.v) amounts to (17.1.i); the definitions of P_j, Q_j preceding (15.5) and Lemma 16.2(b)–(c), and (15.5.v) yield (17.1.ii); also, (17.1.iii) – (17.1.iv) are trivial consequences of (15.8) and (17.1.ii). Next, (17.1.v) follows if one applies D_i to (17.1.i), uses (15.5.iv), (17.1.iii), (17.1.ii), (17.1.iv), and simplifies, with the aid of (17.1.i), the resulting equality

$$(F_{ij} + 3Q)(\mu P + 2\lambda Q) + 2[2\lambda(P^2 - Q^2) - (-1)^i\mu\tau] = 0.$$

Finally, by (17.1.ii) – (17.1.iv) and (15.5.iii) – (15.5.iv), D_i applied to (17.1.v) shows that the left-hand side of (17.1.vi) coincides with

$$3\lambda Q\sigma - 2(F_{ij} + Q)[2\lambda(P^2 - Q^2) - (-1)^i\mu\tau] - 3(\mu P + 2\lambda Q)\sigma/2.$$

Thus, (17.1.v) and (17.1.i) give (17.1.vi). By (14.2.a) and Lemma 14.1(b),

$$(17.2)\qquad \lambda_i = \lambda_j = \lambda \text{ and } F_{ki} = F_{kj} = G_k, \text{ while } P_i = P \text{ and } Q_i = Q.$$

Now, from (15.5.i) and (17.2), along with (13.4.i) and (17.1.iii) – (17.1.iv),

$$\begin{array}{ll} & F_{ik} = (-1)^k P - Q, \qquad D_k F_{ij} = [(-1)^k P - Q - F_{ij}]G_k, \\ (17.3) & D_i F_{ik} = (-1)^k [2PQ + PF_{ij} - 2(-1)^i \tau] - (P^2 + Q^2 + QF_{ij}), \\ & D_i F_{kl} = [(-1)^k P + Q](F_{kl} - G_k). \end{array}$$

By Lemma 16.1, restricting our discussion to an nonempty open subset of U'', we may fix k, l with $\{k,l\} = \{3,4\}$ and $G_l = 0$ everywhere. Then

$$(17.4) \qquad \begin{array}{l} \text{either } G_k \text{ vanishes identically on } U', \text{ or } G_k \neq 0 \\ \text{at all points of some open dense subset of } U', \end{array}$$

as a consequence of real-analyticity, cf. (1.1). Furthermore,

$$(17.5) \qquad \begin{array}{rl} \text{i)} & D_i F_{ij} = -(F_{ij}^2 + G_k^2 + \sigma + \lambda + \mathrm{s}/12), \\ \text{ii)} & PF_{ij} = (-1)^i \tau - \mu/2, \\ \text{iii)} & QF_{ij} + (\sigma - \mathrm{s}/6)/2 = G_k F_{kl}. \end{array}$$

Namely, (17.5.i) is due to the first conclusion of Remark 14.2 with $F_{ji} = 0$, cf. (17.1.ii), and $G_l = 0$. To prove (17.5.ii) – (17.5.iii) we begin by observing that (15.6) and (17.1.ii) – (17.1.iv) easily give

$$\begin{array}{rl} \text{(a)} & D_k G_k + G_k^2 + G_k F_{kl} = 2QF_{ij} + \sigma - \mathrm{s}/6, \\ \text{(b)} & (-1)^k (D_k G_k + G_k^2 - G_k F_{kl}) = 2[PF_{ij} - (-1)^i \tau] + \mu, \\ \text{(c)} & QF_{ij} - G_k F_{kl} + (\sigma - \mathrm{s}/6)/2 = (-1)^k [PF_{ij} - (-1)^i \tau + \mu/2], \end{array}$$

