NONCOMPACTNESS AND MAXIMUM MOBILITY OF TYPE III RICCI-FLAT SELF-DUAL NEUTRAL WALKER FOUR-MANIFOLDS

by ANDRZEJ DERDZINSKI[†]

(Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA)

Abstract

It is shown that a self-dual neutral Einstein four-manifold of Petrov type III, admitting a two-dimensional null parallel distribution compatible with the orientation, cannot be compact or locally homogeneous, and its maximum possible degree of mobility is 3. Díaz-Ramos, García-Río and Vázquez-Lorenzo found a general coordinate form of such manifolds. The present paper also provides a modified version of that coordinate form, valid in a suitably defined generic case and, in a sense, "more canonical" than the usual formulation. Moreover, the local-isometry types of manifolds as above having the degree of mobility equal to 3 are classified. Further results consist in explicit descriptions, first, of the kernel and image of the Killing operator for any torsionfree surface connection with everywhere-nonzero, skew-symmetric Ricci tensor, and, secondly, of a moduli curve for surface connections with the properties just mentioned that are, in addition, locally homogeneous. Finally, hyperbolic plane geometry is used to exhibit examples of codimension-two foliations on compact manifolds of dimensions greater than 2 admitting a transversal torsionfree connection, the Ricci tensor of which is skew-symmetric and nonzero everywhere. No such connection exists on any closed surface, so that there are no analogous examples in dimension 2.

1. Introduction

A traceless endomorphism of a pseudo-Euclidean 3-space is said to be of *Petrov type* III if it is self-adjoint and sends some ordered basis p, q, r to $0, p, q$.

By a *type* III *SDNE manifold* we mean a *self-dual neutral Einstein four-manifold* (M, g) *of Petrov type* III. In other words, (M, g) is assumed to be a self-dual oriented Einstein four-manifold of the neutral metric signature $(--++)$, such that the self-dual Weyl tensor W^+ of (M, g), acting on self-dual 2-forms, is of Petrov type III at every point.

This paper deals with type III SDNE manifolds (M, g) having the *Walker property*, that is, admitting a two-dimensional null parallel distribution which is compatible with the orientation in the sense explained immediately before Remark 2.1. The main results, Theorems 9.3 and 13.1, state that such (M, g) is never compact, while its degree of mobility is at most 3, and so (M, g) cannot be locally homogeneous. In the case where the degree of mobility equals 3, the local-isometry types of (M, g) are explicitly classified (Theorem 16.3).

Questions about type III SDNE manifolds arise for two reasons. First, these are precisely the *type* III *Jordan-Osserman four-manifolds* [**8**, Remark 2.1], a subclass of the class of Jordan-Osserman manifolds, studied by many authors [**11**, **12**]. Secondly, type III SDNE metrics are all curvature homogeneous [**3**, pp. 247–248], so that understanding their structure is a step towards a description of all curvature-homogeneous pseudo-Riemannian Einstein metrics in dimension four.

[†]E-mail: andrzej@math.ohio-state.edu

Of the two main results mentioned above, Theorem 9.3 follows from the divergence formula: as shown in Lemma 9.2, every type III SDNE Walker manifold carries a natural vector field with nonzero constant divergence. The proof of Theorem 13.1 uses, in turn, some conclusions about pairs (Σ, ∇) formed by a surface Σ and a torsionfree connection ∇ on Σ with everywhere-nonzero, skew-symmetric Ricci tensor. Specifically, Sections 12, 14 and 17 contain a characterization of the image and kernel of the *Killing operator* of (Σ, ∇), which sends each 1-form to its symmetrized ∇-covariant derivative. The other conclusion, Theorem 11.4, describes a moduli curve for pairs (Σ, ∇) with the properties just listed that are, in addition, locally homogeneous. (A canonical coordinate form of such locally homogeneous pairs (Σ, ∇) was first found by Kowalski, Opozda and Vlášek [**13**].)

Pairs (Σ, ∇) as above are naturally related to type III SDNE Walker manifolds. Namely, Díaz-Ramos, García-Río and Vázquez-Lorenzo proved in [**8**, Theorem 3.1(ii.3)] that, locally, type III SDNE Walker metrics are nothing else than Patterson and Walker's Riemann-extension metrics associated with triples (Σ, ∇, τ) consisting of any such pair (Σ, ∇) and an arbitrary symmetric 2-tensor τ on Σ. For details, see Section 7. Although τ, unlike Σ and ∇, is not a geometric invariant of the metric g, a canonical choice of τ is possible for a class of type III SDNE Walker manifolds satisfying a general-position requirement introduced in Section 15.

The result of [**8**, Theorem 3.1(ii.3)] implies that every type III SDNE Walker manifold carries a codimension-two foliation admitting a transversal torsionfree connection with everywhere-nonzero, skew-symmetric Ricci tensor. The fact that type III SDNE Walker manifolds are noncompact (Theorem 9.3) cannot be derived just from the presence of such a foliation. Namely, as shown in Proposition 19.2 and Corollary 19.3, foliations with the stated property exist on compact manifolds of all dimensions $n \geq 3$ (though not for $n = 2$).

