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Global Properties of Indefinite Metrics with Parallel Weyl Tensor

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Abstract

This is an exposition of some recent results on ECS manifolds, by which we mean pseudo-Riemannian manifolds of dimensions greater than 3 that are neither conformally flat nor locally symmetric, and have parallel Weyl tensor. All ECS metrics are indefinite. We state two classification theorems, describing the local structure of ECS manifolds, and outline an argument showing that compact ECS manifolds exist in infinitely many dimensions greater than 4. We also discuss some properties of compact manifolds that admit ECS metrics, and provide a list of open questions about compact ECS manifolds.

Introduction

An *ECS manifold* is any pseudo-Riemannian manifold of dimension $n \geq 4$ that has parallel Weyl tensor ($\nabla W = 0$) without being conformally flat ($W = 0$) or locally symmetric ($\nabla R = 0$). ECS manifolds exist in every dimension $n \geq 4$, and their metrics are all indefinite [18, Corollary 3], [4, Theorem 2], cf. [1, Remark 16.75(iii)]. This paper presents selected results on ECS manifolds.

A local classification of ECS metrics, described in §§3 − 4, is naturally divided into two cases, depending on the dimension of a null parallel distribution first introduced by Olszak [16]. In §5 we state a theorem about the existence of compact ECS manifolds in infinitely many dimensions $n \geq 5$. An outline of its proof is given in §9. Sections 6 − 8 contain results on general properties of compact manifolds admitting ECS metrics, and a list of open questions concerning compact ECS manifolds.

Metrics with parallel Weyl tensor were first studied by Chaki and Gupta [2], who referred to them as *conformally symmetric*. Although the latter term was adopted by many authors ([10, 14, 19, 20], to name just a few), it may be misleading, since it appears in the literature with at least two other meanings [13, 15]. This is why, rather than speaking of conformal symmetry, we use the phrases ‘parallel Weyl tensor’ and ‘ECS manifold’ (or, ‘metric’), ‘ECS’ being abbreviated from *essentially conformally symmetric*.

We wish to express our gratitude to David Blair and Ryszard Deszcz for comments concerning terminology and pseudo-symmetry of ECS metrics. We also thank Maciej Dunajski for bringing to our attention the results of [12], and Tadeusz Januszkiewicz for pointing out the fact described in the last paragraph of §8.

1 Parallel Weyl tensor

The symbols R and W always denote the curvature tensor and Weyl conformal tensor of the pseudo-Riemannian metric in question, while ρ, s and ∇ stand for its Ricci tensor, scalar curvature and Levi-Civita connection. In dimensions $n \geq 4$, one has the decomposition

$$\begin{array}{ccccccc} R & = & S & + & E & + & W \\ \text{curvature} & = & \text{scalar} & + & \text{Einstein} & + & \text{Weyl} \end{array} \tag{1.1}$$

of R into its irreducible components under the action of the pseudo-orthogonal group; if $n = 4$, the underlying manifold is oriented, and the metric signature is Riemannian or neutral, W can be further decomposed into its self-dual and anti-self-dual parts. The simplest linear conditions imposed on R are $S = 0$ (scalar-flatness), $W = 0$ (conformal flatness) and $E = 0$ (which defines Einstein metrics). The decomposition of ∇R resulting from (1.1) leads to the analogous conditions $\nabla S = 0$, $\nabla E = 0$ and $\nabla W = 0$. Questions about metrics with parallel Weyl tensor may, consequently, be considered as natural as those concerning constant scalar curvature, or parallel Ricci tensor (including the case of Einstein metrics and their products).

The requirement that the Weyl tensor be parallel is also related to some other conditions that are of independent interest. For instance, all ECS metrics are scalar-flat [5, Theorem 7], which, combined with the second Bianchi identity and the relation $\nabla W = 0$, implies in turn that they have harmonic curvature [1] (in other words, the Ricci tensor satisfies the Codazzi equation).

Furthermore, all ECS manifolds are semisymmetric [5, Theorem 9], and hence pseudo-symmetric [10]. Finally, in terms of the invariant $d \in \{1, 2\}$ appearing in relation (2.2) below, every ECS metric has low cohomogeneity (at most d), an ECS manifold with $d = 1$ is necessarily Ricci-recurrent, and, in dimension four, ECS metrics with $d = 2$ are all self-dual. See §§2 − 3.

