On compact manifolds admitting indefinite metrics with parallel Weyl tensor

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Abstract

Compact pseudo-Riemannian manifolds that have parallel Weyl tensor without being conformally flat or locally symmetric are known to exist in infinitely many dimensions greater than 4. We prove some general topological properties of such manifolds, namely, vanishing of the Euler characteristic and real Pontryagin classes, and infiniteness of the fundamental group. We also show that, in the Lorentzian case, each of them is at least 5-dimensional and admits a two-fold cover which is a bundle over the circle.

Key words: parallel Weyl tensor, conformally symmetric manifold
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Introduction

One calls a pseudo-Riemannian manifold (M,g) of dimension $n \geq 4$ *conformally symmetric* [5] if its Weyl conformal tensor is parallel, and *essentially conformally symmetric* if, in addition, (M,g) is neither conformally flat nor locally symmetric. All essentially conformally symmetric manifolds have indefinite metrics [8, Theorem 2].

The Weyl conformal tensor is one of the three irreducible components of the curvature tensor under the action of the pseudo-orthogonal group, the other two corresponding to the scalar curvature and traceless Ricci tensor. This puts conformally symmetric manifolds on par with two other classes, formed

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by manifolds with constant scalar curvature and, respectively, parallel Ricci tensor, including Einstein spaces.

The local structure of essentially conformally symmetric pseudo-Riemannian metrics is fully understood [11]; they realize any prescribed indefinite signature in every dimension $n \geq 4$. They are also known to exist on some compact manifolds diffeomorphic to torus bundles over the circle [10], where they represent all indefinite metric signatures in all dimensions $n \geq 5$ such that $n \equiv 5 \pmod 3$. Consequently, there arises a natural question of characterizing the compact manifolds that admit such metrics.

The present paper provides a step toward an answer by establishing some necessary conditions. Our first result, except for the claim about $\pi_1 M$, is derived in Section 3 from the Chern-Weil formulae. Infiniteness of $\pi_1 M$ is proved in Section 11: we argue there that, if M were simply connected, it would be a bundle over S^2 with a fibre covered by $\mathbf{R}^{n-2}$, which is impossible for topological reasons.

Theorem A *Let a manifold M of dimension $n \geq 4$ admit an essentially conformally symmetric pseudo-Riemannian metric. The real Pontryagin classes $p_i(M) \in H^{4i}(M, \mathbf{R})$ then vanish for all $i \geq 1$, If, in addition, M is compact, then it has zero Euler characteristic, and its fundamental group is infinite.*

Applied to manifolds M which are both compact and orientable, the assertion about $p_i(M)$ in Theorem A leads to a similar conclusion about the Pontryagin numbers, including the signature of M.

As mentioned above, compact essentially conformally symmetric Lorentzian manifolds exist in infinitely many dimensions n, starting from $n = 5$, while noncompact ones exist in every dimension $n \geq 4$. Our next two results deal with their topological structure in the compact case and with dimension 4.

Theorem B *Let (M, g) be a compact essentially conformally symmetric Lorentzian manifold. Then some two-fold covering manifold $\overline{M}$ of M is the total space of a C^∞ bundle over the circle, the fibre of which admits a flat torsionfree connection with a nonzero parallel vector field.*

Theorem C *Every four-dimensional essentially conformally symmetric Lorentzian manifold is noncompact.*

The fibration $\overline{M} \to S^1$ in Theorem B has an explicit geometric description. Namely, its fibres are the preimages in $\overline{M}$ of the leaves of a parallel distribution $\mathcal{D}^\perp$ on (M, g), which is the orthogonal complement of the *Olszak distribution* $\mathcal{D}$, defined, in any conformally symmetric manifold, by declaring the sections of $\mathcal{D}$ to be the vector fields u such that, for all vector fields v, v', one has $\xi \wedge \Omega = 0$, where $\xi = g(u, \cdot\,)$ and $\Omega = W(v, v', \cdot\,, \cdot\,)$. (By W we denote the Weyl tensor; thus, Ω is a differential 2-form.) Olszak introduced the distribution $\mathcal{D}$ in a more general situation [16], and showed that, on an essentially conformally symmetric manifold, $\mathcal{D}$ is a null parallel distribution of dimension 1 or 2. See Lemma 2.2(i). Olszak's paper [16] may be difficult to obtain; however, a proof of the result just mentioned can also be found in [11, Appendix I].

We derive Theorems B and C in Sections 5 and 9 from the following result, proved in Section 4:

Theorem D *For every essentially conformally symmetric Lorentzian manifold, the Olszak distribution $\mathcal{D}$ is one-dimensional. Passing to a two-fold covering manifold, if necessary, we may assume that $\mathcal{D}$ is trivial as a real line bundle, and then $\mathcal{D}$ is spanned by a global parallel vector field.*

1 Preliminaries

Throughout this paper, all manifolds and bundles, along with sections and connections, are assumed to be of class C^∞. Manifolds (including fibres of bundles) are, by definition, connected. A mapping is always a C^∞ mapping betweeen manifolds.

Remark 1.1 A surjective submersion $\pi : M \to P$ such that the sets $\pi^{-1}(y)$, $y \in P$, are all compact can always be factored as $M \to Q \to P$, with a locally trivial fibration $M \to Q$ having compact (connected) fibres, and a finite covering projection $Q \to P$.

Namely, π itself is a locally trivial fibration, except that each fibre $\pi^{-1}(y)$, rather than being connected (and hence a manifold in our sense), may in general have some finite set Q_y of connected components. This well-known fact becomes clear if one uses the holonomy of any "nonlinear connection" (a distribution in M complementary to the fibres). We now define Q and $Q \to P$ by $Q = \bigcup_{y\in P}(\{y\} \times Q_y)$ and $(y, N) \mapsto y$.

Lemma 1.2 *Suppose that a closed* 1*-form ξ on a compact manifold M is nonzero everywhere and, for some function $\phi : M \to \mathbf{R}$ which is nonzero somewhere, the form $\phi\xi$ is exact. Then M is the total space of a bundle over the circle S^1. In addition, for any functions $t : \widehat{M} \to \mathbf{R}$ on the universal covering manifold of M and $\theta : M \to \mathbf{R}$, such that dt is the pullback of ξ to $\widehat{M}$ and $d\theta = \phi\xi$, and for some $c \in (0, \infty)$,*

(a) *$t : \widehat{M} \to \mathbf{R}$ descends to a bundle projection $M \to \mathbf{R}/c\mathbf{Z} = S^1$,*
(b) *the pullback of θ to $\widehat{M}$ equals the composite $\Lambda(t)$ for some nonconstant*

function $\varLambda : \mathbf{R} \to \mathbf{R}$ *which is periodic, and has* c *as a period,*

(c) $\widehat{M}$ *can be diffeomorphically identified with* $\mathbf{R} \times N$ *for some manifold* N *so as to make* t *coincide with the projection* $\mathbf{R} \times N \to \mathbf{R}$.

Proof. Let $\widehat{g}$ be the pullback to $\widehat{M}$ of any fixed Riemannian metric g on M. The $\widehat{g}$-gradient $\widehat{\nabla}t$ of t gives rise to the vector field $w = \widehat{\nabla}t/\widehat{g}(\widehat{\nabla}t, \widehat{\nabla}t)$, which is complete, being the pullback of the vector field $u/g(u,u)$ on M, for u such that $\xi = g(u,\,\cdot\,)$. A standard argument [15, p. 12] using the flow of w, for which t itself serves as the parameter, yields (c). Denoting by $\widehat{\theta}$ and $\widehat{\phi}$ the pullbacks of θ and ϕ to $\widehat{M}$, we have $\widehat{\nabla}\widehat{\theta} = \widehat{\phi}\widehat{\nabla}t$. Hence $\widehat{\theta}$ is, locally, a function of t. The word 'locally' can in turn be dropped as the level sets of t are connected by (c). Thus, $\widehat{\theta} = \varLambda(t)$ (that is, $\widehat{\theta} = \varLambda \circ t$) for some function $\varLambda : \mathbf{R} \to \mathbf{R}$. Since θ and $\widehat{\theta}$ are nonconstant, so is $\varLambda$.

The invariance of dt under the action of the deck transformation group $\Gamma = \pi_1 M$ implies that $t \circ \alpha = t + \varXi(\alpha)$ for some homomorphism $\varXi : \Gamma \to \mathbf{R}$ and all $\alpha \in \Gamma$. As $\widehat{\theta} = \varLambda(t)$ is Γ-invariant, every nonzero value of $\varXi$ is a period of $\varLambda$. Thus, $\varXi(\Gamma) = c\mathbf{Z}$ for some $c \in (0, \infty)$, and (b) follows. Finally, the surjective submersion $t : \widehat{M} \to \mathbf{R}$ descends to a mapping $M = \widehat{M}/\Gamma \to \mathbf{R}/c\mathbf{Z}$, which must be a surjective submersion as well. In addition, for each $s \in \mathbf{R}$, the preimage of $s + c\mathbf{Z}$ under the latter mapping is connected, as it coincides with the image of $t^{-1}(s) \subset \widehat{M}$ under the covering projection $\widehat{M} \to M$, and $t^{-1}(s) \subset \widehat{M}$ is connected by (c). Remark 1.1 now yields (a). $\square$

Given a connection ∇ in a vector bundle $\mathcal{E}$ over a manifold M, a section ψ of $\mathcal{E}$, and vector fields u, v tangent to M, our sign convention for the curvature tensor $R = R^\nabla$ is

$$R(u,v)\psi \;=\; \nabla_{\!v}\nabla_{\!u}\psi \,-\, \nabla_{\!u}\nabla_{\!v}\psi \,+\, \nabla_{[u,v]}\psi. \tag{1}$$

Such $\nabla, \mathcal{E}, M, u$ and v give rise to the bundle morphism

$$R^\nabla(u,v) : \mathcal{E} \to \mathcal{E} \tag{2}$$

sending a section ψ of $\mathcal{E}$ to $R^\nabla(u,v)\psi = R(u,v)\psi$ defined in (1).