It is unknown whether the conclusion about noncompactness of all type III SDNE Walker manifolds, established in Theorem 9.3(a), remains valid in the non-Walker case. That the two cases differ in global properties other than compactness is exemplified by *vertical completeness*, introduced in Section 20. Specifically, type III SDNE manifolds (M, g) are sometimes vertically complete, yet, according to Theorem 20.1, this can happen only if g is a Walker metric.

2. Preliminaries

Manifolds are by definition connected. All manifolds, bundles, their sections and subbundles, as well as connections and mappings, including bundle morphisms, are assumed to be C^∞-differentiable. A bundle morphism always operates between two bundles with the same base manifold, and acts by identity on the base.

By the *degree of mobility* of a connection ∇ (or, of a pseudo-Riemannian metric g) on a manifold Σ we mean the function assigning to each $y \in \Sigma$ the dimension of the Lie algebra $\mathfrak{a}_y$ (or, $\mathfrak{i}_y$) formed by the germs at y of infinitesimal affine transformations for ∇ (or, respectively, of Killing fields for g). This function is constant when ∇ (or, g) is locally homogeneous.

For a connection ∇ in a real vector bundle over a manifold Σ, sections α of the bundle and vector fields v, w tangent to Σ, our sign convention about the curvature tensor R of ∇ is

$$(2.1) \qquad R(v, w)\alpha = \nabla_w \nabla_v \alpha - \nabla_v \nabla_w \alpha + \nabla_{[v,w]} \alpha.$$

If ∇ is a connection on Σ (that is, in the tangent bundle $T\Sigma$), we treat the covariant derivative ∇w of any vector field w as

$$(2.2) \qquad \text{the bundle morphism } \nabla w : T\Sigma \to T\Sigma \text{ sending each vector field } v \text{ to } \nabla_v w.$$

Given a manifold Σ and a bundle morphism $A : T\Sigma \to T\Sigma$,

$$(2.3) \qquad A^* : T^*\Sigma \to T^*\Sigma \quad \text{denotes the bundle morphism dual to } A,$$

so that A^* sends any 1-form ξ to the composite $A^*\xi = \xi A$ in which $A : T\Sigma \to T\Sigma$ is followed by the morphism ξ from $T\Sigma$ to the product bundle $\Sigma \times \mathbb{R}$.

For the Ricci tensor ρ of a torsionfree connection ∇ on a manifold Σ, and any tangent vector field w, the *Bochner identity* states that

$$(2.4) \qquad \text{a)} \quad \rho(\,\cdot\,, w) = \operatorname{div}[\nabla w] - d[\operatorname{div} w], \quad \text{where} \qquad \text{b)} \quad \operatorname{div} w = \operatorname{tr} \nabla w.$$

Cf. [**9**, formula (4.39) on p. 449]. Here $\operatorname{tr} \nabla w : \Sigma \to \mathbb{R}$ is the pointwise trace of (2.2).

In fact, the coordinate form of (2.4.a), $R_{jk}w^k = w^k{}_{,jk} - w^k{}_{,kj}$, arises by contraction in $l = k$ from the Ricci identity $w^l{}_{,jk} - w^l{}_{,kj} = R_{jks}{}^l w^s$, which in turn is nothing else than (2.1).

For the tensor, exterior and symmetric products, and the exterior derivative of 1-forms β, ξ on a manifold, any tangent vector fields u, w, and any fixed torsionfree connection ∇, we have

$$(2.5) \quad \begin{array}{ll} \text{a)} & [\beta \otimes \xi](u, w) = \beta(u)\xi(w), \quad \beta \wedge \xi = \beta \otimes \xi - \xi \otimes \beta, \quad 2\beta \odot \xi = \beta \otimes \xi + \xi \otimes \beta, \\ \text{b)} & [d\beta](u, w) = d_u[\beta(w)] - d_w[\beta(u)] - \beta([u, w]), \\ \text{c)} & [d\beta](u, w) = [\nabla_u \beta](w) - [\nabla_w \beta](u). \end{array}$$

Since $\mathbb{L}_v = d\iota_v + \iota_v d$ for the Lie derivative $\mathbb{L}_v$ acting on differential forms, it follows that

$$(2.6) \qquad \mathbb{L}_v \zeta = d[\zeta(v, \,\cdot\,)]$$

whenever ζ is a 2-form on a surface and v is a tangent vector field.

Suppose now that E is a finite-dimensional real vector space.

(i) Whenever a subspace E' of E contains the image of an endomorphism T of E, the trace of $T : E \to E$ is obviously equal to the trace of the restriction $T : E' \to E'$.