2 The Olszak distribution

Let a pseudo-Riemannian manifold (M, g) of dimension $n \geq 4$ have parallel Weyl tensor W. The *Olszak distribution* of (M, g), introduced by Olszak [16], is the subbundle $\mathcal{D}$ of TM such that the sections of $\mathcal{D}$ are precisely those vector fields u which satisfy the condition

$$g(u, \cdot\,) \wedge W(v, v', \cdot\,, \cdot\,) \;=\; 0 \tag{2.1}$$

for all vector fields v, v' (exterior multiplication of a 1-form by a 2-form). At every point $x \in M$, the space $\mathcal{D}_x$ thus consists, in addition to the zero vectors, of all vectors that

$\wedge$-divide each 2-form in the image of W_x acting on 2-forms. In other words, $\mathcal{D}_x$ is the subspace of T_xM formed by the vectors that lie in the image of every nonzero 2-form in the image of W_x. (We use g_x to identify 2-forms at x with skew-adjoint endomorphisms of T_xM.)
The distribution $\mathcal{D}$ is obviously parallel. We denote its dimension by d. For any n-dimensional pseudo-Riemannian manifold (M,g) with parallel Weyl tensor, $d \in \{0,1,2,n\}$, while $d=n$ if and only if g is conformally flat. Furthermore,

$$\text{in every ECS manifold, } \mathcal{D} \text{ is a null parallel distribution of dimension } d \in \{1,2\}. \tag{2.2}$$

The facts just stated are due to Olszak [16]. Their proofs an also be found in [8, Lemma 2.1].

Lemma 2.1. *For any pseudo-Riemannian manifold (M,g) with parallel Weyl tensor, one has $d=2$ if and only if $W = \pm\,\omega\otimes\omega$ for some sign $\pm$ and some parallel differential 2-form ω on M, defined, at each point, only up to a sign, and having rank 2 at every point. The image of ω is the Olszak distribution $\mathcal{D}$.*

Proof. This is obvious from [8, Lemma 2.1(iii)] and [6, Lemma 17.1(iii)] along with our characterization of $\mathcal{D}_x$ in terms of images of nonzero 2-forms. □

If (M,g) has parallel Weyl tensor, $d=2$, and $\dim M = 4$, then $\mathcal{D}$, the image of ω (see Lemma 2.1), is a null two-dimensional distribution, and so, by [11, Lemma 37.8], at every $x \in M$ there exists a unique orientation of T_xM for which ω_x is self-dual. Consequently, M must be orientable, and a suitably chosen orientation makes the Weyl tensor self-dual. Dunajski and West [12] constructed various examples of self-dual metrics of the neutral signature $--++$ in dimension four. Some of their metrics have parallel Weyl tensor; for instance, this is the case if Q in [12, formula (6.26)] is quadratic in X.

3 The local structure: case $d=1$

Let the data $I, f, n, V, \langle\,,\rangle, A$ consist of

(i) an open interval $I \subset \mathbf{R}$, a C^∞ function $f : I \to \mathbf{R}$, and an integer $n \geq 4$,

(ii) a real vector space V of dimension $n-2$ with a pseudo-Euclidean inner product $\langle\,,\rangle$,

(iii) a nonzero traceless linear operator $A : V \to V$, self-adjoint relative to $\langle\,,\rangle$.

As in [18, Theorem 3], we use $I, f, n, V, \langle\,,\rangle, A$ to construct the pseudo-Riemannian manifold

$$(I \times \mathbf{R} \times V,\ \kappa\, dt^2 + dt\, ds + \delta) \tag{3.1}$$

of dimension n. Here products of differentials stand for symmetric products, t, s are the Cartesian coordinates on the $I \times \mathbf{R}$ factor, δ is the pullback to $I \times \mathbf{R} \times V$ of the flat pseudo-Riemannian metric on V corresponding to the inner product $\langle\,,\rangle$, and $\kappa : I \times \mathbf{R} \times V \to \mathbf{R}$ is the function with $\kappa(t,s,v) = f(t)\langle v,v\rangle + \langle Av, v\rangle$. Denoting by d the dimension of the Olszak distribution, we can characterize the manifolds (3.1) as follows. A proof can be found in [8, Theorem 4.1]:

Theorem 3.1. *For any* $I, f, n, V, \langle\,,\rangle, A$ *as in* (i) – (iii)*, the pseudo-Riemannian manifold* (3.1) *has parallel Weyl tensor and* $d = 1$*. In particular,* (3.1) *is never conformally flat.*
Conversely, in any pseudo-Riemannian manifold with parallel Weyl tensor and $d = 1$*, every point has a neighborhood isometric to an open subset of a manifold* (3.1) *constructed as above from some such data* $I, f, n, V, \langle\,,\rangle, A$*.*
The manifold (3.1) *is locally symmetric if and only if* f *is constant.*

One calls a pseudo-Riemannian manifold *Ricci-recurrent* if, for every tangent vector field v, the Ricci tensor ρ and the covariant derivative $\nabla_v\rho$ are linearly dependent at each point.
Every ECS metric with $d = 1$ is Ricci-recurrent. In fact, according to [8, Lemma 2.2(b)], $\mathcal{D}$ contains, at each point, the image of the Ricci tensor ρ, and hence also the image of $\nabla_v\rho$ for any vector field v (as $\mathcal{D}$ is parallel). Thus, ρ and $\nabla_v\rho$, being symmetric, must be linearly dependent at every point if $d = 1$. (Note that, in view of (2.2) and the inclusion just mentioned, $\operatorname{rank}\rho \le 2$ at each point of any ECS manifold.)
Theorem 3.1 was inspired by the second author's result [18, Theorem 3], which is a general-position version of Theorem 3.1: rather using the condition $d = 1$, it assumes that the metric is Ricci-recurrent, $\rho\otimes\nabla\rho \ne 0$ at every point, and $f\,df/dt \ne 0$ everywhere in I.
Among ECS manifolds with $d = 2$, in any dimension $n \ge 4$, there are both Ricci-recurrent and non-Ricci-recurrent ones. (See [6, Section 24]; the condition $\operatorname{rank} W = 1$ used there is, by [6, Lemma 17.1(iii)], equivalent to the relation $W = \pm\,\omega\otimes\omega$ in Lemma 2.1, and hence to $d = 2$.)
The local cohomogeneity of any ECS metric is at most equal to the dimension d of the Olszak distribution. If $d = 1$ (or, $d = 2$), this follows from Theorem 3.1, cf. [7, Lemma 2.2] (or, from Theorem 4.1 below, cf. [6, Remark 22.1] and [9, the comment after (f) in Section 10]).

4 The local structure: case $d = 2$

Let $(Q, \mathrm{D}, \zeta, n, \varepsilon, V, \langle\,,\rangle)$ be a septuple formed by

(a) a surface Q with a projectively flat torsionfree connection D,

(b) a D-parallel area form ζ on Q, an integer $n \ge 4$, and a sign factor $\varepsilon = \pm 1$,

(c) a real vector space V of dimension $n - 4$ with a pseudo-Euclidean inner product $\langle\,,\rangle$.

Also, let a twice-contravariant symmetric tensor ϕ on Q satisfy the differential equation

$$\mathrm{div}^{\mathrm{D}}(\mathrm{div}^{\mathrm{D}}\phi) + (\rho^{\mathrm{D}}, \phi) = \varepsilon, \tag{4.1}$$

where ρ^{D} is the Ricci tensor of D (in coordinates: $\phi^{jk}{}_{,jk} + \phi^{jk}R_{jk} = \varepsilon$).
We define τ to be the twice-covariant symmetric tensor field on Q corresponding to ϕ under the isomorphism $TQ \to T^*Q$ provided by ζ. In coordinates, $\tau_{jk} = \zeta_{jl}\zeta_{km}\phi^{lm}$.
Next, let h^{D} be the *Patterson-Walker Riemann extension metric* on T^*Q, obtained [17] by requiring that all vertical and all D-horizontal vectors be h^{D}-null, while $h^{\mathrm{D}}_x(\xi, w) =$

$\xi(d\pi_x w)$ for every $x \in T^*Q$, every vector $w \in T_x T^*Q$, and every vertical vector $\xi \in \operatorname{Ker} d\pi_x = T^*_{\pi(x)}Q$, where $\pi : T^*Q \to Q$ is the bundle projection.
Finally, we denote by δ the constant pseudo-Riemannian metric on V corresponding to the inner product $\langle\,,\rangle$, and let θ stand for the function $V \to \mathbf{R}$ with $\theta(v) = \langle v, v\rangle$.
Our septuple $(Q, \mathrm{D}, \zeta, n, \varepsilon, V, \langle\,,\rangle)$ now gives rise to the pseudo-Riemannian manifold

$$(T^*Q \times V,\ h^{\mathrm{D}} - 2\tau + \delta - \theta\rho^{\mathrm{D}})\,, \tag{4.2}$$

of dimension n, with the metric $h^{\mathrm{D}} - 2\tau + \delta - \theta\rho^{\mathrm{D}}$, where the function θ and covariant tensor fields $h^{\mathrm{D}}, \tau, \rho^{\mathrm{D}}, \delta$ on T^*Q, Q or V are identified with their pullbacks to $T^*Q \times V$. With d again denoting the dimension of the Olszak distribution, we have the following result.