We always denote by ∇ both the Levi-Civita connection of a given pseudo-Riemannian manifold (M,g), and the g-gradient operator. The same symbol ∇ is also used for connections induced by ∇ in ∇-parallel subbundles of TM and their quotients.

A *pseudo-Riemannian fibre metric* γ in a vector bundle $\mathcal{E}$ over a manifold M is, as usual, any family of nondegenerate symmetric bilinear forms γ_x in the fibres $\mathcal{E}_x$ that constitutes a C^∞ section of the symmetric power $(\mathcal{E}^*)^{\odot 2}$.

Remark 1.3 Every simply connected manifold N with a complete flat torsionfree connection ∇ is diffeomorphic to a Euclidean space: the exponential mapping of ∇ at any point $x \in N$ is an affine diffeomorphism $T_xN \to N$. See Auslander and Markus [1, p. 145].

Given a flat connection ∇ in a vector bundle $\mathcal{E}$ of fibre dimension k over a manifold M, we will say that $\mathcal{E}$ *is trivialized by its parallel sections* if the space of ∇-parallel global sections of $\mathcal{E}$ is k-dimensional (or, in other words, ∇ is *globally* flat).

Lemma 1.4 *If a compact manifold N with a flat torsionfree connection ∇ admits a ∇-parallel distribution $\mathcal{L}$ such that both bundles $\mathcal{L}$ and $\mathcal{E} = TN/\mathcal{L}$, with the flat connections induced by ∇, are trivialized by their parallel sections, then ∇ is complete.*

Proof. We denote by $\mathcal{X}$ the space of parallel sections of $\mathcal{L}$, and by $\exp_x$ the exponential mapping of ∇ at $x \in M$. Geodesics tangent to $\mathcal{L}$ at some (or every) point, being integral curves of elements of $\mathcal{X}$, are obviously complete.

Let $\mathcal{Y}$ be a vector subbundle of TN such that $TN = \mathcal{L} \oplus \mathcal{Y}$, and let us fix $w \in V$, where V is the vector space of sections of $\mathcal{Y}$ obtained as the image of the space of parallel sections of $\mathcal{E}$ under the obvious isomorphism $\mathcal{E} \to \mathcal{Y}$. The vector field $v = \nabla_w w$ then is a section of $\mathcal{L}$. (In fact, locally, $w = w' + \widetilde{w}$, for a section w' of $\mathcal{L}$ and a local parallel vector field $\widetilde{w}$, so that $\nabla_w w = \nabla_w w'$, while $\nabla_w w'$ is a section of $\mathcal{L}$.) Any integral curve $\mathbf{R} \ni t \mapsto x(t) \in N$ of w now gives rise to a function $\zeta : \mathbf{R} \to \mathcal{X}$ defined by requiring that $\zeta(t) \in \mathcal{X}$ have the value $v_{x(t)}$ at the point $x(t)$. Any function $\eta : \mathbf{R} \to \mathcal{X}$ with the second derivative $\ddot{\eta} = -\zeta$ leads in turn to the curve $\mathbf{R} \ni t \mapsto y(t) \in N$, given by $y(t) = \exp_{x(t)} \eta(t, x(t))$, where $\eta(t,x) \in T_xM$ is the value of $\eta(t) \in \mathcal{X}$ at $x \in M$. That $t \mapsto y(t)$ is a geodesic is clear: a treating N, locally, as an affine space, we have $y(t) = x(t) + \eta(t)$, and $\ddot{y} = \ddot{x} + \ddot{\eta} = 0$. Since such geodesics realize all initial data, our assertion follows. $\square$

Let $(t,s) \mapsto x(s,t)$ be a fixed *variation of curves* in a pseudo-Riemannian manifold (M,g), that is, an M-valued C^∞ mapping from a rectangle (product of intervals) in the ts-plane. By a *vector field w along the variation* we mean, as usual, a section of the pullback of TM to the rectangle: $w(t,s) \in T_{x(t,s)}M$. Examples are x_s and x_t, which assign to (t,s) the velocity of the curve $t \mapsto x(t,s)$ (or, $s \mapsto x(t,s)$) at s (or t). Further examples are provided by restrictions to the variation of vector fields on M. The partial covariant derivatives of a vector field w along the variation are the vector fields w_t, w_s along the variation, obtained by differentiating w covariantly along the curves

$t \mapsto x(t,s)$ or $s \mapsto x(t,s)$. Skipping parentheses, we write w_{ts}, w_{stt}, etc., rather than $(w_t)_s$, $((w_s)_t)_t$ for higher-order derivatives, as well as x_{ss}, x_{st} instead of $(x_s)_s$, $(x_s)_t$. One always has $w_{ts} = w_{st} + R(x_t, x_s)w$, cf. [12, formula (11.2) on p. 493], and, since the Levi-Civita connection ∇ is torsionfree, $x_{st} = x_{ts}$. Consequently, $x_{ttss} = x_{stts} + [R(x_t, x_s)x_t]_s$, $x_{stts} = x_{stst} + R(x_t, x_s)x_{st}$ and $x_{stst} = x_{sstt} + [R(x_t, x_s)x_s]_t$. Thus, whenever $(t,s) \mapsto x(s,t)$ is a variation of curves in M,

$$\begin{aligned} &\text{a)}\ \ x_{tts} = x_{stt} + R(x_t, x_s)x_t\,, \\ &\text{b)}\ \ x_{ttss} = x_{sstt} + [R(x_t, x_s)x_s]_t + R(x_t, x_s)x_{st} + [R(x_t, x_s)x_t]_s\,. \end{aligned} \tag{3}$$

2 Conformally symmetric manifolds

The Schouten tensor σ and Weyl conformal tensor W of a pseudo-Riemannian manifold (M,g) of dimension $n \geq 4$ are given by the formulae $\sigma = \rho - (2n-2)^{-1}\,\mathrm{s}\,g$ and $W = R - (n-2)^{-1} g \wedge \sigma$, with ρ denoting the Ricci tensor and $\mathrm{s} = \mathrm{tr}_g \rho$ standing for the scalar curvature. Here $\wedge$ is the exterior multiplication of 1-forms valued in 1-forms, which involves the ordinary $\wedge$ as the valuewise multiplication; thus, $g \wedge \sigma$ is a 2-form valued in 2-forms.

Lemma 2.1 *For any essentially conformally symmetric manifold (M,g),*

(a) $R = W + (n-2)^{-1} g \wedge \rho$, *where* $n = \dim M \geq 4$,
(b) *the Ricci tensor ρ satisfies the Codazzi equation, in the sense that the three-times covariant tensor field $\nabla\rho$ is totally symmetric.*

Proof. In any essentially conformally symmetric manifold, $\mathrm{s} = 0$ identically [9, Theorem 7], so that $\sigma = \rho$. This gives (a), and (b) follows since the condition $\nabla W = 0$ implies vanishing of the divergence of W, which, in view of the second Bianchi identity, is equivalent to the Codazzi equation for σ, cf. [12, formula (5.29) on p. 460]. $\square$

Assertion (i) in the next lemma is due to Olszak [16]. Its proof can also be found in [11, Appendix I].

Lemma 2.2 *Let $\mathcal{D}$ be the Olszak distribution of an essentially conformally symmetric manifold (M,g), defined in the Introduction. Then*

(i) *$\mathcal{D}$ is a null parallel distribution of dimension* 1 *or* 2,
(ii) *at every point x, the space $\mathcal{D}_x$ contains the image of the Ricci tensor treated, with the aid of g_x, as a linear operator $T_xM \to T_xM$,*

(iii) $R(v,v',\,\cdot\,,\,\cdot\,) = W(v,v',\,\cdot\,,\,\cdot\,) = 0$ *whenever* v *and* v' *are sections of* $\mathcal{D}^\perp$,
(iv) *of the connections in the vector bundles* $\mathcal{D}$ *and* $\mathcal{E} = \mathcal{D}^\perp/\mathcal{D}$, *induced by the Levi-Civita connection of* g, *the latter is always flat, and the former is flat if* $\mathcal{D}$ *is one-dimensional.*

Proof. See [11, Lemmas 2.1(ii) and 2.2]. $\square$

For a k times covariant tensor field B on a pseudo-Riemannian manifold (M,g), $k \geq 1$, and a point $x \in M$, we denote by $\operatorname{Ker} B_x$ the subspace of T_xM formed by all vectors v with $B_x(v,\,\cdot\,,\ldots,\,\cdot\,) = 0$. Its orthogonal complement $(\operatorname{Ker} B_x)^\perp$ is the *image* of B_x, that is, the subspace of T_xM spanned by vectors $u \in T_xM$ such that $g(u,\,\cdot\,) = B(\,\cdot\,,u_2,\ldots,u_k)$ for some $u_2,\ldots,u_k \in T_xM$. If B is parallel, the spaces $\operatorname{Ker} B_x$ and $(\operatorname{Ker} B_x)^\perp$ form parallel distributions $\operatorname{Ker} B$ and $(\operatorname{Ker} B)^\perp$ on M.

Remark 2.3 Given an essentially conformally symmetric manifold (M,g) such that the Olszak distribution $\mathcal{D}$ is 2-dimensional, we have $W = \varepsilon\omega\otimes\omega$ for some $\varepsilon = \pm 1$ and a parallel differential 2-form ω with $\operatorname{rank}\omega = 2$, defined, at each point of M, only up to a sign. In addition,

$$\mathcal{D} \,=\, (\operatorname{Ker}\omega)^\perp, \tag{4}$$

In fact, if $x \in M$ and u,v,v' are vector fields chosen so that $u_x \in \mathcal{D}_x \smallsetminus \{0\}$ and $\varOmega_x \neq 0$, for $\varOmega = W(v,v',\,\cdot\,,\,\cdot\,)$, then the 2-form $\varOmega_x$ is $\wedge$-divisible by ξ_x, where $\xi = g(u,\,\cdot\,)$ (cf. the definition of $\mathcal{D}$). Thus, if $\mathcal{D}$ is 2-dimensional, the image of the Weyl tensor W_x acting on exterior 2-forms is spanned by $\xi\wedge\xi'$, where $\xi = g(u,\,\cdot\,)$ and $\xi' = g(u',\,\cdot\,)$ for any basis u,u' of $\mathcal{D}_x$. Since W_x acting on 2-forms is self-adjoint, our claim follows, (4) being immediate as $(\operatorname{Ker}\omega)^\perp$ is the image of ω.