(ii) Given an m-form $\zeta \in [E^*]^{\wedge m}$, where $m = \dim E$, and any endomorphism T of E, the sum $\zeta(Tv_1, v_2, \ldots, v_m) + \zeta(v_1, Tv_2, v_3, \ldots, v_m) + \ldots + \zeta(v_1, v_2, \ldots, v_{m-1}, Tv_m)$ equals $\zeta(v_1, v_2, \ldots, v_m) \operatorname{tr} T$, for any $v_1, \ldots, v_m \in E$. One sees this using the matrix of T in the basis $v_1, \ldots, v_m$, if $v_1, \ldots, v_m$ are linearly independent, and noting that both sides vanish for reasons of skew-symmetry, if $v_1, \ldots, v_m$ are linearly dependent.

(iii) If $\dim E = 2$ and a trilinear mapping $(v, v', v'') \mapsto \chi(v, v', v'')$ from E into any vector space is skew-symmetric in v', v'', then $\chi(v, v', v'')$ summed cyclically over v, v', v'' yields 0. In fact, the cyclic sum depends on v, v' and v'' skew-symmetrically, so that it vanishes as E is two-dimensional.

It is well-known (see, e.g., [**9**, Proposition 37.1(i) on p. 638]) that any null two-dimensional subspace S in a pseudo-Euclidean 4-space E of the neutral signature $(--++)$ naturally distinguishes an orientation of E, namely, the one which, for some/any basis u, v of S, makes the bivector $u \wedge v$ self-dual. This makes it meaningful to say that a null distribution of dimension 2 on a pseudo-Riemannian four-manifold (M, g) of the neutral metric signature is, or is not, *compatible* with a prescribed orientation of M.

Remark 2.1. Let $\mathcal{V}$ be an integrable distribution on a manifold M. The maximal integral manifolds of $\mathcal{V}$ will be simply referred to as the *leaves of* $\mathcal{V}$. (They are the leaves of the foliation on M, the tangent bundle of which is $\mathcal{V}$.) We will also speak of *sections of* $\mathcal{V}$, treating $\mathcal{V}$ as a vector subbundle of TM. Finally, by a $\mathcal{V}$-*projectable local vector field* in M

we will mean any vector field w defined on a nonempty open set $U \subset M$ and such that, whenever v is a section of $\mathcal{V}$ defined on U, so is $[w,v]$. If, in addition,

(2.7)   the leaves of $\mathcal{V}$ restricted to U are all contractible and constitute
           the fibres of a bundle projection $\pi : U \to \Sigma$ over some manifold Σ,

then $\mathcal{V}$-projectability of a vector field w defined on U is equivalent to its π-projectability. (This is easily seen in suitable local coordinates.)

3. The Codazzi and Killing operators

By a *k-tensor* on a manifold Σ we always mean a k times covariant tensor field on Σ. For instance, the curvature 4-tensor $\bar{R}$ of a pseudo-Riemannian manifold (M, g) is characterized by $\bar{R}(u, v, u', v') = g(\bar{R}(u, v)u', v')$, where u, v, u', v' are any tangent vector fields and $\bar{R}$ on the right-hand side is defined as in (2.1) for the Levi-Civita connection $\overline{\nabla}$ of g.

A connection ∇ on a manifold Σ gives rise to two first-order linear differential operators that will repeatedly appear in our discussion. One is the *Codazzi operator* d^{∇}, sending each symmetric 2-tensor τ on Σ to the 3-tensor equal to twice the skew-symmetrization of the ∇-covariant derivative of τ in the first two arguments. The other is the *Killing operator* $\mathcal{L}$, which sends each 1-form ξ on Σ to the symmetric 2-tensor obtained by symmetrizing the ∇-covariant derivative of ξ. Explicitly, for any tangent vector fields u, v,

(3.1)   a) $[d^{\nabla}\tau](u, v, \cdot) = [\nabla_u \tau](v, \cdot) - [\nabla_v \tau](u, \cdot)$,   b) $2[\mathcal{L}\xi](u, v) = [\nabla_u \xi](v) + [\nabla_v \xi](u)$.

We denote by $\operatorname{Ker}\mathcal{L}$ the space of all C^{∞}-differentiable 1-forms ξ with $\mathcal{L}\xi = 0$.

If ∇ is torsionfree, the second covariant derivative $\nabla\nabla\xi$ of any 1-form ξ on Σ and the tensor field $\tau = \mathcal{L}\xi$ satisfy the following well-known relation, immediate from the Ricci and Bianchi identities, cf. [**7**, the bottom of p. 572], in which both sides are 2-tensors:

(3.2) $\nabla_v \nabla\xi = -\xi[R(\cdot, \cdot)v] + [d^{\nabla}\tau](\cdot, \cdot, v) + \nabla_v \tau$ whenever $\tau = \mathcal{L}\xi$.

Here v stands for an arbitrary vector field, d^{∇} denotes the Codazzi operator with (3.1.a), and $\nabla\xi$ is treated as a 2-tensor acting on vector fields u, v by $[\nabla\xi](u, v) = [\nabla_u \xi](v)$. In coordinates, (3.2) reads $\xi_{j,kl} = -R_{kjl}{}^s \xi_s + \tau_{jl,k} - \tau_{kl,j} + \tau_{kj,l}$, with $\tau_{kj} = (\xi_{j,k} + \xi_{k,j})/2$.