Theorem 4.1. *The pseudo-Riemannian manifold* (4.2) *obtained as above from any given septuple* $(Q, \mathrm{D}, \zeta,\ n, \varepsilon, V, \langle\,,\rangle)$ *with* (a) – (c) *has parallel Weyl tensor and* $d = 2$.
Conversely, in any pseudo-Riemannian manifold with parallel Weyl tensor and $d = 2$, *every point has a neighborhood isometric to an open subset of a manifold* (4.2) *constructed from some septuple* $(Q, \mathrm{D}, \zeta, n, \varepsilon, V, \langle\,,\rangle)$ *as in* (a) – (c).
The manifold (4.2) *is never conformally flat, and it is locally symmetric if and only if the Ricci tensor* ρ^{D} *is* D*-parallel.*

Our septuple $(Q, \mathrm{D}, \zeta, n, \varepsilon, V, \langle\,,\rangle)$ of parameters does not include ϕ, even though the metric in (4.2) evidently depends on τ (and hence on ϕ). This is justified by the fact that, locally, ϕ with (4.1) always exists, and, for any fixed $(Q, \mathrm{D}, \zeta, n, \varepsilon, V, \langle\,,\rangle)$, the manifolds (4.2) corresponding to two choices of ϕ are, locally, isometric to each other [6, Remark 22.1].
In dimension four, $V = \{0\}$ and $\langle\,,\rangle = 0$, so that the septuple $(Q, \mathrm{D}, \zeta, n, \varepsilon, V, \langle\,,\rangle)$ with (a) – (c) may be replaced by a quadruple $(Q, \mathrm{D}, \zeta, \varepsilon)$ consisting of a surface Q, a projectively flat torsionfree connection D on Q, a D-parallel area form ζ on Q, and a sign factor $\varepsilon = \pm 1$. The pair (4.2) then becomes the pseudo-Riemannian four-manifold $(T^*Q,\ h^{\mathrm{D}} - 2\tau)$.

5 Compact ECS manifolds: existence

Theorem 5.1. *In every dimension* $n \geq 5$ *with* $n \equiv 5 \pmod 3$, *there exists a compact ECS manifold of any prescribed indefinite metric signature, diffeomorphic to a nontrivial torus bundle over the circle.*

A proof of Theorem 5.1 is outlined in §9. For a more detailed argument, see [7].
The compact ECS manifolds that are shown to exist in §9 have further properties, not included in the statement of Theorem 5.1. Some of these properties are mentioned in §8 (items II, IV and V). The first four steps of the proof in §9 might *a priori* produce ECS metrics on bundles over the circle, the fibre of which is either a torus, or a 2-step nilmanifold admitting a complete flat torsionfree connection with a nonzero parallel vector field. However, our particular existence argument realizes only the torus as the fibre. See [7, Remarks 4.1(iv) and 6.2].

6 Compact ECS manifolds: properties

First, the existence of an ECS metric on a given compact manifold imposes specific restrictions on the fundamental group, Euler characteristic, and real Pontryagin classes:

Theorem 6.1. *If a compact manifold* M *of dimension* $n \geq 4$ *admits an ECS metric, then* $\pi_1 M$ *is infinite,* $\chi(M) = 0$*, and* $p_i(M) = 0$ *in* $H^{4i}(M, \mathbf{R})$ *for all* $i \geq 1$*.*

Here $\chi(M)$ and $p_i(M)$ vanish since so do the Gauss-Bonnet-Chern integrand and the Pontryagin forms [9]. For proofs of infiniteness of $\pi_1 M$ and the next two theorems, see [9] as well.