Next, at any point x of any essentially conformally symmetric manifold,

$$\text{a)}\quad (\operatorname{Ker}\rho_x)^\perp \subset \mathcal{D}_x \subset \mathcal{D}_x^\perp \subset \operatorname{Ker}\rho_x, \qquad \text{b)}\quad \mathcal{D} \subset \operatorname{Ker} W, \tag{5}$$

where $\mathcal{D}$ is the Olszak distribution and ρ denotes the Ricci tensor. Namely, the first inclusion in (a) follows from Lemma 2.2(ii) (as $(\operatorname{Ker}\rho_x)^\perp$ is the image of ρ_x), the second from Lemma 2.2(i), and the third from the first. For (b), we consider the two possible values $d \in \{1,2\}$ of the dimension of $\mathcal{D}$ (see Lemma 2.2(i)). If $d = 2$, we have $\mathcal{D} = (\operatorname{Ker} W)^\perp$, that is, $\mathcal{D}$ equals the image of W (which coincides with the image $(\operatorname{Ker}\omega)^\perp$ of ω, cf. (4)), while $\mathcal{D} \subset \mathcal{D}^\perp$ by Lemma 2.2(i). Now let $d = 1$. As W and $\mathcal{D}^\perp$ are both parallel, it suffices to establish (b) at any fixed $x \in M$ with $\rho_x \neq 0$. (Note that g is not Ricci-flat.)

Now the image $(\operatorname{Ker}\rho_x)^\perp$ of ρ_x is contained in $\operatorname{Ker} W_x$, cf. [9, Theorem 8(c) on p. 22], while $\mathcal{D}_x = (\operatorname{Ker}\rho_x)^\perp$ by Lemma 2.2(ii), which yields (b).

3 Proof of Theorem A, first part

The phrase 'up to a factor' means, in this section, *up to a nonzero constant factor, which may depend on the dimensions involved.*

Given a pseudo-Euclidean inner product $\langle\,,\rangle$ in an oriented real vector space V of even dimension $r = 2m$, let Θ be the volume form, with $\Theta(e_1,\dots,e_r) = 1$ for any positive-oriented orthonormal basis $e_1,\dots,e_r$ of V. We denote by Pf the *Pfaffian* function of $\langle\,,\rangle$, assigning to an m-tuple of linear operators $S_j : V \to V$, which are all skew-adjoint relative to $\langle\,,\rangle$, the value $s = \operatorname{Pf}(S_1,\dots,S_m) \in \mathbf{R}$ such that $\zeta_1 \wedge \dots \wedge \zeta_m = s\Theta$, with the 2-forms ζ_j characterized by $\zeta_j(u,v) = \langle S_j u, v\rangle$ for all $u, v \in V$.

Lemma 3.1 *For S_j as above, $\operatorname{Pf}(S_1,\dots,S_m) = 0$ if $\bigcap_{j=1}^m \operatorname{Ker} S_j \neq \{0\}$.*

Proof. Our ζ_j are pullbacks to V of some 2-forms in the space V/V', where $V' = \bigcap_{j=1}^m \operatorname{Ker} S_j$. Hence $\zeta_1 \wedge \dots \wedge \zeta_m = 0$ if $\dim(V/V') < 2m$. □

Given an oriented real vector bundle $\mathcal{E}$ of fibre dimension $r \geq 1$ over a manifold M, let a pair (∇,γ) consist of a connection ∇ and a ∇-parallel pseudo-Riemannian fibre metric γ in $\mathcal{E}$. The *Euler form* of (∇,γ) then is the differential r-form on M equal to 0, when r is odd, and for even r obtained, up to a factor, by skew-symmetrization of the r times covariant tensor field $v_1,\dots,v_r$ to $\operatorname{Pf}(R^\nabla(v_1,v_2),\dots,R^\nabla(v_{r-1},v_r))$, cf. (2), with Pf as above for $V = \mathcal{E}_x$, $x \in M$.

The Euler form of (∇,γ) is closed, and represents in cohomology the real Euler class of the oriented bundle $\mathcal{E}$. See [2,6,14,4].

Similarly, the real Pontryagin classes $p_i(\mathcal{E}) \in H^{4i}(M,\mathbf{R})$ of a real vector bundle $\mathcal{E}$ over a manifold M are the cohomology classes of the Pontryagin forms of any connection ∇ in $\mathcal{E}$, given by explicit formulae involving the curvature tensor $R = R^\nabla$. To prove vanishing of the Pontryagin forms (and classes) under some specific assumptions, one may instead use what we call here the *generating forms*, the cohomology classes of which form another set of generators for the Pontryagin algebra (the subalgebra of $H^*(M,\mathbf{R})$ generated by all $p_i(\mathcal{E})$). The ith generating form of ∇, for any integer $i \geq 1$, is the differential $4i$-form on M obtained, up to a factor, as the skew-symmetrization of the $4i$ times covariant tensor field sending vector fields $v_1, \ldots, v_{4i}$ to $\mathrm{tr}\,[R^\nabla(v_1, v_2) \circ \ldots \circ R^\nabla(v_{4i-1}, v_{4i})]$, with $R^\nabla(v, v')$ as in (2). See [7].

In the case where ∇ is the Levi-Civita connection of a pseudo-Riemannian manifold (M, g) and $\mathcal{E} = TM$, we speak of the *Euler form* and *generating forms of* (M, g).

Theorem A (minus the claim about $\pi_1 M$) is immediate from the following local result.

Lemma 3.2 *The Euler form and all Pontryagin forms of any oriented essentially conformally symmetric manifold* (M, g) *are identically zero.*

Proof. Given a point $x \in M$ and an integer i with $1 \le i \le n/4$, where $n = \dim M$, we set $B_j = W_x(u_{2j-1}, u_{2j}) : T_xM \to T_xM$ for linearly independent vectors $u_1, \ldots, u_{4i}$ in T_xM, and $b = \mathrm{tr}\,(B_1 \circ \ldots \circ B_{2i})$. (Notation as in (2), with the Weyl tensor W instead of R.) For vanishing of the Pontryagin forms, it suffices to prove that $b = 0$, since, as shown by Avez [3], in the definition of generating forms of (M, g) one may replace the curvature tensor $R = R^\nabla$ by W. We are thus allowed to choose x at which the Ricci tensor ρ_x is nonzero: W is parallel, and hence so is the ith generating form. For any fixed $v' \in T_xM$ with $\rho_x(v', \cdot\,) \ne 0$, Lemma 2.2(ii) implies that each B_j, when treated (with the aid of g_x) as a 2-form at x, is $\wedge$-divisible by $\rho_x(v', \cdot\,)$. In other words, $B_j : T_xM \to T_xM$ equals $g_x(w_j, \cdot\,)v - \rho_x(v', \cdot\,)w_j$ for some $w_j \in T_xM$ and the unique $v \in T_xM$ with $g_x(v, \cdot\,) = \rho_x(v', \cdot\,)$. Furthermore, $\mathcal{D}_x \subset \mathrm{Ker}\, W_x \subset \mathrm{Ker}\, B_j$ and $\mathcal{D}_x \subset \mathrm{Ker}\, \rho_x$ (see (5)), so that, by Lemma 2.2(ii), $v \in \mathcal{D}_x$ and $w_j \in \mathcal{D}_x^\perp$ (as w_j lies in the image $(\mathrm{Ker}\, B_j)^\perp$ of B_j). As $\mathcal{D}$ is null (Lemma 2.2(i)), $B_1 \circ B_2 = -g_x(w_1, w_2)\rho_x(v', \cdot\,)v$, $\mathrm{tr}\,(B_1 \circ B_2) = 0$ and $B_1 \circ B_2 \circ B_3 = 0$ if $i > 1$, which implies that $b = 0$ both for $i = 1$ and $i > 1$.

Now let n be even. Given $x \in M$, we set $s = \mathrm{Pf}\,(S_1, \ldots, S_m)$, where $m = n/2$ and $S_j = R_x(e_{2j-1}, e_{2j}) : T_xM \to T_xM$ for a basis $e_1, \ldots, e_n$ of T_xM containing a basis of $\mathcal{D}_x^\perp$ (where $\mathcal{D}$ is the Olszak distribution). To obtain vanishing of the Euler form, we need to show that $s = 0$. First, $s = 0$ if, for each $j \in \{1, \ldots, m\}$, at least one of the vectors e_{2j-1}, e_{2j} lies in $\mathcal{D}_x^\perp$. In fact, by Lemma 2.1(a), $R_x(v, u', u, \cdot\,) = 0$ whenever $u \in \mathcal{D}_x$, $v \in \mathcal{D}_x^\perp$ and $u' \in T_xM$, as $\mathcal{D}_x \subset \mathrm{Ker}\, W_x$ and $(g \wedge \rho)_x(v, u', u, \cdot\,) = 0$ in view of (5); thus, $\bigcap_{j=1}^m \mathrm{Ker}\, S_j$ contains the subspace $\mathcal{D}_x \ne \{0\}$ (cf. Lemma 2.2(i)), and Lemma 3.1 shows that $s = 0$.

In the remaining case, $e_{2j-1}, e_{2j} \in \mathcal{D}_x^\perp$ for some $j \in \{1, \ldots, m\}$. Namely, if d denotes the dimension of $\mathcal{D}$, then $\mathcal{D}^\perp$ is $(2m - d)$-dimensional, with $d \le 2 \le m$ by Lemma 2.2(i). Among $e_1, \ldots, e_{2m}$ there are $2m - d \ge m$ elements of $\mathcal{D}_x^\perp$, so that one of the m sets $\Sigma_j = \{e_{2j-1}, e_{2j}\}$ must be contained in $\mathcal{D}_x^\perp$ (or else $\Sigma_j \cap \mathcal{D}_x^\perp$ would, for each $j = 1, \ldots, m$, have exactly one element, leading to a case which we already excluded).