4. Riemann extensions

Let $M = T^*\Sigma$ be the total space of the cotangent bundle of a manifold Σ carrying a torsionfree connection ∇, and let $\pi : T^*\Sigma \to \Sigma$ denote the bundle projection. Following Patterson and Walker [**16**, p. 26], by a *Riemann extension metric* associated with ∇ we mean any 2-tensor on M having the form $g = g^{\nabla} + 2\pi^*\tau$, where τ is a symmetric 2-tensor on Σ, and g^{∇} stands for the pseudo-Riemannian metric on $T^*\Sigma$ defined by requiring that all vertical and all ∇-horizontal vectors be g^{∇}-null, while $g_x^{\nabla}(\xi, w) = \xi(d\pi_x w)$ for any $x \in M = T^*\Sigma$, any $w \in T_x M$, and any vertical vector $\xi \in \operatorname{Ker} d\pi_x = T_y^*\Sigma$, with $y = \pi(x)$. Such g is clearly a pseudo-Riemannian metric of the neutral signature. In local coordinates y^j, q_j for $T^*\Sigma$ arising from a coordinate system y^j for Σ in which ∇ has the components Γ_{kl}^j,

(4.1) $g = 2dq_j \odot dy^j + 2(\tau_{kl} - q_j \Gamma_{kl}^j)\, dy^k \odot dy^l$,

cf. [**16**, formula (28)], with the symmetric multiplication $\odot$ given by (2.5.a).

We use the term 'Riemann extension' narrowly, as dictated by the specific applications described in Section 7. Wider classes of Riemann extensions have been discussed by many authors, for instance, in [**16**], [**1**] and, most recently, [**4**].

Part (a) of the following lemma goes back to Patterson and Walker [**16**, §8].

Lemma 4.1. *Let Σ, ∇ and τ have the properties listed above.*

(a) *Any 1-form ξ on Σ gives rise to a diffeomorphism $K_\xi : M \to M$, acting as the translation by ξ_y in the fibre $T_y^*\Sigma$ of $M = T^*\Sigma$, for every $y \in \Sigma$, and the K_ξ-pullback of $g^\nabla + 2\pi^*\tau$ is $g^\nabla + 2\pi^*(\tau + \mathcal{L}\xi)$, with $\mathcal{L}$ as in (3.1.b).*

(b) *In particular, for any 1-form ξ on Σ, the metrics $g^\nabla + 2\pi^*\tau$ and $g^\nabla + 2\pi^*(\tau + \mathcal{L}\xi)$ on M are isometric to each other.*

(c) *If $\Theta : T^*\Sigma \to T^*\Sigma$ is a vector-bundle isomorphism and the Θ-pullback of $g^\nabla + 2\pi^*\tau$, restricted to some nonempty open set in $T^*\Sigma$ interseting $T_y^*\Sigma$ for each $y \in \Sigma$, coincides with $g^\nabla + 2\pi^*\tau'$ for some symmetric 2-tensor τ' on Σ, then $\Theta = \mathrm{Id}$ and $\tau' = \tau$.*

Proof. This is immediate if one replaces the ingredients $q_j, y^j, \tau_{kl}, \Gamma_{kl}^j$ of formula (4.1) with their pullbacks under K_ξ (or, under Θ), that is, with $q_j + \xi_j$ (or, $\Theta_j^l q_l$), y^j, τ_{kl} and Γ_{kl}^j. $\square$

To avoid confusion caused by the presence of both upper and lower indices in y^j, q_j, we fix a nonsingular square matrix $[g_{j\lambda}]$ of constants, and replace y^j, q_j with the new coordinates y^j, x^λ, related to the old ones by $q_j = g_{j\lambda}x^\lambda$. Keeping the Greek indices λ, μ always separate from the Roman indices j, k, l, p, q, s, even though both sets of indices range over $\{1, \ldots, \dim\Sigma\}$, we easily verify that the components of $g = g^\nabla + 2\pi^*\tau$, its reciprocal metric, the Levi-Civita connection $\overline{\nabla}$ of g, and its curvature 4-tensor $\bar{R}$, in the coordinates y^j, x^λ, are given by