Theorem 6.2. *Every four-dimensional Lorentzian ECS manifold is noncompact.*

Theorem 6.3. *Let* (M, g) *be a compact Lorentzian ECS manifold. Then some two-fold covering manifold of* M *is the total space of a* C^∞ *bundle over the circle, the fibre of which admits a flat torsionfree connection with a nonzero parallel vector field.*

If (M, g) is a compact ECS manifold and $d \in \{1, 2\}$ is the dimension of its Olszak distribution (§2), then $TM = H \oplus H^+ \oplus H^-$ for some vector subbundles H, H^+ and H^- of TM such that H^+ is spacelike, H^- is timelike, and both $H^\pm$ have the fibre dimension d.
In fact, we obtain H^+ and H^- by decomposing TM into a spacelike and a timelike subbundle, and then projecting the null distribution $\mathcal{D}$ onto the summands.

7 Some open questions

I. Do compact ECS manifolds exist in dimension four?

II. Does any torus admit an ECS metric? More generally, does there exist a compact ECS manifold with an Abelian fundamental group?

III. Are there compact ECS manifolds of dimensions $n \geq 5$ other than $n = 3j + 2$, $j \in \mathbf{Z}$?

IV. Can a compact ECS manifold be locally homogeneous?

V. Must the Olszak distribution of a compact ECS manifold be one-dimensional? More generally, are all compact ECS manifolds Ricci-recurrent?

8 Comments on Questions I – V in §7

I. If the answer to Question I is 'yes' and compact four-dimensional ECS manifolds do exist, they all must have the neutral metric signature $--++$. In fact, they can be neither Riemannian [4, Theorem 2], nor Lorentzian (Theorem 6.2). See also item V below.

II. None of the known compact ECS manifolds admits a finite covering by a manifold with an Abelian fundamental group.

III. The matrices (9.3) acting in $\mathbf{R}^3$ lead to a translation operator that has the property required in STEP 3 of §9. The jth Cartesian-power extension of (9.3) does the same in $\mathbf{R}^{3j}$, which is why our argument yields compact ECS manifolds of dimensions $n = 3j+2$. A "building block" $\mathbf{R}^m$, $m \geq 4$, instead of $\mathbf{R}^3$, with operators analogous to (9.3), would lead to examples in other dimensions, and it seems reasonable to expect such operators to exist, although they might be harder to find than those appearing in (9.3).

IV. Locally homogeneous ECS manifolds exist [3] in every dimension $n \geq 4$. However, none of the compact ECS manifolds arising from the argument in §9 is locally homogeneous.

V. Every known compact ECS manifold has a one-dimensional Olszak distribution, and is thereforee Ricci-recurrent (§3). One might try to answer Question I in the affirmative and, simultaneously, the first part of Question V in the negative, by proceeding as follows. We begin by choosing a quadruple $(Q, \mathrm{D}, \zeta, \varepsilon)$ as at the end of §4, with a *closed* surface Q. (According to [6, Section 23], such quadruples exist, and realize all diffeomorphic types of closed surfaces Q.) As a next step, we need to find a discrete group Γ of isometries of the four-dimensional ECS manifold $(T^*Q,\ h^{\mathrm{D}} - 2\tau)$, acting on T^*Q properly discontinuously with a compact quotient. There is a topological restriction: Γ as above cannot exist unless Q is diffeomorphic to the torus T^2 or the Klein bottle K^2 (see below). We do not know if such Γ exists either for T^2 or for K^2, even though, on T^2, the connections in question are relatively well understood, cf. [6, the discussion following Remark 7.2].

The "topological restriction" mentioned under V can be phrased as follows.
If Q is a compact manifold and some group Γ of diffeomorphisms of T^*Q acts on T^*Q properly discontinuously with a compact quotient, then $\chi(Q) = 0$. In fact, we may assume that Q is orientable (by passing, if necessary, to a two-fold orientable covering and replacing Γ with a $\mathbf{Z}_2$ extension). Since the inclusion of the zero section Q in T^*Q is a homotopy equivalence, a generator $[Q]$ of $H_m(Q, \mathbf{Z}) \approx \mathbf{Z}$, for $m = \dim Q$, is also a generator of $H_m(T^*Q, \mathbf{Z})$, and so its image $F_*[Q]$ under any diffeomorphism $F : T^*Q \to T^*Q$ equals $\pm[Q]$. We now choose $F \in \Gamma$ such that Q and $F(Q)$ are disjoint: due to compactness of Q and proper discontinuity of the action, this is the case for all but finitely many $F \in \Gamma$. Thus, $\chi(Q) = [Q] \cdot [Q] = \pm[Q] \cdot F_*[Q] = 0$, where $\cdot$ is the intersection form in $H_m(T^*Q, \mathbf{Z})$. (The same argument remains valid if T^*Q is replaced by the total space of an orientable real vector bundle of fibre dimension n over a compact orientable n-dimensional manifold Q, the conclusion being now that the Euler number of the vector bundle must vanish.)