Now that $e_{2j-1}, e_{2j} \in \mathcal{D}_x^\perp$ for some j, Lemma 2.2(iii) gives $S_j = 0$ and, consequently, $s = 0$. $\square$

4 Proof of Theorem D

In any essentially conformally symmetric manifold (M, g) such that the Olszak distribution $\mathcal{D}$ is one-dimensional, setting $\mathcal{E} = \mathcal{D}^\perp/\mathcal{D}$, one has a vector-bundle morphism

$$\varPhi : (\mathcal{D}^*)^{\otimes 2} \to (\mathcal{E}^*)^{\otimes 2} \tag{6}$$

defined as follows. We declare $\varPhi_x(\lambda \otimes \lambda') : \mathcal{E}_x \times \mathcal{E}_x \to \mathbf{R}$, for $x \in M$ and $\lambda, \lambda' \in \mathcal{D}_x^*$, to be the symmetric bilinear form sending the cosets $v + \mathcal{D}_x$ and $v' + \mathcal{D}_x$ of vectors $v, v' \in \mathcal{D}_x^\perp$ to $W_x(v, u, u', v')$, where $u, u' \in T_xM$ are any vectors with $\lambda = g_x(u, \cdot\,)$ and $\lambda' = g_x(u', \cdot\,)$ on $\mathcal{D}_x$. Note that, as $\mathcal{D}_x \subset \operatorname{Ker} W_x$ by (5.b), the value $W_x(v, u, u', v')$ depends just on the $\mathcal{D}_x$-cosets, rather than the vectors v, v' themselves, while, by Lemma 2.2(iii), $W_x(v, u, u', v')$ is not affected by how u and u' were chosen: two such choices of either vector differ by an element of $\mathcal{D}_x^\perp$.

Remark 4.1 If (M, g) is essentially conformally symmetric and $\mathcal{D}$ is one-dimensional, the Levi-Civita connection of g induces flat connections in both bundles $(\mathcal{D}^*)^{\otimes 2}$ and $(\mathcal{E}^*)^{\otimes 2}$ (by Lemma 2.2(iv)), and the morphism $\varPhi$ is

(a) parallel relative to those connections,
(b) nonzero, and hence injective, at every point $x \in M$.

Here (a) states that $\varPhi$, viewed as a section of the bundle $\operatorname{Hom}((\mathcal{D}^*)^{\otimes 2}, (\mathcal{E}^*)^{\otimes 2})$, is parallel relative to the induced flat connection, or, equivalently, that the $\varPhi$-image of any parallel local section of $(\mathcal{D}^*)^{\otimes 2}$ is parallel in $(\mathcal{E}^*)^{\otimes 2}$.

Assertion (a) is obvious from naturality of $\varPhi$, since W is parallel. To verify (b), note that if we had $W_x(v, u, u, v') = 0$ for a fixed $u \in T_xM \smallsetminus \mathcal{D}_x^\perp$ and all $v, v' \in \mathcal{D}_x^\perp$, the components of W_x in a basis consisting of u and a basis of $\mathcal{D}_x^\perp$ would vanish (Lemma 2.2(iii)), even though (M, g) is not conformally flat.

Remark 4.2 Given an essentially conformally symmetric manifold (M, g), let $\mathcal{D}$ and $\mathcal{E}$ be as in (6). Since $\mathcal{D}$ is the g-nullspace subbundle of $\mathcal{D}^\perp$ (cf. Lemma 2.2(i)), the metric g, restricted to $\mathcal{D}^\perp$, descends to a pseudo-Riemannian fibre metric γ on $\mathcal{E}$. Clearly, γ is parallel relative to the connection induced by the Levi-Civita connection ∇ of g. Being ∇-parallel, $\mathcal{D}^\perp$ is integrable and has totally geodesic leaves, and the Levi-Civita connection of g induces on each leaf a torsionfree connection, which is flat in view of Lemma 2.2(iii).

If the sign pattern of g is (i_-, i_+), with i_- minuses and i_+ pluses, then γ has the sign pattern $(i_- - d, i_+ - d)$, where d the dimension of the distribution $\mathcal{D}$. This is clear if one chooses a subspace V of $\mathcal{D}_x^\perp$ with $\mathcal{D}_x^\perp = \mathcal{D}_x \oplus V$, for any $x \in M$, and notes that $T_xM = V \oplus V^\perp$, while V and $V^\perp$ have the sign patterns equal to that of γ and, respectively, (d,d) (as $\mathcal{D}_x \subset V^\perp$).

We can now prove Theorem D. Let (M,g) be essentially conformally symmetric and Lorentzian, and let d be the dimension of $\mathcal{D}$. Then $d = 1$ by Lemma 2.2(i), since $d \le 1$ due to the Lorentzian sign pattern $-+\ldots+$. (In fact, T_xM contains $V \oplus \mathcal{D}_x$, for any $x \in M$ and any codimension-one subspace $V \subset T_xM$ on which g_x is positive definite.)

The parallel injective morphism $\varPhi$ in (6) now gives rise to a fibre norm $|\,|$ in the line bundle $\mathcal{D}$, which is parallel (invariant under parallel transports). Namely, for $x \in M$ and $u \in \mathcal{D}_x \smallsetminus \{0\}$, we set $|u| = |\varPhi_x(\lambda \otimes \lambda)|^{-1/2}$, where $\lambda \in \mathcal{D}_x^*$ is chosen so that $\lambda(u) = 1$, and the latter $|\,|$ is the fibre norm in $(\mathcal{E}^*)^{\otimes 2}$ corresponding to the fibre metric γ in $\mathcal{E}$. Note that γ is positive definite as $d = 1$ (see Remark 4.2). Since a $|\,|$-unit section of $\mathcal{D}$ is parallel, this proves Theorem D.

5 Proof of Theorem B

Let an essentially conformally symmetric manifold (M,g) satisfy one of the following two conditions:

(i) M is simply connected and the Olszak distribution $\mathcal{D}$ is one-dimensional,
(ii) g is Lorentzian and $\mathcal{D}$ is trivial as a real line bundle, cf. Theorem D.

Then there exist functions ψ, ϕ and a vector field u on M such that

(a) u is parallel, nonzero, and spans $\mathcal{D}$,
(b) the 1-form $\xi = g(u,\,\cdot\,)$ is parallel, the Ricci tensor ρ equals $\psi\,\xi \otimes \xi$, and $d\psi = \phi\,\xi$,
(c) ϕ is nonconstant if M is compact.

Still assuming (i) or (ii), we define a vector-bundle morphism $A : \mathcal{E} \to \mathcal{E}$ by requiring that $\gamma_x(A_x\eta,\,\cdot\,) = [\varPhi_x(\lambda \otimes \lambda)](\eta,\,\cdot\,)$ for $x \in M$ and $\eta \in \mathcal{E}_x$ with $\lambda \in \mathcal{D}_x^*$ such that $\lambda(u_x) = 1$. (Notation of Lemma 2.2(iv), Remark 4.2 and (6).) Then

(d) A is ∇-parallel as a section of $\mathrm{Hom}(\mathcal{E},\mathcal{E})$, nonzero, self-adjoint relative to γ and traceless at every point,

(e) in the case where $A_x : \mathcal{E}_x \to \mathcal{E}_x$ has $n-2 = \dim \mathcal{E}_x$ distinct eigenvalues at some/every point $x \in M$, for $n = \dim M \geq 4$, the bundle $\mathcal{E}$ over M is an orthogonal direct sum of ∇-parallel real-line subbundles.

Under the assumption (i), there exists a function $t : M \to \mathbf{R}$ such that, for $\xi = g(u, \cdot\,)$,

(f) $\xi = dt$ (or, equivalently, $u = \nabla t$) and ψ is, locally, a function of t.

In fact, u in (a) exists in view of Lemma 2.2(iv) and Theorem D, while (b), for some ψ and ϕ, follows from Lemma 2.2(ii), since $\nabla\rho = d\psi \otimes \xi \otimes \xi$ is totally symmetric by Lemma 2.1(b). Now, if ϕ were constant and M compact, $d\psi$ would be parallel (as ξ is), and so, being zero somewhere, $d\psi$ would vanish identically. However, ψ is nonconstant, since ρ cannot be parallel: g is conformally symmetric but not locally symmetric. This contradiction proves (c). Next, (d) holds in view of Remark 4.1 and Theorem D, with tracelessness of A due to vanishing of the contractions of W. Assertion (e) is now immediate, the subbundles in question being the eigenspace bundles of A. Finally, ξ is parallel, and hence closed, so that (b) implies (f).

We can now prove Theorem B. Let (M, g) be a compact essentially conformally symmetric Lorentzian manifold. Theorem D allows us to assume that (M, g) admits a global parallel vector field u spanning the one-dimensional null parallel distribution $\mathcal{D}$. Condition (ii) above is therefore satisfied, which implies (b), while the function ϕ in (b) is nonconstant by (c). The assertion of Theorem B is thus immediate from Lemmas 1.2 and 2.2(iii).

Remark 5.1 Let (M, g) be any compact essentially conformally symmetric Lorentzian manifold such that the Olszak distribution $\mathcal{D}$ is trivial as a real line bundle. Choosing ξ, ϕ, ψ as in (a) – (c), and t as in (f) (where, for t to exist, we use instead of (M, g) its universal covering manifold $(\widehat{M}, \widehat{g})$), we see that ξ, ϕ, ψ and t satisfy all the hypotheses of Lemma 1.2. Consequently, *they satisfy the conclusions of Lemma* 1.2 *as well.* This proves the claims immediately following Theorem C in the Introduction.

6 Examples

Suppose that we are given a nonconstant C^∞ function $f : \mathbf{R} \to \mathbf{R}$, a real vector space V of dimension $n-2 \geq 2$ with a pseudo-Euclidean inner product $\langle\,,\rangle$, and a nonzero traceless linear operator $A : V \to V$, self-adjoint relative to $\langle\,,\rangle$. Following [17], we use such data to define a pseudo-Riemannian metric

$\widehat{g} = \kappa\,dt^2 + dt\,ds + h$ on the manifold $\widehat{M} = \mathbf{R}^2 \times V$, diffeomorphic to $\mathbf{R}^n$, where products of differentials stand for symmetric products, t, s are the Cartesian coordinates on the $\mathbf{R}^2$ factor, h denotes the pullback to $\widehat{M}$ of the flat pseudo-Riemannian metric on V corresponding to the inner product $\langle\,,\rangle$, and $\kappa : \widehat{M} \to \mathbf{R}$ is given by $\kappa(t,s,v) = f(t)\langle v,v\rangle + \langle Av,v\rangle$.

