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On Compact Riemannian Manifolds with Harmonic Curvature*

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1. Introduction

A Riemannian manifold M is said to have *harmonic curvature* if $\delta R = 0$ (in local coordinates, $\nabla^i R_{hijk} = 0$), R being the curvature tensor of M and δR its formal divergence. In view of the second Bianchi identity, this happens if and only if the Ricci tensor S of M satisfies the Codazzi equation $\nabla_k S_{hj} = \nabla_j S_{hk}$. Thus, every Riemannian manifold with parallel Ricci tensor has harmonic curvature. The question whether the converse statement is true for *compact* manifolds, raised by Bourguignon in [3], was answered in the negative in [5] by giving explicit counterexamples in dimension four. Moreover, [5] contains a classification theorem for a certain class of compact Riemannian four-manifolds with harmonic curvature.

The aim of this paper is to extend the results of [5] (description of examples and the classification theorem) to the case of arbitrary dimension $n \geqq 3$. More precisely, we construct, in Sect. 3, a family of compact Riemannian manifolds with harmonic curvature, the Ricci tensor of which is not parallel and has less than three distinct eigenvalues at any point, and prove that every analytic manifold with these properties is covered isometrically by one of our examples (Theorem 2). For $n \geqq 5$, this classification is highly inefficient, since it contains, as a kind of parameter, an arbitrary compact $(n-1)$-dimensional Einstein manifold of positive scalar curvature.

2. Preliminaries

Given Riemannian manifolds (M, h^M) and (N, h^N) and a positive function F on M, one defines [7, 2] the *F-warped product* $M \times_F N$ of M and N to be the Riemannian

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manifold $(M \times N,\ h^M \times {}_F h^N)$, where

$$(h^M \times {}_F h^N)_{(x,y)}(u+X, v+Y) = h^M_x(u,v) + F(x) h^N_y(X,Y)$$

for $u, v \in T_x M$, $X, Y \in T_y N$.

For warped products $M \times_F N$ with $\dim M = 1$, an explicit computation gives the following

Lemma 1 [5, Remark 1 and Lemma 4]. *Let I be an interval of $\mathbb{R}$, considered with its stardard metric, F a positive C^∞ function on I and (N,h) an $(n-1)$-dimensional Riemannian manifold. Denoting by g the F-warped product metric of $I \times_F N$ and by S its Ricci tensor, and letting the indices i, j, k run through $1, \dots, n-1$, we have*

(i) *Given a product chart $t = x^0, x^1, \dots, x^{n-1}$ for $I \times N$ with $g_{00} = 1$, $g_{0i} = 0$ and $g_{ij} = F h_{ij}$, the components of S and ∇S are given by*

$$\begin{aligned} S_{00} &= \frac{1-n}{4}[2q'' + (q')^2],\ S_{0i} = 0, \\ S_{ij} &= \varrho_{ij} - \tfrac{1}{4} e^q [2q'' + (n-1)(q')^2] h_{ij}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \nabla_0 S_{00} &= \frac{1-n}{2}[q''' + q'q''],\ \nabla_0 S_{i0} = \nabla_i S_{00} = 0, \\ \nabla_0 S_{ij} &= -q'\varrho_{ij} - \tfrac{1}{2} e^q [q''' + (n-1) q'q''] h_{ij}, \\ \nabla_i S_{0j} &= -\tfrac{1}{2} q' \varrho_{ij} + \frac{2-n}{4} e^q q'q'' h_{ij},\ \nabla_k S_{ij} = D_k \varrho_{ij}, \end{aligned} \tag{2}$$

where $q = \log F$ and D, ϱ denote the Riemannian connection and Ricci tensor of (N,h), respectively, while the components of h, ϱ and $D\varrho$ are considered with respect to the chart $x^1, \dots, x^{n-1}$ of N.

(ii) *If F is non-constant and $n \geqq 3$, then $I \times_F N$ has harmonic curvature if and only if (N,h) is an Einstein space and the positive function $\varphi = F^{n/4}$ on I satisfies the ordinary differential equation*

$$\varphi'' - \frac{nk}{4(n-1)} \varphi^{1-4/n} = p\varphi \tag{3}$$

for some real number p, k being the constant scalar curvature of N.