9 Proof of Theorem 5.1 (an outline)

STEP 1. Consider an n-dimensional pseudo-Riemannian ECS manifold (3.1) constructed from some $I, f, n, V, \langle\,,\rangle, A$ with (i) – (iii) in §3 such that $I = \mathbf{R}$, while f is nonconstant (cf. Theorem 3.1) and periodic, for some period $p > 0$. Let $\mathcal{E}$ be the vector space of all C^∞ solutions $u : \mathbf{R} \to V$ to the differential equation $\ddot{u}(t) = f(t)u(t) + Au(t)$. The set $\mathrm{G} = \mathbf{Z} \times \mathbf{R} \times \mathcal{E}$ has a group structure such that, setting $(k, q, u) \cdot (t, s, v) =$

$(t+kp, s+q-\langle\dot u(t), 2v+u(t)\rangle, v+u(t))$, for $(k,q,u)\in \mathrm{G}$ and $(t,s,v)\in \mathbf{R}^2\times V$, we define an action of G on $\mathbf{R}^2\times V$ from the left. The action of G consists of isometries of our manifold (3.1). See [7, Lemma 2.2].

STEP 2. Question: when does a manifold (3.1) selected as in STEP 1, with the corresponding $f, n, V, \langle\,,\rangle, A, p, \mathcal{E}, \mathrm{G}$ and $I = \mathrm{R}$, lead to a compact ECS manifold that arises as the quotient of (3.1) under a discrete subgroup Γ of G acting on $\mathbf{R}^2\times V$ properly discontinuously? Answer: such Γ exists if and only if some C^∞ curve $\mathbf{R}\ni t\mapsto B(t)\in \mathrm{End}(V)$, periodic of period p, satisfies the differential equation $\dot B(t)+[B(t)]^2 = f(t)+A$ (briefly, $\dot B + B^2 = f + A$) and, at the same time, certain additional conditions, listed in [7, Theorem 6.1(ii)], hold for some lattice Σ in the vector space $\mathcal{W} = \mathbf{R}\times\mathcal{L}$, where $\mathcal{L}$ is the space of all C^∞ functions $u:\mathbf{R}\to V$ with $\dot u(t) = B(t)u(t)$. (Thus, $\mathcal{L}\subset\mathcal{E}$.) The most important of these additional conditions is an arithmetic property of the *translation operator* $T:\mathcal{L}\to\mathcal{L}$ acting by $(Tu)(t) = u(t-p)$. Namely, we require the existence of a linear functional $\varphi\in\mathcal{L}^*$ such that $\Psi(\Sigma)=\Sigma$ for the operator $\Psi:\mathcal{W}\to\mathcal{W}$ given by $\Psi(r,u) = (r+\varphi(u), Tu)$.

STEP 3. Observe that, in STEP 2, a lattice Σ with $\Psi(\Sigma)=\Sigma$ for some functional φ exists if and only if $\det T = \pm 1$ and the matrix of T in some basis of $\mathcal{L}$ consists of integers, cf. [7, the end of Section 1]. Thus, the main part of our task is to find B with $\dot B + B^2 = f + A$ such that the corresponding translation operator T has the property just stated.

STEP 4. Fix integers k and l with $4 < k < l \le k^2/4$. (Such l exists for any given $k > 4$.) It is an easy exercise [7, Lemma 1.3] to verify that the polynomial $P(\lambda) = -\lambda^3 + k\lambda^2 - l\lambda + 1$ then has real roots λ,μ,ν with $0<\lambda<\mu<\nu$, $\lambda<1<\nu$, $\lambda\mu<1<\mu\nu$ and $\lambda\nu\ne 1$.