$$
\begin{aligned}
(4.2) \quad & g_{jk} = 2(\tau_{jk} - g_{s\lambda}x^\lambda\Gamma_{jk}^s), \quad g_{\lambda\mu} = 0, \quad g_{\lambda j} = g_{j\lambda} \text{ (the fixed constants)}, \\
& g^{jk} = 0, \quad [g^{j\lambda}] = [g_{j\lambda}]^{-1}, \quad g^{\lambda\mu} = -g^{j\lambda}g^{k\mu}g_{jk}, \\
& \bar{\Gamma}_{\lambda\mu}^{\cdot} = 0, \quad \bar{\Gamma}_{\lambda\cdot}^{j} = 0, \quad \bar{\Gamma}_{jk}^l = \Gamma_{jk}^l, \quad \bar{\Gamma}_{j\lambda}^\mu = -g_{s\lambda}g^{k\mu}\Gamma_{jk}^s, \\
& g_{k\mu}\bar{\Gamma}_{jl}^\mu = g_{s\lambda}x^\lambda(R_{lkj}{}^s - \partial_j\Gamma_{lk}^s + \Gamma_{kp}^s\Gamma_{jl}^p + \Gamma_{lp}^s\Gamma_{jk}^p) + \tau_{lk,j} + (d^\nabla\tau)_{lkj}, \\
& \bar{R}_{\lambda\mu\cdot\cdot} = \bar{R}_{\lambda\cdot\mu\cdot} = 0, \quad \bar{R}_{jkl\lambda} = g_{s\lambda}R_{jkl}{}^s, \\
& \bar{R}_{jklp} = g_{s\lambda}x^\lambda(R_{lpj}{}^s{}_{,k} - R_{lpk}{}^s{}_{,j} + \Gamma_{jq}^sR_{lpk}{}^q - \Gamma_{kq}^sR_{lpj}{}^q + \Gamma_{lq}^sR_{jkp}{}^q - \Gamma_{pq}^sR_{jkl}{}^q) \\
& \quad + R_{kjp}{}^s\tau_{sl} - R_{kjl}{}^s\tau_{sp} + (d^\nabla\tau)_{lpj,k} - (d^\nabla\tau)_{lpk,j}.
\end{aligned}
$$

Here the dots stand for indices of either kind, Γ_{jk}^l and $R_{jkl}{}^s$ for the components of ∇ and its curvature tensor R in the coordinates y^j, the commas for ∇-covariant derivatives, d^∇ for the Codazzi operator of ∇ (see (3.1.a)), and $[g_{j\lambda}]^{-1}$ for the matrix inverse of $[g_{j\lambda}]$.

One easily obtains the following conclusion, due to Patterson and Walker [**16**, p. 26]:

Lemma 4.2. *For $g = g^\nabla + 2\pi^*\tau$ defined as above, the following three conditions are equivalent:*

(i) *g is an Einstein metric,*

(ii) *g is Ricci-flat,*

(iii) *the Ricci tensor ρ of ∇ is skew-symmetric at every point.*

Proof. By (4.2), the Ricci tensor $\bar{\rho}$ of g has the components $\bar{\rho}_{jk} = g^{s\lambda}\bar{R}_{jsk\lambda} + g^{\lambda s}\bar{R}_{j\lambda ks} = \rho_{jk} + \rho_{kj}$ and $\bar{\rho}_{j\lambda} = \bar{\rho}_{\lambda\mu} = 0$, and the so scalar curvature of g is $g^{jk}\bar{\rho}_{jk} = 0$. $\square$

The relations $g_{\lambda\mu} = \bar{\Gamma}_{\lambda\mu}^{\cdot} = \bar{\Gamma}_{\lambda\cdot}^{j} = \bar{R}_{\lambda\cdot\mu\cdot} = 0$ in (4.2) state that the vertical distribution $\mathcal{V} = \mathrm{Ker}\,d\pi$ is g-null, g-parallel, and satisfies the following curvature condition (cf. Remark 2.1):

$$(4.3) \qquad \bar{R}(u, \cdot, v, \cdot) = 0 \text{ for any sections } u, v \text{ of } \mathcal{V}.$$

The properties just listed form an intrinsic local characterization of Riemann extension metrics, which is a result of Afifi [**1**], stated below as Theorem 4.5. We preceed it by a more general discussion, beginning with a lemma phrased in the language of Remark 2.1:

Lemma 4.3. *Let an m-dimensional null parallel distribution $\mathcal{V}$ on a pseudo-Riemannian manifold (M, g) with $\dim \mathrm{M} = 2m$ satisfy condition (2.7) for $U = \mathrm{M}$.*

   (a) *The requirement that $\pi^*\xi = g(v, \cdot)$ defines a natural bijective correspondence between sections v of $\mathcal{V}$ parallel along $\mathcal{V}$ and sections ξ of $T^*\Sigma$.*
   (b) *If (4.3) holds as well, then there exists a unique torsionfree connection ∇ on Σ such that, for any π-projectable vector fields u, u' on M, the covariant derivative $\overline{\nabla}_u u'$, relative to the Levi-Civita connection $\overline{\nabla}$ of g, is π-projectable onto the vector field $\nabla_w w'$ on Σ, where w, w' are the π-images of u and u'.*