Let $\mathcal{E}$ be the vector space of all C^∞ solutions $u : \mathbf{R} \to V$ to the differential equation $\ddot{u}(t) = f(t)u(t) + Au(t)$, and let $\mathbf{P}$ be the additive group of all $p \in \mathbf{R}$ with $f(t+p) = f(t)$ for every real t. The set $\mathrm{G} = \mathbf{P} \times \mathbf{R} \times \mathcal{E}$ has a unique group structure such that, for $(p,q,u) \in \mathrm{G}$ and $(t,s,v) \in \widehat{M} = \mathbf{R}^2 \times V$, the formula $(p,q,u)\cdot(t,s,v) = (t+p,\, s+q-\langle\dot{u}(t), 2v+u(t)\rangle,\, v+u(t))$ describes a group action of G on $\widehat{M}$. See [10, Section 2].

Lemma 6.1 *For any choice of the above data $f, n, V, \langle\,,\rangle$ and A,*

(i) *the metric $\widehat{g}$ is essentially conformally symmetric,*
(ii) *the sign pattern of $\widehat{g}$ arises from that of $\langle\,,\rangle$ by adding one plus and one minus,*
(iii) *the group G acts on $(\widehat{M},\widehat{g})$ by isometries,*
(iv) *if $n = 4$ and the metric $\widehat{g}$ is Lorentzian,*
   a) G is a subgroup of finite index in the full isometry group of $(\widehat{M},\widehat{g})$,
   b) $(\widehat{M},\widehat{g})$ is not the universal covering space of any compact pseudo-Riemannian manifold.

Proof. For (i), see [17, Theorem 3] or [10, Lemma 2.1], while (ii) is obvious, and (iii) is immediate from [10, Lemma 2.2].

Generally, the index $\mathrm{ind}(\mathrm{G}',\mathrm{G})$ of a subgroup G in a group G' is the cardinality of the quotient set G'/G consisting of all left cosets of G in G'. If $\mathrm{H}' \subset \mathrm{G}'$ is a subgroup such that $\mathrm{G}' = \mathrm{H}'\mathrm{G}$, then, for $\mathrm{H} = \mathrm{G} \cap \mathrm{H}'$, the inclusion mapping $\mathrm{H}' \to \mathrm{G}'$ clearly induces a bijection $\mathrm{H}'/\mathrm{H} \to \mathrm{G}'/\mathrm{G}$, and so $\mathrm{ind}(\mathrm{G}',\mathrm{G}) = \mathrm{ind}(\mathrm{H}',\mathrm{H})$. Here are two special cases in which groups G', G and H satisfy the above assumption, and hence the conclusion:

(I) $\mathrm{H}' = \mathrm{Ker}\,\Pi$ for a group homomorphism $\Pi : \mathrm{G}' \to \mathrm{K}$ with $\Pi(\mathrm{G}) = \mathrm{K}$,
(II) G' acts from the left on a set Y, the action restricted to G is transitive, and H' is the isotropy subgroup of G' at some $y \in Y$.

Let $\widehat{g}$ now be Lorentzian, and let G' denote the group of those isometries of $(\widehat{M},\widehat{g})$ which preserve the 1-form dt. The Ricci tensor of $\widehat{g}$ is given by $\rho = (2-n)f(t)\,dt \otimes dt$, and dt is parallel. (See [17, p. 93], where the sign convention for ρ is the opposite of ours.) Thus, by Lemma 2.2(ii), the Olszak distribution $\mathcal{D}$ is spanned by the null parallel vector field $u = \nabla t$. Since u can be naturally normalized with the aid of a parallel fibre norm in $\mathcal{D}$ (see the end of Section 4), isometries of $(\widehat{M},\widehat{g})$ leave dt invariant up to a sign, so that the full isometry group of $(\widehat{M},\widehat{g})$ contains G' as a subgroup of index at most 2, and (iv-a) will follow if we show that $\mathrm{ind}(\mathrm{G}',\mathrm{G})$ is finite.

As G' preserves dt and $\rho=(2-n)f(t)\,dt\otimes dt$, we have $t\circ\alpha=t+\mathit{\Pi}(\alpha)$ for some homomorphism $\mathit{\Pi}:\mathrm{G}'\to\mathbf{R}$ and all $\alpha\in\mathrm{G}'$, while the function $f(t):\widehat{M}\to\mathbf{R}$ is G'-invariant. Thus, $\mathit{\Pi}(\mathrm{G}')$ coincides with the additive group $\mathbf{P}$ defined earlier in this section, and so $\mathrm{ind}(\mathrm{G}',\mathrm{G})=\mathrm{ind}(\mathrm{Ker}\,\mathit{\Pi},\mathrm{G}\cap\mathrm{Ker}\,\mathit{\Pi})$, in view of (I) above for $\mathrm{K}=\mathbf{P}$ and $\mathit{\Pi}:\mathrm{G}'\to\mathbf{P}$. Next, for any fixed $t\in\mathbf{R}$, the action of $\mathrm{Ker}\,\mathit{\Pi}$ leaves the affine subspace $\widehat{M}_t=\{t\}\times\mathbf{R}\times V$ of $\widehat{M}$ invariant, and the restriction of this action to $\mathrm{G}\cap\mathrm{Ker}\,\mathit{\Pi}$ is transitive on $\widehat{M}_t$, since it consists of affine transformations realizing all translational parts in $\{0\}\times\mathbf{R}\times V$. Consequently, (II) gives $\mathrm{ind}(\mathrm{G}',\mathrm{G})=\mathrm{ind}(\mathrm{H}',\mathrm{H})$, where H' is the isotropy subgroup of G' at any fixed $x\in\widehat{M}$, and $\mathrm{H}=\mathrm{G}\cap\mathrm{H}'$.

On the other hand, $\mathrm{ind}(\mathrm{H}',\mathrm{H})\le 4$ if $n=4$. Namely, the infinitesimal action of H' on $T_x\widehat{M}=\mathbf{R}^2\times V$ preserves dt_x and the vector $u_x=(0,1/2,0)$, along with the subspace $u_x^\perp=\mathcal{D}_x^\perp$. Hence the action descends to $\mathcal{E}_x=\mathcal{D}_x^\perp/\mathcal{D}_x$, where it preserves the inner product γ_x (see Remark 4.2) and commutes with the operator $A_x:\mathcal{E}_x\to\mathcal{E}_x$ which, under the obvious identification $\mathcal{E}_x\approx V$, coincides with our $A:V\to V$. (See [17, the description of W on p. 93].) The differentials at x of elements of H', acting on $T_x\widehat{M}=\mathbf{R}^2\times V$, thus have the form $(t,s,v)\mapsto(t,s+\varphi(v),Lv)$, where $L:V\to V$ is a linear isometry commuting with A, and $\varphi\in V^*$. As elements of the subgroup H realize all $\varphi\in V^*$, and have $L=\mathrm{Id}$, it follows that $\mathrm{ind}(\mathrm{H}',\mathrm{H})\le 4$ (see Remark 6.2 below), which yields (iv-a).

Finally, to prove (iv-b), we may suppose that, on the contrary, some group Γ of isometries of $(\widehat{M},\widehat{g})$ acts on $\widehat{M}$ properly discontinuously, producing a compact quotient manifold. The same is then true for the subgroup $\Gamma\cap\mathrm{G}$ of Γ (as $\Gamma\cap\mathrm{G}$ is of finite index in Γ, by (iv-a)), which in turn contradicts [10, Theorem 7.3]. Note that periodicity of f as a function of t, required in [10], follows from Lemma 1.2(b), cf. Remark 5.1. $\square$

Remark 6.2 For a nonzero traceless self-adjoint linear endomorphism A of a pseudo-Euclidean plane V, there may exist at most four linear isometries $L:V\to V$ commuting with A. This is clear when A is diagonalizable, since L must then send an orthonormal basis (v,w) diagonalizing A to $(\pm v,\pm w)$ or $(\pm v,\mp w)$. On the other hand, if a linear isometry L commutes with A and A is non-diagonalizable (so that V is Lorentzian), we have $L=\pm\mathrm{Id}$. In fact, let $L\ne\pm\mathrm{Id}$. The two null lines in V are interchanged by L (if they were preserved, L would be diagonalizable, implying the same for A). However, choosing a basis (v,w) of null vectors with $Lv=w$ and $Lw=v$, we would then again diagonalize L (and hence A), this time with the eigenvectors $v\pm w$.

7 A classification theorem

In the following theorem, t denotes any fixed function $\widehat{M} \to \mathbf{R}$ such that $u = \nabla t$ is a global parallel vector field spanning the Olszak distribution $\mathcal{D}$. Such t exists according to (f) in Section 5, cf. Theorem D.

Theorem 7.1 *Let* $(\widehat{M}, \widehat{g})$ *be a simply connected essentially conformally symmetric Lorentzian manifold of dimension* $n \geq 4$ *such that the leaves of the parallel distribution* $\mathcal{D}^\perp$ *are all complete and the function* $t : \widehat{M} \to \mathbf{R}$ *satisfies condition* (c) *in Lemma* 1.2. *Then, up to an isometry,* $(\widehat{M}, \widehat{g})$ *is one of the manifolds constructed in Section* 6*.*

Our proof of Theorem 7.1, given in Section 9, uses the facts presented below.