Remark 1. Let F be a non-constant positive function on an interval I such that, for some $(n-1)$-dimensional $(n \geqq 3)$ Riemannian manifold N, the warped product $I \times_F N$ has harmonic curvature. By Lemma 1 (ii), N is an Einstein space and $\varphi = F^{n/4}$ satisfies (3) with p, k as above. Using (2) it is now easy to verify that the Ricci tensor of $I \times_F N$ is parallel if and only if F is given by one of the following formulae:

$$F(t) = \frac{k(t-t_0)^2}{(n-1)(n-2)}, \qquad \text{for} \quad p = 0,\ k > 0,$$

$$F(t) = \exp(4\varepsilon(t-t_0)p^{1/2}/n),\ \varepsilon = \pm 1, \qquad \text{for} \quad p > 0,\ k = 0,$$

or

$$F(t)=\begin{cases} |A|\sin^2(2(t-t_0)|p|^{1/2}/n), & \text{for} \quad p<0, k>0, \\ |A|\operatorname{ch}^2(2(t-t_0)p^{1/2}/n), & \text{for} \quad p>0, k<0, \\ |A|\operatorname{sh}^2(2(t-t_0)p^{1/2}/n), & \text{for} \quad p>0, k>0, \end{cases}$$

where $A=\dfrac{n^2k}{4p(n-1)(n-2)}$ and t_0 is a real parameter. In fact, in view of (2), condition $\nabla S=0$ is equivalent to $q'''+q'q''=0$ and $(n-1)(n-2)q''+2ke^{-q}=0$, which, in terms of the function $v=\sqrt{F}=e^{q/2}$, means precisely that $v''=cv$ and $(n-1)(n-2)[(v')^2-cv^2]=k$, for some constant c. Using (3) with $\varphi=v^{n/2}$, we see that $c=4p/n^2$. Our assertion can now be obtained by explicitly solving the equation $v''=cv$.

Lemma 1 suggests that, in order to use the warped product construction to obtain *compact* manifolds with harmonic curvature, it is necessary to examine the global behaviour of the solutions of (3). In this connection, we can prove the following

Theorem 1. *For real numbers k, p and an integer $n\geqq 3$, we have*

(i) *If $k>0$ and $p<0$, then* (3) *possesses some 2-parameter family of non-constant positive periodic solutions, defined everywhere in* $\mathbb{R}$.

(ii) *Conversely, if* (3) *has at least one non-constant positive periodic solution, defined on* $\mathbb{R}$, *then $k>0$ and $p<0$.*

Proof. Set $B=n^2k/(4(n-1)(n-2))$. Multiplying both sides of (3) by φ' it is easy to see that, apart from constant solutions, (3) is equivalent to

$$(\varphi')^2=B\varphi^{2-4/n}+p\varphi^2-c, \tag{4}$$

where c runs through $\mathbb{R}$. Setting $H=\varphi^{2/n}$, we can rewrite (4) in the form

$$\frac{n^2}{4}H^{n-2}(H')^2=pH^n+BH^{n-2}-c. \tag{5}$$

Denote by P the polynomial $P(y)=py^n+By^{n-2}$.

If $k>0$ and $p<0$, then $B>0$ and there exists exactly one $y_0>0$ with $P'(y_0)=0$. Moreover, $P(y_0)=\max\limits_{y>0}P(y)>0$. The relations $P(0)=0$ and $P'(y)\neq 0$ for $y>0$, $y\neq y_0$, imply now that, for any real constant $c\in(0,P(y_0))$, the polynomial $P-c$ has precisely two positive roots a_c, b_c, both simple, which satisfy $a_c<y_0<b_c$ and, viewed as functions of c, are strictly monotone and analytic. Moreover, $P-c>0$ everywhere in (a_c,b_c). The function Φ_c on the closed interval $I_c=[a_c,b_c]$, given by

$$\Phi_c(y)=\frac{n}{2}\int_{a_c}^{y}\left(\frac{u^{n-2}}{P(u)-c}\right)^{1/2}du,$$

is well-defined and continuous on the whole interval I_c (which follows from the fact that a_c, b_c are simple roots of $P-c$) and analytic in its interior. Moreover, $\lim\limits_{y\to a_c(+)}\Phi_c'(y)=\lim\limits_{y\to b_c(-)}\Phi_c'(y)=\infty$. Thus, Φ_c maps I_c homeomorphically onto $[0,T_c]$, where $T_c=\Phi_c(b_c)>0$. The inverse mapping $H=H_c\colon[0,T_c]\to I_c$ is analytic in $(0,T_c)$, satisfies (5), and $\lim_{t\to 0(+)} H_c'(t) = \lim_{t\to T_c(-)} H_c'(t) = 0$. We can now extend H_c to a function on $\mathbb{R}$, denoted again by H_c and defined by