Proof. The pullback $\pi^*\xi$ of any given section ξ of $T^*\Sigma$ determines ξ uniquely (since π is a submersion), and equals $g(v, \cdot)$ for a unique vector field v on M, which defines an injective assignment $\xi \mapsto v$. For sections u of $\mathcal{V} = \mathrm{Ker}\, d\pi$ we have $(\pi^*\xi)(u) = 0$ (so that v is a section of $\mathcal{V}^\perp = \mathcal{V}$) and $g(\nabla_u v, \cdot) = \nabla_u(\pi^*\xi) = 0$ (which gives $\nabla_u v = 0$), as one sees noting that, since $\mathcal{V}$ is parallel, $[\nabla_u(\pi^*\xi)](w) = -(\pi^*\xi)(\nabla_u w) = -(\pi^*\xi)(\nabla_w u) = 0$ for any π-projectable vector field w on M, in view of the Leibniz rule and Remark 2.1. Finally, the assignment $\xi \mapsto v$ is surjective: for a section v of $\mathcal{V}$ parallel along $\mathcal{V}$, we define a section ξ of $T^*\Sigma$ by $\xi(\widetilde{w}) = g(v, w)$, for any vector field $\widetilde{w}$ on Σ, where w is any π-projectable vector field on M with the π-image $\widetilde{w}$. Since $\mathcal{V} = \mathrm{Ker}\, d\pi$ is a g-null distribution, $\xi(\widetilde{w})$ does not depend on the choice of w. Also, for any section u of $\mathcal{V}$, we have $\nabla_u v = 0$, and so $d_u[\xi(\widetilde{w})] = d_u[g(v, w)] = g(v, \nabla_u w) = g(v, \nabla_w u)$ in view of Remark 2.1, which in turn vanishes, as $\mathcal{V}$ is parallel and null. Therefore, $\xi(\widetilde{w})$ may be treated as a function $\Sigma \to \mathbb{R}$, and so ξ is well defined. This proves (a).

   Assuming (4.3), let us fix u, u' as in (b) and a section v of $\mathcal{V}$. Thus, $\overline{\nabla}_v u \approx 0$, where $\approx$ means *differ by a section of $\mathcal{V}$*. (In fact, $[v, u] \approx 0$, cf. Remark 2.1, and $\overline{\nabla}_u v \approx 0$ since $\mathcal{V}$ is parallel.) Also, $\bar{R}(v, u)u'$ is, by (4.3), a section of $\mathcal{V}^\perp = \mathcal{V}$. This, along with (2.1) and Remark 2.1, gives $[v, \overline{\nabla}_u u'] \approx \overline{\nabla}_v \overline{\nabla}_u u' \approx 0$, so that $\overline{\nabla}_u u'$ is π-projectable, and (b) follows. $\square$

For $\mathcal{V}$ as in Lemma 4.3, assertion (b) describes a transversal connection ∇, in the sense of Molino [**14**], for the foliation on M tangent to $\mathcal{V}$. (See also Section 19.) We will refer to ∇ as the *transversal connection* on Σ, corresponding to g and $\mathcal{V}$.

   Patterson and Walker [**16**, p. 26] were the first to observe that, in the case where g is a Riemann extension metric $g^\nabla + 2\pi^*\tau$ on $T^*\Sigma$, both Σ and ∇ (though not τ) are, locally, determined just by g and $\mathcal{V} = \mathrm{Ker}\, d\pi$.

   Using (4.2), we now describe vertical Killing fields in Riemann extensions, cf. [**17**]:

Lemma 4.4. *Under the assumptions of Lemma 4.3, a section v of $\mathcal{V}$ is a Killing field for (M, g) if and only if $\mathcal{L}\xi = 0$, where ξ corresponds to v as in Lemma 4.3(a) and $\mathcal{L}$ is the Killing operator, with (3.1.b), of the connection ∇ described in Lemma 4.3(b).*

   In fact, since $v^j = 0$ and $\xi_j = g_{j\lambda}v^\lambda$, (4.2) gives $v_j = \xi_j$ and $v_\lambda = 0$ (with g-lowered indices), which implies the Lie-derivative relation $\mathbb{L}_v g = 2\pi^*(\mathcal{L}\xi)$.

   As mentioned earlier, the following intrinsic local characterization of Riemann extension metrics is a special case of a result of Afifi [**1**].

Theorem 4.5. *Let an m-dimensional null parallel distribution $\mathcal{V}$ on a pseudo-Riemannian manifold (M, g) with $\dim M = 2m$ satisfy the curvature condition* (4.3). *Then, for every point $x \in M$, there exist a manifold Σ of dimension m, a torsionfree connection ∇ on Σ, a symmetric 2-tensor τ on Σ, and a diffeomorphism of a neighborhood of x onto an open subset of $T^*\Sigma$, which sends*

(i) *g to the Riemann extension metric $g^\nabla + 2\pi^*\tau$,*
(ii) *$\mathcal{V}$ to the vertical distribution $\mathrm{Ker}\, d\pi$ of the bundle projection $\pi : T^*\Sigma \to \Sigma$,*
(iii) *the transversal connection described in Lemma 4.3(b) to ∇.*

Proof. By Lemma 4.3(a), the leaves of $\mathcal{V}$ coincide, in a neighborhood of any given point $x \in M$, with the fibres of an affine bundle which, through any fixed choice of a zero section, becomes identified with the vector bundle $T^*\Sigma$ for a local leaf space Σ of $\mathcal{V}$. Let ∇ be the transversal connection on Σ, corresponding to g and $\mathcal{V}$ as in Lemma 4.3(b). Clearly,

$$(4.4) \qquad (g - g^\nabla)(v, \cdot) = 0 \quad \text{for every section } v \text{ of } \mathcal{V}.$$