Let $(\widehat{M}, \widehat{g})$ be a simply connected essentially conformally symmetric manifold such that the Olszak distribution $\mathcal{D}$ is one-dimensional. If v, v' are sections of $\mathcal{D}^\perp$ and u is a fixed nonzero parallel section of $\mathcal{D}$, while u' is any vector field, then

$$R(u', v)v' \;=\; g(u', u)[\gamma(A\underline{v}, \underline{v}') + fg(v, v')]u, \tag{7}$$

where $f : \widehat{M} \to \mathbf{R}$ is given by $f = (2 - n)^{-1}\psi$, for $n = \dim \widehat{M} \geq 4$, with ψ and A defined as in Section 5, and $\underline{v}$ denotes the image of v under the quotient-projection morphism $\mathcal{D}^\perp \to \mathcal{E} = \mathcal{D}^\perp/\mathcal{D}$. (Thus, f is characterized by $\rho = (2 - n)f\,\xi \otimes \xi$, for the parallel 1-form $\xi = g(u, \cdot\,) = dt$.)

In fact, by Lemma 2.2(iii), $R(u', v)v'$ is orthogonal to $\mathcal{D}^\perp$, and hence equals a function times u. As both sides of (7) are linear in u', (7) will follow if, under the assumption $g(u', u) = 1$, applying $g(u', \cdot\,)$ to both sides we obtain the same value. This last conclusion is in turn immediate from Lemma 2.1(a) combined with the definition of A in Section 5, since $\rho = (2 - n)f\,\xi \otimes \xi$. (By (5.a), $\rho(v, \cdot\,) = \rho(v', \cdot\,) = 0$.)

Remark 7.2 Let $(\widehat{M}, \widehat{g})$ satisfy the assumptions of Theorem 7.1. We say that a curve $I \to \widehat{M}$, defined on an interval $I \subset \mathbf{R}$, is *parametrized by the function* $t : \widehat{M} \to \mathbf{R}$ (chosen at the beginning of this section) if t sends the image of the curve diffeomorphically onto I. Such a curve may be written as $I \ni t \mapsto y(t) \in \widehat{M}$, with t serving as the parameter.

- (a) Up to an affine re-parametrization, every geodesic in $(\widehat{M}, \widehat{g})$, not tangent to the distribution $\mathcal{D}^\perp$, is parametrized by the function t.
- (b) If a curve $\mathbf{R} \ni t \mapsto y(t) \in \widehat{M}$ is parametrized by the function t, then so is every curve $t \mapsto x(t, s)$ in the variation given by $x(t, s) = \exp_{y(t)} sw(t)$, where $\mathbf{R} \ni t \mapsto w(t) \in \mathcal{D}^\perp_{y(t)}$ is any vector field along the original curve,

tangent to $\mathcal{D}^\perp$.

(c) For any curve $\mathbf{R} \ni t \mapsto y(t) \in \widehat{M}$ is parametrized by the function t, we have $g(\dot y, u) = 1$, for $u = \nabla t$, and $\nabla_{\dot y}\dot y$ is tangent to $\mathcal{D}^\perp$.

In fact, (a) and (b) follow since $\nabla dt = 0$, so that t restricted to any geodesic is an affine function of the geodesic parameter, while the leaves of $\mathcal{D}^\perp$ are totally geodesic (Remark 4.2), and t is constant on each leaf as $\mathcal{D}^\perp = \operatorname{Ker} dt$. Now (c) is immediate as $\mathcal{D}^\perp = u^\perp$ and $u = \nabla t$ is parallel: differentiating $g(\dot y, u) = 1$, we get $g(\nabla_{\dot y}\dot y, u) = 0$.

The following lemma is a crucial step in proving Theorem 7.1.

Lemma 7.3 *Under the assumptions of Theorem* 7.1, $(\widehat{M}, \widehat{g})$ *is complete.*

Proof. Using (c) in Lemma 1.2, we may fix a curve $\mathbf{R} \ni t \mapsto y(t) \in \widehat{M}$ parametrized by the function t (cf. Remark 7.2), and consider the differential equation

$$\nabla_{\dot y}\nabla_{\dot y} w + R(\dot y, w)\dot y + \nabla_{\dot y}\dot y \;=\; -Q(w)u/4 \tag{8}$$

imposed on vector fields w along the curve which are tangent to $\mathcal{D}^\perp$. Here $u = \nabla t$ and $Q(w) = 3[\gamma(A\underline{w}, \underline{w})]^{\cdot} + 3f\,[g(w,w)]^{\cdot} + 2\dot f g(w,w)$, with $f, A, \underline{w}$ as in (7), and $[\;]^{\cdot} = d/dt$. (Thus, $\dot f = df(y(t))/dt$.) By Remark 7.2(c), both sides of (8) are tangent to $\mathcal{D}^\perp$, that is, orthogonal to u, as $\nabla u = 0$ and so $R(\,\cdot\,,\,\cdot\,,\,\cdot\,, u) = 0$.

Every solution w to (8) can be defined on the whole real line. Namely, this is true, due to linearity, for solutions $\widetilde{w}$ (tangent to $\mathcal{D}^\perp$) of the equation $\nabla_{\dot y}\nabla_{\dot y}\widetilde{w} + R(\dot y, \widetilde{w})\dot y + \nabla_{\dot y}\dot y = 0$. Using any such $\widetilde{w}$ and any function $\mu : \mathbf{R} \to \mathbf{R}$ with the second derivative $\ddot\mu = Q(\widetilde{w})/4$, we now get a solution $w = \widetilde{w} - \mu u$ to (8), defined on $\mathbf{R}$. (Note that $Q(w) = Q(\widetilde{w})$, and $R(\dot y, u)\dot y = 0$ since $\nabla u = 0$.)

Any solution w to (8), defined on $\mathbf{R}$, leads to the variation of curves in $\widehat{M}$ given by $x(t,s) = \exp_{y(t)} sw(t)$. Let v be the vector field along the variation such that $v_s = 0$ for all (t,s), and $v = \nabla_{\dot y}\dot y$ at $s = 0$ (notation as in (3)). We have

$$\text{i)}\;\; x_{tt} + (s-1)[v - sQ(x_s)u/4] = 0, \qquad \text{ii)}\;\; [Q(x_s)]_s = 0. \tag{9}$$

where the subscripts now also stand for partial derivatives of functions of (t,s), and $Q(x_s) = 3[\gamma(A\underline{x_s}, \underline{x_s})]_t + 3f\,[g(x_s,x_s)]_t + 2f_t g(x_s,x_s)$. Before proving (9), note that

$$
\begin{aligned}
&\text{a)}\quad g(x_t,u)=1\,, \qquad \text{b)}\quad x_{ss}=0\,,\\
&\text{c)}\quad x_s, x_{tt}\ \text{ and }\ x_{st}=x_{ts}\ \text{ are all tangent to }\ \mathcal{D}^\perp.
\end{aligned}
\tag{10}
$$

In fact, (10.a) and the claim about x_{tt} in (10.c) are immediate from clauses (b), (c) in Remark 7.2. Next, $x_{ss}=0$ and x_s is tangent to $\mathcal{D}^\perp$, as $\mathcal{D}^\perp$ has totally geodesic leaves (Remark 4.2), while the curves $s\mapsto x(t,s)$ are geodesics, tangent to $\mathcal{D}^\perp$ at $s=0$. Finally, x_{st} is tangent to $\mathcal{D}^\perp$, since so is x_s and $\mathcal{D}^\perp$ is parallel.

Furthermore, for the covariant derivatives R_t, R_s of the curvature tensor,

$$
\text{a)}\ \ R_t(x_t,x_s)x_s=f_t\,g(x_s,x_s)\,u\,, \qquad \text{b)}\ \ R_s=0\,, \qquad \text{c)}\ \ f_s=0\,. \tag{11}
$$

Namely, by (f) in Section 5, f is a function of t, while $t_s=0$ as the curves $s\mapsto x(t,s)$ are tangent to $\mathcal{D}^\perp=\operatorname{Ker} dt$, and (11.c) follows. In view of the frelations $\rho=(2-n)f(t)\,dt\otimes dt$, $\nabla W=0$ and $\nabla dt=0$, equalities (11.a) and (11.b) are immediate from (11.c) and Lemma 2.1(a).

We can now prove (9). Relation (9.ii) is obvious from (11.c), as g,γ and A are parallel, so that (10.b) yields $[\gamma(A\underline{x_s},\underline{x_s})]_s=[g(x_s,x_s)]_s=0$. Denoting by $\widetilde{v}$ the left-hand side of (9.i), we get $\widetilde{v}=\widetilde{v}_s=0$ at $s=0$ (from (3.a), (8) and (9.ii) with $u_s=v_s=0$). Finally, $\widetilde{v}_{ss}=0$ for all (t,s), since (3.b) and (7) give $x_{ttss}=Q(x_s)u/2$. More precisely, according to (3.b) with $x_{ss}=0$ (see (10.b)), x_{ttss} equals the sum of three curvature terms, so that, using the Leibniz rule with $x_{ts}=x_{st}$, we obtain $x_{ttss}=3R(x_t,x_s)x_{st}+R_t(x_t,x_s)x_s$, all the other terms being zero as a consequence of Lemma 2.2(iii) combined with (10.b,c), and (11.b). Using (7) with (10.a,c) and (11.a), we now get $x_{ttss}=Q(x_s)u/2$ and $\widetilde{v}_{ss}=0$.

Thus, $\widetilde{v}_s$ must be identically zero, as it is parallel in the s direction and vanishes at $s=0$. For the same reason, $\widetilde{v}=0$ for all (t,s), which yields (9.i).

By (9.i), $x_{tt}=0$ when $s=1$, so that the curve $t\mapsto x(t,1)$ is a geodesic defined on $\mathbf{R}$. Such geodesics realize all initial conditions $(x,\dot{x})$ with the velocities $\dot{x}$ for which $g_x(u_x,\dot{x})=1$ (the normalization being due to the fact that they are parametrized by the function t, cf. Remark 7.2). Namely, we can realize $(x,\dot{x})$ by the curve $t\mapsto y(t)$ chosen above, and then use the solution w to (8) with the zero initial conditions. $\square$

8 Proof of Theorem 7.1

Suppose that $(\widehat{M}, \widehat{g})$ satisfies the assumptions of Theorem 7.1. The data required for the construction in Section 6 can be introduced as follows. For u chosen at the beginning of Section 7, we define f as in the lines following (7). According to (f) in Section 5, f is, locally, a function of t. The word 'locally' may be dropped as we are assuming condition (c) in Lemma 1.2, and hence the level sets of t are connected. Next, we let V be the space of all parallel sections of $\mathcal{E}$, so that $\dim V = n-2$ by Lemma 2.2(iv). Finally, the pseudo-Euclidean inner product $\langle\,,\rangle$ in V and $A : V \to V$ are the objects induced, in an obvious manner, by the fibre metric γ on $\mathcal{E}$ and the bundle morphism $A : \mathcal{E} \to \mathcal{E}$, both of which are parallel (see Remark 4.2 and (d) in Section 5).