$$H_c(t) = \begin{cases} H_c(t-2mT_c), & \text{if } 2m \leqq t/T_c \leqq 2m+1,\, m\in\mathbb{Z}, \\ H_c(2mT_c - t), & \text{if } 2m-1 \leqq t/T_c \leqq 2m,\, m\in\mathbb{Z}. \end{cases}$$

It is easy to see that H_c is of class C^1 on $\mathbb{R}$, periodic with minimal period $2T_c > 0$ and

$$\max H_c = b_c,\ \min H_c = H_c(0) = a_c > 0. \tag{6}$$

Moreover, outside of the discrete set $\mathbb{Z}T_c$, $H = H_c$ is analytic and satisfies (5). Thus, the non-constant positive periodic function $\varphi = \varphi_c = H_c^{n/2}$ of class C^1 is a solution of (3) in $\mathbb{R} - \mathbb{Z}T_c$. Consequently, it is of class C^2 and therefore analytic, and satisfies (3) everywhere. The functions $\varphi_{c,r}$, where $\varphi_{c,r}(t) = \varphi_c(t+r)$ form a family of non-constant positive periodic solutions of (3), depending essentially, in view of (6), on two real parameters $c \in (0, P(y_0))$ and $r \in (-T_c, T_c)$. This yields (i).

To prove (ii), suppose that $\varphi: \mathbb{R} \to \mathbb{R}$ is a non-constant positive periodic solution of (3). Then the function $H = \varphi^{2/n}$ satisfies (5) for a suitably chosen c. Since H is periodic, we have $0 < a = \min H < b = \max H$ and, by (5), the polynomial $P - c$ satisfies $P(a) - c = P(b) - c = 0$ and $P(y) - c \geqq 0$ for $a < y < b$. Moreover, (4) yields $B^2 + p^2 > 0$. Choosing $y_0 \in (a,b)$ to be a local maximum of P, we have $0 = y_0^{3-n} P'(y_0) = npy_0^2 + (n-2)B$, so that $pB < 0$, and, for $n \geqq 4$, $0 \geqq y_0^{4-n} P''(y_0) = n(n-1)py_0^2 + (n-2)(n-3)B = 2(2-n)B$, while for $n = 3$, $0 \geqq P''(y_0) = 6py_0$. Consequently, $p < 0$ and $B > 0$, i.e., $k > 0$. This completes the proof.

For a later application, we also prove

Lemma 2. *Suppose that $n \geqq 3$, (N, h) is an $(n-1)$-dimensional Einstein manifold of (constant) scalar curvature k and p a real number. Let F be a non-constant positive function on $\mathbb{R}$ such that $\varphi = F^{n/4}$ is a solution of* (3), *G the group of all translations of $\mathbb{R}$ leaving F invariant and $M = \mathbb{R} \times_F N$. If the Ricci tensor of M is not parallel, then*

(i) *The group $G \times \mathrm{Isom}\,N$, acting on the underlying manifold $\mathbb{R} \times N$ of M in the product manner, is contained in $\mathrm{Isom}\,M$ as a subgroup of finite index. If N is compact, then this is also true for $G \times \mathrm{Isom}^0 N$, $\mathrm{Isom}^0 N$ being the identity component of $\mathrm{Isom}\,N$.*

(ii) *If $\mathbb{R} \times_F N$ covers isometrically a compact manifold, then $k > 0$, $p < 0$ and G is non-trivial, i.e., F is periodic.*