(c) arising when one subtracts (b) multiplied by $(-1)^k$ from (a). Next, applying (13.4.v) to the triple (k,i,j) rather than (i,k,l), one obtains $D_k D_i F_{ij} + (F_{ki} + F_{kj}) D_i F_{ij} = F_{kj} D_i F_{ik} + (F_{ik} - F_{ij}) H_{lj}$. Equivalently, due to (17.2), $D_k D_i F_{ij} = [(F_{ik} - F_{ij}) F_{ik} + D_i F_{ik} - 2 D_i F_{ij}] G_k$, as (13.1) and (17.2) give $H_{lj} = F_{ik} G_k$. With F_{ik}, $D_i F_{ik}$, $D_i F_{ij}$ replaced by the expressions in (17.3) and (17.5.i), this shows that $D_k D_i F_{ij}$ equals $2G_k$ times

$$F_{ij}^2 + G_k^2 - (-1)^{i+k} \tau + \lambda + \sigma + \mathrm{s}/12.$$

Simultaneously, by (17.5.i), $D_k D_i F_{ij} = -D_k (F_{ij}^2 + G_k^2 + \sigma + \lambda + \mathrm{s}/12)$ which, evaluated via (17.3), (a), (15.3.c) and (4.3), equals $2G_k$ times

$$F_{ij}^2 + G_k^2 - QF_{ij} + G_k F_{kl} - (-1)^k [PF_{ij} + \mu/2] + \lambda + (\sigma - \mathrm{s}/3)/2.$$

Equating the two displayed expression, one easily gets

$$(17.6) \quad [QF_{ij} - G_k F_{kl} + (\sigma - \mathrm{s}/6)/2 + (-1)^k \{PF_{ij} - (-1)^i \tau + \mu/2\}] G_k = 0.$$

In the first case of (17.4), (a) – (b) clearly yield (17.5.ii) and (17.5.iii). In the second case, the two equal sides of (c) are, by (17.6), also each other's opposites, and so both vanish, proving (17.5.ii) – (17.5.iii).

Lemma 17.1. *Each of the following seven functions*:

$$(-1)^k P - Q, \quad 2(-1)^i \tau + 3(-1)^k \sigma, \quad \lambda, \quad \mu, \quad P, \quad Q, \quad D_i \mu$$

is nonzero everywhere in some open dense subset of U'.

Proof. Due to real-analyticity, cf. (1.1), it suffices to show that none of the seven functions can vanish on a nonempty open subset U'' of U'. For $\lambda, D_i\mu$ (and, consequently, μ) this is clear from (15.7) and Lemma 16.2(a). If P were identically zero on U'', so would be τ, and hence μ, by (17.1.iii) and and (17.5.ii), contrary to what we just showed about μ. Thus, $Q \neq 0$ on U'' from (17.1.i) with $\mu P \neq 0$. In view of (17.1.i), vanishing of $(-1)^k P - Q$ on on U'' would give $2(-1)^k \lambda P = 2\lambda Q = -\mu P$ on U'' and, as $P \neq 0$, the equality $\mu = -2(-1)^k\lambda$ would follow, even though $D_i\mu \neq 0 = D_i\lambda$, cf. (17.1.ii). Finally, suppose that $2(-1)^i\tau + 3(-1)^k\sigma = 0$ on U''. Using, successively, (8.2.b), (15.1.c) and (15.1.b), we now get $\sigma_{ij} - \sigma_{il} = \sigma_{ij} + (\sigma_{ij} + \sigma_{ik}) = 2\sigma + \sigma_{ik} = 2\sigma + (-1)^{i+k}\tau - \sigma/2 = (-1)^k[2(-1)^i\tau + 3(-1)^k\sigma]/2 = 0$, so that $\sigma_{ij} = \sigma_{il}$, which contradicts (8.2.a). □