We now fix a null geodesic $\mathbf{R} \ni t \mapsto x(t) \in \widehat{M}$ parametrized by the function t, which exists in view of Lemma 7.3 and Remark 7.2(a). As $g(\dot{x}, u) = 1$ (see Remark 7.2(c)), the plane P in $T_{x(0)}\widehat{M}$ spanned by the null vectors $\dot{x}(0)$ and $u_{x(0)}$ is g_x-nondegenerate, with the sign pattern $-+$, so that $T_{x(0)}\widehat{M} = P \oplus \widetilde{V}$, for $\widetilde{V} = P^\perp$.

We define a mapping $F : \mathbf{R}^2 \times V \to \widehat{M}$ by $F(t,s,v) = \exp_{x(t)}(\tilde{v}(t) + s u_{x(t)}/2)$, for the parallel vector field $t \mapsto \tilde{v}(t) \in T_{x(t)}\widehat{M}$ with $\tilde{v}(0) = \mathrm{pr}\, v_{x(0)}$, where pr is the orthogonal projection $T_{x(0)}\widehat{M} \to \widetilde{V}$ (and so pr, restricted to $\mathcal{D}_x^\perp$ for $x = x(0)$, descends to the quotient $\mathcal{E}_x = \mathcal{D}_x^\perp/\mathcal{D}_x$, forming an isomorphism $\mathcal{E}_x \to \widetilde{V}$).

That F is a diffeomorphism can be seen as follows. The manifold N in Lemma 1.2(c) is simply connected, since so is $\widehat{M}$. Therefore, each leaf of $\mathcal{D}^\perp$ (level set of t), with its complete flat torsionfree connection (Remark 4.2), is the diffeomorphic image of its tangent space at any point under the exponential mapping, cf. Remark 1.3.

Finally, according to [11, Lemma 5.1] $F^*\widehat{g}$ coincides with the metric $\kappa\, dt^2 + dt ds + h$ on $\mathbf{R}^2 \times V$, constructed from the data described above as in Section 6.

9 Proof of Theorem C

In the following lemma, the bundle morphism $A : \mathcal{E} \to \mathcal{E}$ defined in Section 5 makes sense even without assuming that the line bundle $\mathcal{D}$ is trivial. In fact, A depends quadratically on our fixed nonzero parallel section u of $\mathcal{D}$, which, for nontrivial $\mathcal{D}$, is still well-defined, locally, up to a sign. (Cf. Theorem D.)

Lemma 9.1 *Let (M,g) be a compact essentially conformally symmetric Lorentzian manifold of dimension $n \geq 4$ such that, at some/every point $x \in M$, the nonzero traceless self-adjoint endomorphism $A_x : \mathcal{E}_x \to \mathcal{E}_x$ has $n-2 = \dim \mathcal{E}_x$ distinct eigenvalues. Then the leaves of the parallel distribution $\mathcal{D}^\perp$ are all complete.*

In fact, in view of Lemma 2.2(iv) and (e) in Section 5, passing to a suitable finite covering of M we may assume that both vector bundles $\mathcal{D}$ and $\mathcal{E}$ over M are trivialized by their parallel sections. Our assertion now follows from Lemma 1.4 applied to $\mathcal{L}$ which is the restriction of $\mathcal{D}$ to any leaf N of $\mathcal{D}^\perp$.

Remark 9.2 For $n = 4$, the assumption about eigenvalues in Lemma 9.1 is redundant: it follows from the other stated properties of A_x, since γ_x is positive definite (Remark 4.2).

We now proceed to prove Theorem C. Suppose that (M,g) is a compact essentially conformally symmetric Lorentzian four-manifold. Its universal covering $(\widehat{M},\widehat{g})$ then satisfies the assumptions of Theorem 7.1: the leaves of $\mathcal{D}^\perp$ are complete by Lemma 9.1 (cf. Remark 9.2), while condition (c) in Lemma 1.2 holds for t in view of Remark 5.1.

The conclusion of Theorem 7.1 now contradicts Lemma 6.1(iv-b).

10 Vector bundles related to Killing fields

Throughout this section, (M,g) stands for a fixed essentially conformally symmetric pseudo-Riemannian manifold of dimension $n \geq 4$ such that the Olszak distribution $\mathcal{D}$ is two-dimensional, and ω is the 2-form described in Remark 2.3. We assume that ω is single-valued, rather than being defined just up to a sign, which can always be achieved by passing to a two-fold covering manifold, if necessary.

In addition to $\mathcal{D}$ and $\mathcal{D}^\perp$, we consider here the real vector bundle $\mathcal{Y}$ over M, the sections of which are the differential 2-forms ζ such that $\zeta(v,\,\cdot\,) = 0$ for every section v of $\mathcal{D}$. Thus, $\mathcal{Y}$ is a subbundle of $(T^*M)^{\wedge 2}$, isomorphic to $([(TM)/\mathcal{D}]^*)^{\wedge 2}$, and ω is a section of $\mathcal{Y}$ (as $\mathcal{D} \subset \operatorname{Ker}\omega$ by (5.a) and (4). Let $\mathcal{L}$ be the real-line subbundle of $\mathcal{Y}$ spanned by ω. The Levi-Civita connection of g induces connections, all denoted by ∇, in each of the bundles $\mathcal{D}, \mathcal{D}^\perp, \mathcal{Y}$ and $\mathcal{L}$.

The formula $\mathring{\nabla}_u(v,\zeta) = (\nabla_u v - \zeta u, \nabla_u\zeta - (n-2)^{-1} g(v,\,\cdot\,) \wedge \rho(u,\,\cdot\,))$, with ρ standing for the Ricci tensor of g, clearly defines a connection $\mathring{\nabla}$ in the vector bundle $\mathcal{D}^\perp \oplus \mathcal{Y}$. (Here ζu is the unique vector field with $g(\zeta u,\,\cdot\,) = \zeta(u,\,\cdot\,)$, and similarly for the symbols $\rho u, \omega_x$ appearing below.) The following facts will be needed in the next section.

(a) The connection $\mathring{\nabla}$ in $\mathcal{D}^\perp \oplus \mathcal{Y}$ is flat.
(b) The subbundle $\mathcal{D} \oplus \mathcal{L}$ of $\mathcal{D}^\perp \oplus \mathcal{Y}$ is $\mathring{\nabla}$-parallel.
(c) For every $\mathring{\nabla}$-parallel section (v, ζ) of $\mathcal{D}^\perp \oplus \mathcal{Y}$, the g-covariant derivative ∇v is ∇-parallel along $\mathcal{D}^\perp$.

If M is also simply connected, then, for the space $\mathcal{V}$ of all $\mathring{\nabla}$-parallel sections of $\mathcal{D} \oplus \mathcal{L}$,

(d) $\dim \mathcal{V} = 3$,
(e) there exists a unique C^∞ mapping $F : M \to \mathcal{V} \smallsetminus \{0\}$ such that $F(x)$, for $x \in M$, is characterized by $F(x) = (v, \zeta)$ with $v_x = 0$ and $\zeta_x = \omega_x$,
(f) for F as in (e) and any $x \in M$, the differential $dF_x : T_xM \to \mathcal{V}$ has rank 2, while the image $dF_x(T_xM)$ and $F(x)$ together span $\mathcal{V}$,
(g) the leaves of $\mathcal{D}^\perp$ are the connected components of nonempty F-preimages of points of $\mathcal{V}$,
(h) for every leaf N of $\mathcal{D}^\perp$, the tangent bundle TN is trivialized by its ∇-parallel sections (cf. Remark 4.2).

We will not use the easily-verified fact that the assignment $v \mapsto (v, \zeta)$, with ζ characterized by $\zeta(u, \cdot\,) = g(\nabla_u v, \cdot\,)$ for all vector fields u, is a linear isomorphism between the space of all Killing fields v on (M, g) tangent to $\mathcal{D}^\perp$, and the space of all $\mathring{\nabla}$-parallel sections (v, ζ) of $\mathcal{Y}$.

Let $\mathring{R}, R$ and $\hat{R}$ be the curvature tensors of $\mathring{\nabla}$, the Levi-Civita connection ∇ of g and, respectively, the connection induced by ∇ in $[T^*M]^{\wedge 2}$. For arbitrary vector fields u, u' on M, (1) yields $\mathring{R}(u, u')(v, \zeta) = (v', \zeta')$, where $v' = R(u, u')v - (n-2)^{-1}[g(v, u)\rho u' - g(v, u')\rho u]$ and

$$\zeta' = \hat{R}(u, u')\zeta + (n-2)^{-1}[\zeta(u, \cdot\,) \wedge \rho(u', \cdot\,) - \zeta(u', \cdot\,) \wedge \rho(u, \cdot\,)].$$

(By Lemma 2.1(b), $(\nabla_u \rho)(u', \cdot\,)$ is symmetric in u, u'.) Now Lemma 2.1(a) gives $g(v', \cdot\,) = W(u, u', v, \cdot\,)$. (Note that $\rho(v, \cdot\,) = 0$ in view of (5.a), since v is a section of $\mathcal{D}^\perp$.) Consequently, $v' = 0$, as v is a section of $\mathcal{D}^\perp = \operatorname{Ker} \omega \subset \operatorname{Ker} W$, by (4) with $W = \varepsilon\omega \otimes \omega$. Furthermore, by the Ricci identity, $[\hat{R}(u, u')\zeta](w, w') = \zeta(R(u', u)w, w') + \zeta(w, R(u', u)w')$ for any vector fields w, w'. Replacing R here by the expression in Lemma 2.1(a), we get $\zeta' = 0$. (In fact, numerous terms vanish since the images of W_x and ρ_x at any point x are contained in $\mathcal{D}_x \subset \operatorname{Ker} \zeta_x$, cf. (4) and (5.a).) This proves (a).