If $\overline{D}$ is the Levi-Civita connection of any pseudo-Riemannian metric h on a neighborhood of x and v, w are vector fields, $d_v[h(w,w)]/2 = h(\overline{D}_v w, w) = h(\overline{D}_w v, w) + h([v,w], w)$, and so

$$(4.5) \qquad d_v[h(w,w)]/2 = d_w[h(v,w)] - h(v, \overline{D}_w w) + h([v,w], w).$$

Our two choices of h are $h = g$ and $h = g^\nabla$. If v is a section of $\mathcal{V}$ parallel along $\mathcal{V}$ and w is $\mathcal{V}$-projectable (Remark 2.1), then each term on the right-hand side of (4.5) is the same for $h = g$ as it is for $h = g^\nabla$. In the case of $d_w[h(v,w)]$ and $h([v,w], w)$ this is obvious from (4.4) since, according to Remark 2.1, $[v,w]$ is a section of $\mathcal{V}$. On the other hand, the term $h(v, \overline{D}_w w)$, for either choice of h, equals $\xi(\nabla_{\pi w}(\pi w))$, where ξ corresponds to v as in Lemma 4.3(a). (That ∇ is also the transversal connection for g^∇ is immediate from the formula $\overline{\Gamma}^l_{jk} = \Gamma^l_{jk}$ in (4.2) with $\tau = 0$.)

Subtracting the two versions of (4.5), for $h = g$ and $h = g^\nabla$, and using (4.4), we see that $g - g^\nabla = 2\pi^*\tau$ for some symmetric 2-tensor τ on Σ, which completes the proof. $\qquad\square$

The following lemma describes local isometries between two Riemann extension metrics, sending one vertical distribution onto the other. We use the symbol π for both bundle projections $T^*\Sigma \to \Sigma$ and $T^*S \to S$, the meaning of K_ξ is the same as in Lemma 4.1(a), $\mathcal{L}$ is the Killing operator for ∇, given by (3.1.b), while $F^* : T^*S \to T^*\Sigma$ denotes the diffeomorphism induced by $F : \Sigma \to S$, and, at the same time, F^*t stands for the F-pullback of the 2-tensor t.

Lemma 4.6. *Suppose that a triple (Σ, ∇, τ) consists of a manifold Σ with a torsionfree connection ∇ and a symmetric 2-tensor τ on Σ. Let (S, D, t) be another such triple.*

(i) *For any diffeomorphism $F : \Sigma \to S$ sending ∇ to D and any 1-form ξ on Σ such that $F^*t = \tau + \mathcal{L}\xi$, the composite $\Phi = K_\xi \circ F^*$ is an isometry of $(T^*S, g^D + 2\pi^*t)$ onto $(T^*\Sigma, g^\nabla + 2\pi^*\tau)$ sending one vertical distribution onto the other.*
(ii) *Conversely, if $x \in T^*S$ and Φ an isometry of a connected neighborhood of x in $(T^*S, g^D + 2\pi^*t)$ onto an open submanifold of $(T^*\Sigma, g^\nabla + 2\pi^*\tau)$, sending one vertical distribution onto the other, then Φ restricted to some neighborhood of x equals $K_\xi \circ F^*$, where $F : \Sigma' \to S'$ and ξ are defined on Σ' and have the properties listed in* (i)*, for some open submanifolds $\Sigma' \subset \Sigma$ and $S' \subset S$.*

Proof. If $F^*D = \nabla$, the F^*-pullbacks of g^∇ and π^*F^*t are g^D and π^*t. Thus, (i) is immediate from Lemma 4.1(a).

Conversely, given x and Φ as in (ii), we may assume (2.7) for suitable neighborhoods U of x in $T^*\mathrm{S}$ and U' of $\Phi(x)$ in $T^*\Sigma$ with some base manifolds which are open connected sets $\mathrm{S}' \subset \mathrm{S}$ and $\Sigma' \subset \Sigma$, in such a way that $\pi \circ \Phi = F^{-1} \circ \pi$ for some diffeomorphism $F : \Sigma' \to \mathrm{S}'$.

In view of Lemma 4.3 and the formula $\bar{\Gamma}^l_{jk} = \Gamma^l_{jk}$ in (4.2), the affine structures of the fibres in $T^*\mathrm{S}$ and $T^*\Sigma$, as well as the connections D and ∇ are local geometric invariants associated with the metrics $g^{\mathrm{D}} + 2\pi^* t$, $g^\nabla + 2\pi^* \tau$ and the respective vertical distributions. Therefore, $F^*\mathrm{D} = \nabla$ and, on a neighborhood of x, we have $\Phi = \Theta \circ \mathrm{K}_\xi \circ F^*$ for some vector-bundle isomorphism $\Theta : T^*\Sigma' \to T^*\Sigma'$ and some 1-form ξ on Σ'. Now F^* pushes $g^{\mathrm{D}} + 2\pi^* t$ forward onto the metric $g^\nabla + 2\pi^* F^* t$, which, according to Lemma 4.1(a), is pushed forward by K_ξ onto $g^\nabla + 2\pi^*(F^* t - \mathcal{L}\xi)$. Since Φ is an isometry, this last metric is the Θ-pullback of $g^\nabla + 2\pi^*\tau$. By Lemma 4.1(c), $\Theta = \mathrm{Id}$ and $F^* t - \mathcal{L}\xi = \tau$, which completes the proof. $\qquad\square$