Lemma 17.2. *The second case of* (17.4) *cannot occur.*

Proof. Let us assume that, on the contrary, $G_k \neq 0$ everywhere in a dense open set, and apply D_i to (17.5.iii), using (17.5.i), (17.1.iv), (15.5.ii) with (17.2), (4.3), (17.1.ii) and (17.3). We consequently get

$$\begin{aligned}(-1)^k(G_k - F_{kl})PG_k + [QF_{ij} - G_kF_{kl} + (\sigma - \mathrm{s}/6)/2]Q \\ + [PF_{ij} - (-1)^i\tau + \mu/2]P - (\mu P + 2\lambda Q)/2 = 0\end{aligned}$$

which, by (17.5.iii), (17.5.ii) and (17.1.i), amounts to $(G_k - F_{kl})PG_k = 0$. As Lemma 17.1 now gives $PG_k \neq 0$, one has $G_k = F_{kl}$. Replacing, in (15.9), the triple (P_i, Q_i, F_{kl}) with (P, Q, G_k), we easily obtain the equality $[(-1)^kP - Q][2(-1)^i\tau + 3(-1)^k\sigma]G_k\mu = 0$. This contradicts Lemma 17.1. □

Lemma 17.2 and the line preceding (17.4) give $G_k = G_l = 0$, and so, by (15.3.c), Lemma 16.2(b), and the first claim in (15.7)

$$\lambda \text{ is a nonzero constant,} \tag{17.7}$$

if U'' is connected (or replaced by a connected component). Also,

$$\begin{array}{ll} & \text{a)}\;\; \mu P + 2\lambda Q = 0, \\ & \text{b)}\;\; PF_{ij} = (-1)^i\tau - \mu/2, \\ & \text{c)}\;\; QF_{ij} = -(\sigma - \mathrm{s}/6)/2, \\ (17.8) & \text{d)}\;\; 2(-1)^i(2F_{ij}\mu - 5\lambda P)\tau = 3\lambda Q\sigma, \\ & \text{e)}\;\; [(-1)^i\tau - \mu/2]\mu = (\sigma - \mathrm{s}/6)\lambda, \\ & \text{f)}\;\; (-1)^i(F_{ij}\mu - 2\lambda P)\tau = (2\lambda + \sigma - \mathrm{s}/6)\lambda Q, \\ & \text{g)}\;\; (-1)^iF_{ij}\mu\tau = (10\lambda + 2\sigma - 5\mathrm{s}/6)\lambda Q, \\ & \text{h)}\;\; 4(-1)^i\lambda\tau + (8\lambda + \sigma - 2\mathrm{s}/3)\mu = 0. \end{array}$$

In fact, (17.8.a) – (17.8.d) are just certain parts of (17.1) and (17.5), with $G_k = 0$ in (17.5). Equality (17.8.e) arises in turn due to (17.8.a), if one adds (17.8.b) multiplied by μ to (17.8.c) times 2λ. Let us now apply D_i to (17.8.e), using (17.7), (4.3), (15.5) and (17.2). The resulting relation $2(-1)^i(F_{ij}\mu - 2\lambda P)\tau + (2\lambda + 3\sigma/2)(\mu P + 2\lambda Q) - 2(2\lambda + \sigma - \mathrm{s}/6)\lambda Q + 2\{(\sigma - \mathrm{s}/6)\lambda - [(-1)^i\tau - \mu/2]\mu\}Q = 0$ becomes (17.8.f) when combined with (17.8.a) and (17.8.e), while (17.8.g) is just the side-by-side difference of (17.8.f) multiplied by 5 and (17.8.d). Next, subtracting (17.8.d) from 4 times (17.8.f) and cancelling the factor λ, as allowed due to (17.2), we get $2(-1)^i P\tau = (8\lambda + \sigma - 2\mathrm{s}/3)Q$. Multiplying this by μ and then replacing μP with $-2\lambda Q$, cf. (17.8.a), we see that Q times the left-hand side of (17.8.h) equals zero, and so (17.8.h) follows, since $Q \neq 0$ according to Lemma 17.1.