Next, if (v, ζ) is a section of $\mathcal{D} \oplus \mathcal{L}$, then, for any vector field u, so are $(\nabla_u v, \nabla_u \zeta)$ (since $\mathcal{D}$ and ω are parallel), and $(\zeta u, (n-2)^{-1} g(v, \cdot\,) \wedge \rho(u, \cdot\,)$ (as $\mathcal{D}$ is the image of ω by (4), and ζ equals a function times ω, while

v and ρu are sections of $\mathcal{D}$, cf. Lemma 2.2(ii)). Consequently, $\mathring{\nabla}_u(v,\zeta)$ is a section of $\mathcal{D}\oplus\mathcal{L}$ as well, and (b) follows.

By the definition of $\mathring{\nabla}$, if (v,ζ) is a $\mathring{\nabla}$-parallel section of $\mathcal{D}^\perp\oplus\mathcal{Y}$, then $\nabla_u\zeta = 0$ for every section u of $\mathcal{D}^\perp$, as one then has $\rho(u,\,\cdot\,) = 0$ by (5.a). Hence ζ is ∇-parallel along $\mathcal{D}^\perp$, which yields (c).

Now let M be simply connected. Assertion (d) is obvious from (a) and (b), as 3 is the fibre dimension of $\mathcal{D}\oplus\mathcal{L}$. Similarly, (e) is due to the existence of a unique parallel section of $\mathcal{D}\oplus\mathcal{L}$ with a prescribed value at x.

To prove (f), we first observe that dF_x sends any $u\in T_xM$ to $(\dot v,\dot\zeta)\in\mathcal{V}$ characterized by $\dot v_x = -\omega_x u$ and $\dot\zeta_x = 0$. In fact, let $t\mapsto x(t)$ be a curve in M, and let us set $(v(t),\zeta(t)) = F(x(t))$. Suppressing the dependence on t, and differentiating, covariantly along the curve, both the relation $v_x = 0$ and the equality which states that $(\nabla v)_x$ corresponds via g_x to ω_x, we get $\dot v_x+\omega_x(\dot x,\,\cdot\,) = 0$ and $(\nabla\dot v)_x = 0$, as required. (The second covariant derivative of the Killing field v at x depends linearly on v_x, due to a well-known identity, cf. [12, formula (17.4) on p. 536], and so $\nabla(\nabla v) = 0$ at x, since $v_x = 0$.) As $\operatorname{rank}\omega = 2$ (see Remark 2.3), this implies (f).

As a consequence of (c), F is constant along $\mathcal{D}^\perp$. (Note that $\nabla\omega = 0$ and, by (4), $\mathcal{D}^\perp = \operatorname{Ker}\omega$.) Now (g) is immediate from (f).

Finally, in view of (a), given a leaf N of $\mathcal{D}^\perp$, a point $x\in N$, and a vector $w\in T_xN = \mathcal{D}_x^\perp$, we may choose a $\mathring{\nabla}$-parallel section (v,ζ) of $\mathcal{D}^\perp\oplus\mathcal{Y}$ satisfying the initial conditions $v_x = w$ and $\zeta_x = 0$ (that is, $(\nabla v)_x = 0$). Thus, v is tangent to N and, by (c), ∇-parallel along N, which proves (h).

11 Proof of the second part of Theorem A

We need the following two simple facts from topology.

Lemma 11.1 *If the fundamental group Γ of a compact k-dimensional manifold P is Abelian and the universal covering manifold of P is diffeomorphic to $\mathbf{R}^k$, then Γ is isomorphic to $\mathbf{Z}^k$.*

Proof. As Γ is torsionfree by Smith's theorem [13, p. 287], and finitely generated, it is isomorphic to $\mathbf{Z}^r$ for some integer $r\ge 1$. The $K(\mathbf{Z}^r,1)$ space P must have the homotopy type of the r-torus [18, pp. 93–95], so that $r = k$, since both r and k are equal to the highest integer m with $H_m(P,\mathbf{Z}_2)\ne\{0\}$. $\square$

Lemma 11.2 *If $M \to S^2$ is a fibration and its fibre N is a compact manifold of dimension $k \geq 2$ with a universal covering space diffeomorphic to $\mathbf{R}^k$, then the fundamental group of M is infinite.*

Proof. Since $\mathbf{Z} = \pi_2 S^2 \to \pi_1 N \to \pi_1 M$ is a part of the homotopy exact sequence of the fibration $M \to S^2$, if $\pi_1 M$ were finite, the image Γ of $\mathbf{Z} = \pi_2 S^2$ would be a cyclic subgroup of finite index in $\pi_1 N$. The manifold $P = \mathbf{R}^k/\Gamma$, forming a finite covering space of $N = \mathbf{R}^k/\pi_1 N$, would be compact, which, as $k \geq 2$, would contradict Lemma 11.1. $\square$

We now assume that (M, g) is a compact simply connected essentially conformally symmetric manifold. As we show below, this assumption leads to a contradiction, which proves the claim about $\pi_1 M$ in Theorem A.

Let $d \in \{1, 2\}$ be the dimension of the Olszak distribution $\mathcal{D}$ (Lemma 2.2(i)).

If $d = 1$, Lemma 2.2(iv) implies the existence of a nonzero global parallel vector field u spanning $\mathcal{D}$. The 1-form $\xi = g(u, \cdot\,)$, being parallel, is closed, so that $\xi = dt$ for some function t. As dt is parallel, $dt \neq 0$ everywhere, which contradicts compactness of M.

Now let $d = 2$, and let $\mathcal{V}, F$ be as in (d) – (g), Section 10. For a fixed Euclidean norm $|\,|$ in the 3-space $\mathcal{V}$, the formula $\pi(x) = F(x)/|F(x)|$ defines a submersion $\pi : M \to S^2$ valued in the unit sphere $S^2 = \{w \in \mathcal{V} : |w| = 1\}$. As $\pi(M) \subset S^2$ is both compact and open, the submersion π is surjective, so that π is a fibration (Remark 1.1). The fibres of π thus are the leaves of $\mathcal{D}^\perp$ (see (g) in Section 10). If a fixed fibre N is endowed with the flat torsionfree connection mentioned in Remark 4.2, then, according to (h) in Section 10, TN is trivialized by its parallel sections. Lemma 1.4 (for $\mathcal{L} = TN$) and Remark 1.3 now imply that the universal covering manifold of N is diffeomorphic to $\mathbf{R}^{n-2}$. Since $\pi_1 M$ was assumed to be trivial, and $n - 2 \geq 2$, this contradicts Lemma 11.2, thus completing the proof of Theorem A.

12 Further remarks

This section consists of three separate comments, indicating how some results presented above might be strengthened.

First, Theorem 7.1, with essentially the same proof, remains valid if, in its assumptions, condition (c) in Lemma 1.2 and completeness of the leaves of $\mathcal{D}^\perp$ are replaced by completeness of $\widehat{g}$.

Secondly, the argument that we used to show nonexistence of compact four-dimensional essentially conformally symmetric Lorentzian manifolds can be minimally modified so as to yield the following classification theorem: *If a compact essentially conformally symmetric Lorentzian manifold* (M, g) *of any dimension* $n \geq 4$ *satisfies the assumption about distinct eigenvalues made in Lemma* 9.1*, then the pseudo-Riemannian universal covering* $(\widehat{M}, \widehat{g})$ *of* (M, g) *coincides, up to an isometry, with one of the manifolds constructed in Section* 6*, and the fundamental group of* M*, treated as a group of isometries of* $(\widehat{M}, \widehat{g})$*, has a finite-index subgroup contained in the group* G *defined in Section* 6.

Finally, in Section 11 we showed that $\pi_1 M$ is infinite for any compact essentially conformally symmetric manifold (M, g), using separate arguments for the cases $d = 1$ and $d = 2$, where d is the dimension of the Olszak distribution $\mathcal{D}$. If $d = 1$, we obtain the stronger conclusion $b_1(M) \geq 1$ from the following lemma applied to $\mathcal{F} = \mathcal{D}^\perp$ along with a vector-bundle isomorphism $\mathcal{D} \to [(TM)/\mathcal{F}]^*$ provided by g, cf. Lemma 2.2(iv).

Lemma 12.1 *Let a compact manifold* M *with a torsionfree connection* ∇ *admit a codimension-one* ∇*-parallel distribution* $\mathcal{F}$ *such that the connection induced by* ∇ *in the line bundle* $(TM)/\mathcal{F}$ *is flat. Then the first Betti number* $b_1(M)$ *is positive, and, consequently,* M *has an infinite fundamental group.*

Proof. The dual bundle of $(TM)/\mathcal{F}$ may be identified with the real-line subbundle $\mathcal{L}$ of T^*M such that the sections of $\mathcal{L}$ are precisely those 1-forms ξ on M with $\xi(w) = 0$ for every section w of $\mathcal{F}$. Clearly, $\mathcal{L}$ is ∇-parallel.

Suppose, on the contrary, that $b_1(M) = 0$, so that the homology group $H_1(M, \mathbf{Z})$ is finite. Replacing M by a two-fold covering manifold, if necessary, we may assume that $\mathcal{L}$ is spanned by a global nonzero parallel 1-form ξ. In fact, the connection in $\mathcal{L}$ induced by ∇ is flat, and so its holonomy representation, with any fixed base point $x \in M$, is valued in the multiplicative group $\mathbf{R} \smallsetminus \{0\}$. Since $\mathbf{R} \smallsetminus \{0\}$ is Abelian, the holonomy representation is a composite $\pi_1 M \to H_1(M, \mathbf{Z}) \to \mathbf{R} \smallsetminus \{0\}$, and its image must, due to finiteness of $H_1(M, \mathbf{Z})$, be contained in $\{1, -1\}$.

As ∇ is torsionfree and $b_1(M) = 0$, the parallel 1-form ξ is closed, and hence $\xi = dt$ is nonzero everywhere, for some function t. Since this contradicts compactness of M, our assertion follows. $\square$

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