STEP 5. Fix $p\in(0,\infty)$ and denote by $\mathcal{F}_p$ the set of all septuples $(\alpha,\beta,\gamma,f,a,b,c)$ consisting of C^∞ functions $\alpha,\beta,\gamma,f:\mathbf{R}\to\mathbf{R}$ of the variable t, periodic of period p, and three distinct real constants a,b,c with $a+b+c=0$, of which b is the smallest, such that, with $(\ )^\cdot = d/dt$,

$$\dot\alpha+\alpha^2 = f+a, \qquad \dot\beta+\beta^2 = f+b, \qquad \dot\gamma+\gamma^2 = f+c, \tag{9.1}$$

and $\alpha>\beta>\gamma$. Let $\mathcal{C}$ be the subset of $\mathcal{F}_p$ formed by all $(\alpha,\beta,\gamma,f,a,b,c)$ with *constant* α,β,γ and f. Define a mapping $\mathrm{spec}:\mathcal{F}_p\to\mathbf{R}^3$ by $\mathrm{spec}(\alpha,\beta,\gamma,f,a,b,c) = (\lambda,\mu,\nu)$, where

$$(\lambda,\mu,\nu) \;=\; (\exp[-\textstyle\int_0^p\alpha(t)\,dt],\ \exp[-\int_0^p\beta(t)\,dt],\ \exp[-\int_0^p\gamma(t)\,dt]). \tag{9.2}$$

STEP 6. Show that $\mathrm{spec}(\mathcal{F}_p\smallsetminus\mathcal{C}) = \mathcal{U}$, where $\mathcal{U}\subset\mathbf{R}^3$ is the open set of all (λ,μ,ν) with $0<\lambda<\mu<\nu$, $\lambda<1<\nu$, $\lambda\mu<1<\mu\nu$ and $\lambda\nu\ne1$. This is done by reducing the number of unknown functions in a septuple with (9.1). Specifically, to solve the two-equation system $\dot\alpha+\alpha^2 = f+a$, $\dot\beta+\beta^2=f+b$ with $\alpha>\beta$ (treating our a,b as fixed), one sets $\rho=\alpha-\beta$ and $\psi=\alpha+\beta$. As $\psi = (a-b-\dot\rho)/\rho$, we may reconstruct α and β from ρ. Solutions (α,β) of the two-equation system are thus in a bijective correspondence with arbitrary C^∞ functions $\rho:\mathbf{R}\to(0,\infty)$, periodic of period p. Similarly, a solution to the last two equations in (9.1) is represented by a single arbitrary positive function analogous to ρ, which we denote by σ. Our ρ and σ are subject to a single differential equation, stating that β reconstructed from ρ is the same as β reconstructed from σ. The function $\log(\sigma/\rho)$ may, however, be completely arbitrary (aside from being of class C^∞ and periodic of period p); see [7, Lemma 9.6]. Expressing the triple (9.2) in terms of the new unknown function $\log(\sigma/\rho)$, we now verify that $\operatorname{spec}(\mathcal{F}_p \smallsetminus \mathcal{C}) = \mathcal{U}$.

STEP 7. Using (λ, μ, ν) obtained in STEP 4, apply STEP 6 to pick $(\alpha, \beta, \gamma, f, a, b, c) \in \mathcal{F}_p \smallsetminus \mathcal{C}$ with $\operatorname{spec}(\alpha, \beta, \gamma, f, a, b, c) = (\lambda, \mu, \nu)$, and then set

$$B(t) = \begin{bmatrix} \alpha(t) & 0 & 0 \\ 0 & \beta(t) & 0 \\ 0 & 0 & \gamma(t) \end{bmatrix}, \qquad A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}. \tag{9.3}$$

As a consequence of (9.1), we now have $\dot{B} + B^2 = f + A$. Treating the matrices (9.3) as endomorphisms of the space $\mathbf{R}^3$ endowed with the standard pseudo-Euclidean inner product (of any signature), and noting that all $B(t)$ commute with one another, we easily conclude that the spectrum of the corresponding translation operator (STEP 2) is given by (9.2), and hence coincides with our fixed (λ, μ, ν). Thus, $P(\lambda)$ appearing in STEP 4 is the characteristic polynomial of T, and, consequently, T has the property required in STEP 3 (see [7, the end of Section 1]). Also, f is nonconstant since $(\alpha, \beta, \gamma, f, a, b, c) \notin \mathcal{C}$, cf. [7, Remark 9.1]. The same conclusions hold if the endomorphisms (9.3) are replaced by their jth Cartesian powers acting in $V = \mathbf{R}^{3j}$, $j \geq 1$. According to STEP 3, this completes the proof.

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