Proof. Formula (1) together with $\varrho_{ij} = \dfrac{k}{n-1} h_{ij}$ shows that the Ricci tensor S of M has at most two distinct eigenvalues at any point, and, since S is not a multiple of the metric, it must have exactly two eigenvalues almost everywhere. Moreover, again by (1), the S-eigenspace decomposition of TM coincides with the product decomposition of $T(\mathbb{R} \times N)$. The latter is therefore invariant under the action of $\mathrm{Isom}\,M$, i.e., every isometry of M is a Cartesian product $\theta \times \eta$ of a diffeomorphism θ of $\mathbb{R}$ with a diffeomorphism η of N. The definition of warped product yields $(\theta \times \eta)^*(g \times_F h) = \theta^* g \times_{F\circ\theta} \eta^* h$, g being the standard metric of $\mathbb{R}$. Thus $\theta \in \mathrm{Isom}\,\mathbb{R}$, $F \circ \theta = \tau F$ and $\eta^* h = \tau^{-1} h$ for some real number $\tau > 0$. Since $\varphi = F^{n/4}$ satisfies $\varphi \circ \theta = \tau^{n/4} \varphi$ with $\theta \in \mathrm{Isom}\,\mathbb{R}$, it is easy to deduce from (3) together with Remark 1 that $\tau=1$, i.e., θ leaves F invariant and $\eta\in\operatorname{Isom}N$. The group of all isometries of M which preserve the orientation of the line subbundle of TM corresponding to the $\mathbb{R}$-factor of $\mathbb{R}\times N$ is, clearly, a subgroup of index 1 or 2 in $\operatorname{Isom}M$, which proves the first statement of (i). Now, if N is compact, (i) follows from the fact that $\operatorname{Isom}N/\operatorname{Isom}^0N$ is finite. To prove (ii), suppose that $M_0=M/\Gamma_0$ is a compact manifold covered isometrically by M, Γ_0 being a discrete subgroup of $\operatorname{Isom}M$. Since, by (i), the subgroup $\Gamma_1=\Gamma_0\cap(G\times\operatorname{Isom}N)$ is of finite index in Γ_0, the obvious map $M_1=M/\Gamma_1\to M/\Gamma_0=M_0$ is a finite covering and hence M_1 is compact. If G were trivial, all elements of Γ_1 would be of the form $(0,Q)$ [notation as in (i)] and the product projection $M=\mathbb{R}\times_F N\to\mathbb{R}$ would give rise to an unbounded function on M_1, contradicting its compactness. Therefore F must be periodic, and, by Theorem 1 (ii), $k>0$ and $p<0$, which completes the proof.

3. Construction of Examples and a Classification Theorem

We are now in a position to describe a family of compact Riemannian manifolds with harmonic curvature and non-parallel Ricci tensor, extending the construction given in [5] for the case of dimension four.

Suppose we are given an integer $n\geqq 3$, a compact $(n-1)$-dimensional Einstein manifold N with (constant) scalar curvature $k>0$, a non-constant, positive, periodic (with minimal period $2T>0$) function F on $\mathbb{R}$ such that $\varphi=F^{n/4}$ satisfies (3) for some real number $p<0$ (for the existence, see Theorem 1), a positive integer m and an isometry $Q\in\operatorname{Isom}^0N$ of the identity component of $\operatorname{Isom}N$. We define the n-dimensional Riemannian manifold $M=M^n_{(N,F,Q,m)}$ as follows. The infinite cyclic subgroup $\Gamma_{(m,F,Q)}$ of $\operatorname{Isom}(\mathbb{R}\times_F N)$, generated by $(2mT,Q)$ (notation of Lemma 2), acts on $\mathbb{R}\times_F N$ properly discontinuously. Therefore we may set

$$M=M^n_{(N,F,Q,m)}=(\mathbb{R}\times_F N)/\Gamma_{(m,F,Q)}\,.$$

The assignment $M\ni(t,y)\bmod\Gamma_{(m,F,Q)}\to\exp\dfrac{\pi it}{mT}\in S^1$ is a fibre bundle projection with fibre N, which is trivial since Q is isotopic to the identity transformation of N. Consequently, $M=M^n_{(N,F,Q,m)}$ is diffeomorphic to $S^1\times N$ and, as a Riemannian manifold, it has harmonic curvature [Lemma 1 (ii)]. The Ricci tensor S of M has, by (1), less than three eigenvalues at any point. Finally, $\nabla S\neq 0$, since, in view of Remark 1, the functions F for which S is parallel cannot be non-constant, positive and periodic on $\mathbb{R}$.

The manifolds $M^n_{(N,F,Q,m)}$ are characterized by the above properties as follows.