Now (17.1.vi) times μ, with μP replaced by $-2\lambda Q$ as above, reads $4(-1)^i F_{ij}\mu^2\tau = [3\mu\sigma - 20(-1)^i\lambda\tau]\lambda Q$, while (17.8.g) multiplied by 4μ yields $4(-1)^i F_{ij}\mu^2\tau = (10\lambda + 2\sigma - 5\mathrm{s}/6)\mu\lambda Q$. As $Q \neq 0$ in Lemma 17.1, equating the two right-hand sides, we easily get $\mu\sigma = 20(-1)^i\lambda\tau + (10\lambda - 5\mathrm{s}/6)\mu$. However, from (17.8.h), $\mu\sigma = -4(-1)^i\lambda\tau - (8\lambda - 2\mathrm{s}/3)\mu$. Equating, again, the two new right-hand sides, we see that, due to (4.3) and (17.7) (that is, constancy of both λ and s), $\tau = c\mu$ for some constant c, and so $\mu\sigma$ is a constant multiple of μ as well. By Lemma 17.1, σ must be constant. We have four further equalities:

$$
\begin{aligned}
&\text{(i)} && 2(-1)^i P\tau \,-\, 3Q\sigma \,=\, 0,\\
&\text{(ii)} && \mu P \,+\, 2\lambda Q \,=\, 0,\\
&\text{(iii)} && 4(-1)^i\lambda\tau \,+\, 3\mu\sigma \,=\, 0,\\
&\text{(iv)} && (10\lambda \,+\, \sigma \,-\, 2\mathrm{s}/3)\mu^2 \,+\, 4(\sigma \,-\, \mathrm{s}/6)\lambda^2 \,=\, 0.
\end{aligned}
$$

Here (i) follows from constancy of σ, by (15.5.ii) and (17.2), while (ii) is nothing else than (17.1.i), repeated here for convenience, and (iii) amounts to vanishing of the determinant of the system (i) – (ii), due to nontriviality of its solution (P,Q) (see Lemma 17.1). Finally, (iv) is the result of subtracting (17.8.e) multiplied by 4λ from (17.8.h) multiplied by μ.

Using (iii) to replace $4(-1)^i\lambda\tau$ in (17.8.h) with $-3\mu\sigma$, and then cancelling the factor μ, cf. Lemma 17.1, we see that $\sigma = 4\lambda - \mathrm{s}/3$, which allows us to rewrite (iv) as $(14\lambda - \mathrm{s})\mu^2 + 2(8\lambda - \mathrm{s})\lambda^2 = 0$ By (17.7) and (4.3), this last equality implies that μ is constant, which contradicts the assertion about $D_i\mu$ in Lemma 17.1.

Appendix: The other curvature components in Theorem 10.1

Whenever $\{i,j\} = \{1,2\}$ and $\{k,l\} = \{3,4\}$, with y_l as in (8.7),

$$
\begin{aligned}
R_{ijij} \,&=\, \frac{D_iD_i\lambda_i + D_jD_j\lambda_i}{2\lambda_i} \,-\, 3\,\frac{(D_i\lambda_i)^2 + (D_j\lambda_i)^2}{4\lambda_i^2}\\
&\qquad\qquad\qquad\qquad -\, \frac{(D_k\lambda_i)^2 + (D_l\lambda_i)^2 + y_k^2 + y_l^2}{\lambda_i^2 - \lambda_k^2},\\
R_{ikij} \,&=\, \frac{D_kD_j\lambda_i}{2\lambda_i} \,+\, (\lambda_k - 2\lambda_i)\frac{(D_k\lambda_i)D_j\lambda_i}{2\lambda_i^2(\lambda_i - \lambda_k)} \,+\, \frac{(\lambda_i + 3\lambda_k)y_l D_i\lambda_i}{4\lambda_i^2(\lambda_i + \lambda_k)} \,-\, \frac{D_i y_l}{2\lambda_i},\\
R_{klij} \,&=\, D_l\frac{y_l}{2\lambda_i} \,-\, D_k\frac{y_k}{2\lambda_i}, \qquad R_{klik} \,=\, R_{klkl} \,=\, 0.
\end{aligned}
$$

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