Theorem 2. *Let M be a compact, analytic, n-dimensional ($n\geqq 3$) Riemannian manifold with harmonic curvature. If the Ricci tensor S of M is not parallel and has less than three distinct eigenvalues at each point, then M is covered isometrically by one of the manifolds $M^n_{(N,F,Q,m)}$ defined above.*

Proof. Since M is not Einstein, the set U of all points at which S has exactly two distinct eigenvalues is non-empty. Therefore U is an open dense subset of M and, by Theorem 1 (i) of [5], each point x of U has a neighbourhood isometric to a warped product $I \times_F V$, where I is an open interval of $\mathbb{R}$, V an $(n-1)$-dimensional Einstein space and F a non-constant positive function on I such that $\varphi = F^{n/4}$ satisfies (3), k being the constant scalar curvature of V and p a real number. Viewing $I \times_F V$ as embedded in M, it is clear from (1) that, in a neighbourhood of x, the S-eigenspace decomposition of TM coincides with the product decomposition of $T(I \times_F V)$. Again by (1), the eigenvalues of S are constant along each submanifold of the form $\{t\} \times V$, $t \in I$. The same is true for the r-th elementary symmetric function $\sigma_r(S) = \sum_{\alpha_1 < \ldots < \alpha_r} \lambda_{\alpha_1} \ldots \lambda_{\alpha_r}$, $r = 1, \ldots, n$, of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of S (taken at any point together with their multiplicities). The functions $\sigma_r(S)$ are well-defined and analytic on M, and one of them is non-constant. In fact, constancy of all $\sigma_r(S)$ would imply constancy of the eigenvalues of S, which, by the argument of Lemma 3 of [5], would yield $\nabla S = 0$. Choosing r so that $h = \sigma_r(S)$ is not constant, we may clearly find $x \in U$ such that $h(x)$ is not a critical value of h. Hence the connected component of $h^{-1}(h(x))$, passing through x, is a compact analytic submanifold of M, containing $\{t\} \times V$ for some $t \in I$. The corresponding embedding $V \to h^{-1}(h(x))$ is isometric (up to a constant factor), in view of the definition of warped product. Consequently, V is isometric to an open submanifold of some compact $(n-1)$-dimensional Einstein manifold Y.

To prove that our function F can be extended from I to a positive analytic function on $\mathbb{R}$, note that the function S_{00} occurring in (1), viewed as depending on the parameter $t = x^0 \in I$, has an analytic extension to the whole real line. In fact, since, for a fixed $y \in V$, the curve $I \ni t \to \gamma(t) = (t, y) \in I \times_F V$ is a geodesic (for arbitrary F), we have $S_{00}(t) = S(\dot\gamma(t), \dot\gamma(t))$, and the existence of the extension of S_{00} to $\mathbb{R}$ follows from the fact that M is complete. On the other hand, the function $u = F^{1/2}$ satisfies [in view of (3) with $\varphi = F^{n/4}$ and (1)] the second order linear differential equation

$$u'' = \frac{1}{1-n} S_{00} u. \tag{7}$$

Therefore u, as well as $F = u^2$, has an analytic extension to $\mathbb{R}$. We claim that $u \neq 0$ everywhere in $\mathbb{R}$. In fact, from (3) we obtain, with $u = \varphi^{2/n}$,

$$uu'' = \frac{k}{2(n-1)} - \frac{n-2}{2}(u')^2 + \frac{2p}{n} u^2. \tag{8}$$

If we had, for some $t_0 \in \mathbb{R}$,

$$u(t_0) = 0, \tag{9}$$

then (8) would imply $k = (n-1)(n-2)(u'(t_0))^2 \geqq 0$. Moreover, since u is not identically zero, the linear equation (7) would yield $u'(t_0) \neq 0$ and, consequently, $k > 0$. In this case, we claim that the differential equation (8) with initial condition (9) has exactly two analytic solutions on $\mathbb{R}$ (for arbitrary fixed parameter p), which differ only by sign. In fact, letting $u(t) = \sum_{m \geqq 1} a_m (t - t_0)^m$ be a local power series expansion for such a solution u, we obtain from (8), by a direct calculation, $a_1^2 = k/((n-1)(n-2)) \neq 0$ and $a_2 = 0$, while for $m \geqq 2$,

$$0 = (n+m-2)(m+1)a_1 a_{m+1} + \frac{1}{2} \sum_{1<r<m-1} (r+1)[(n-2)(m+1)-(n-4)r]a_{r+1}a_{m-r+1} - \frac{2p}{n} \sum_{0<r<m} a_r a_{m-r},$$

which is nothing but a recursion formula for a_m in terms of a_1. The only solutions of (8) and (9) must therefore be given by the formula

$$\pm u(t) = \begin{cases} (k/((n-1)(n-2)))^{1/2}(t-t_0), & \text{if } p=0 \\ |A|^{1/2} \sin(2(t-t_0)|p|^{1/2}/n), & \text{if } p<0 \\ |A|^{1/2} \operatorname{sh}(2(t-t_0)p^{1/2}/n), & \text{if } p>0 \end{cases} \tag{10}$$

$\left(\text{where } A = \dfrac{n^2 k}{4p(n-1)(n-2)}\right)$, which is easily seen to define a solution of (8) and (9). However, if our function u is given by (10), then $F = u^2$ and Remark 1 yield $\nabla S = 0$, contradicting our hypothesis. Consequently, (9) is not satisfied by any real t_0 and the analytic function $F = u^2$ is positive everywhere in $\mathbb{R}$.

By the first part of this proof, the Riemannian manifolds M and $\mathbb{R} \times_F Y$ (the latter being complete by Lemma 7.2 of [2, p. 23]), possess isometric open submanifolds. The extension theorem for analytic isometries [6, p. 252] implies now that the Riemannian universal covering $\tilde{M}$ of M is isometric to $\mathbb{R} \times_F N$, N being the universal covering manifold of Y. Consequently, up to an isometry we have $M = \mathbb{R} \times_F N/\Gamma$ for some discrete subgroup Γ of $\mathrm{Isom}(\mathbb{R} \times_F N)$. Compactness of M together with Lemma 2 (ii) implies now that $k>0$, $p<0$ and F is periodic, with some minimal period $2T>0$. Therefore, the simply connected manifold N is compact, since it carries a complete Einstein metric of positive scalar curvature k. Now, in view of Lemma 2 (i) (and in the notations thereof), the discrete group $\Gamma_1 = \Gamma \cap (G \times \mathrm{Isom}^0 N)$ is of finite index in Γ. Moreover, Γ_1 must contain an element of $G \times \mathrm{Isom}^0 N$ of the form $(2mT, Q)$ with some integer $m>0$, for otherwise the manifold $M_1 = \mathbb{R} \times_F N/\Gamma_1$ would be non-compact (it would admit the unbounded function determined by the projection $\mathbb{R} \times_F N \to \mathbb{R}$) and, at the same time, it would finitely cover the compact manifold $M = \mathbb{R} \times_F N/\Gamma$. Consequently, $\Gamma \supset \Gamma_{(m,F,Q)}$ (notation of the beginning of this section), i.e., we have the obvious map

$$M^n_{(N,F,Q,m)} = \mathbb{R} \times_F N/\Gamma_{(m,F,Q)} \to \mathbb{R} \times_F N/\Gamma = M,$$

which is a finite isometric covering. This completes the proof.

Remark 2. For any n-dimensional ($n \geqq 3$) Riemannian manifold (M, g), the second Bianchi identity yields the well-known divergence formulae $\delta S = \frac{1}{2}\nabla K$, $\delta R = dS$ and, consequently,

$$\delta W = \frac{n-3}{n-2} d\left(S - \frac{K}{2(n-1)} g\right), \tag{11}$$

K being the scalar curvature of (M, g) and W its Weyl conformal curvature tensor. Combining the first formula above with the Codazzi equation for S, one sees easily that K is constant whenever M has harmonic curvature (i.e. $\delta R = dS = 0$). On the other hand, (11) implies that every manifold with constant scalar curvature, which is locally conformally Euclidean (for $n \geqq 4$, this is equivalent to $W=0$) must have harmonic curvature. (This relates the problem of finding new compact manifolds with harmonic curvature and $W=0$ to a special case of Yamabe's conjecture, cf. [1, 8, 4].) However, the manifolds $M^n_{(N,F,Q,m)}$ with harmonic curvature, described above, do not satisfy $W=0$ unless N is a space of constant curvature. In fact, each of these manifolds looks locally like the warped product $\mathbb{R} \times_F N$ which is, in turn, locally conformally equivalent to the ordinary product $\mathbb{R} \times N$. It is now easy to see that, provided $\dim N \geqq 3$, $\mathbb{R} \times N$ satisfies $W=0$ if and only if N has constant sectional curvature.

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