5. RSTS connections

By an *RSTS connection* we mean a 'Ricci skew-symmetric torsionfree surface connection' or, more precisely, a torsionfree connection ∇ on a surface Σ such that the Ricci tensor of ∇ is skew-symmetric at every point of Σ.

Wong [**20**, Theorem 4.2] found a canonical coordinate form of RSTS connections. A simplified version of Wong's result can be phrased as follows.

Theorem 5.1. *A torsionfree connection ∇ on a surface Σ has skew-symmetric Ricci tensor if and only if, on some neighborhood of any point of Σ, there exist coordinates in which the component functions of ∇ are $\Gamma^1_{11} = -\partial_1\varphi$, $\Gamma^2_{22} = \partial_2\varphi$ for a function φ, and $\Gamma^l_{jk} = 0$ unless $j = k = l$. The Ricci tensor ρ of ∇ then is given by $\rho_{12} = -\partial_1\partial_2\varphi$.*

Proof. See [**5**, Section 6]. $\qquad\square$

All general local properties of RSTS connections could in principle be derived from Theorem 5.1. However, such derivations are often tedious, which is why in this and the following sections direct arguments will be used.

Denoting by R and ρ the curvature and Ricci tensors of any RSTS connection ∇, we have

$$(5.1) \qquad \mathrm{a)} \quad R(u,v)v' = \rho(u,v)v', \qquad \mathrm{b)} \quad \beta \wedge [\rho(u,\cdot)] = \beta(u)\rho,$$

for all tangent vector fields u, v, v' and 1-forms β (notations of (2.5.a)). In fact, (5.1.a) is a well-known special case of the fact that the Ricci tensor of a torsionfree surface connection uniquely determines its curvature tensor. (See, for instance, [**5**, Lemma 4.1].) That both sides of (5.1.b) agree on any given pair (v, v') of vector fields is in turn obvious from (2.5.a), along with (iii) in Section 2 applied to the expression $\beta(v)\rho(u,v')$, trilinear in v, u, v'.

In the remainder of this section we assume that ∇ is a torsionfree connection on a surface Σ and its Ricci tensor ρ, in addition to being skew-symmetric, is nonzero everywhere.

Since ρ trivializes the bundle $[T^*\Sigma]^{\wedge 2}$, there exist a unique 1-form ϕ, called the *recurrence 1-form* of ∇, and a unique vector field w on Σ such that

$$(5.2) \qquad \mathrm{i)} \quad \nabla\rho = \phi \otimes \rho, \qquad \mathrm{ii)} \quad \phi = \rho(w, \cdot), \qquad \mathrm{iii)} \quad \phi(w) = 0, \qquad \mathrm{iv)} \quad d\phi = 2\rho.$$

(Relation (5.2.iv) is an easy consequence of the Ricci identity; see [**5**, formula (8.1)].) Furthermore, for w defined by (5.2.ii) and any vector field v on Σ,

$$(5.3) \qquad \mathrm{a)} \quad \mathrm{div}\, w = 2, \qquad \mathrm{b)} \quad d[\rho(v, \cdot)] = [\mathrm{div}\, v + \phi(v)]\rho.$$

In fact, if u, u' are arbitrary vector fields, $(\nabla_u[\rho(v, \cdot)])(u') = \phi(u)\rho(v, u') + \rho(\nabla_u v, u')$ by (5.2.i), so that (2.5.c) and (5.1.b) yield (5.3.b), since, according to (ii) in Section 2 and (2.4.b), skew-symmetrizing $\rho(\nabla_u v, u')$ in u, u' we obtain one-half of $\mathrm{div}\, v$ times $\rho(u, u')$. Now (5.3.a) follows if one applies (5.3.b) to $v = w$, using (5.2.iii), (5.2.ii) and (5.2.iv). By (5.3.a),

$$(5.4) \qquad \text{the set } \Sigma' \subset \Sigma \text{ on which } w \neq 0 \text{ is open and dense in } \Sigma.$$

For Σ and ∇ as above, still assuming that the Ricci tensor ρ is skew-symmetric and $\rho \neq 0$ everywhere, we define a vector-bundle morphism $Q : T\Sigma \to T\Sigma$ by

$$(5.5) \qquad \text{i) } Q = 4 + \nabla w + 3\phi \otimes w/4, \quad \text{ so that } \quad \text{ii) } \mathrm{tr}\, Q = 10.$$

Here $\nabla w : T\Sigma \to T\Sigma$ as in (2.2), 4 means 4 times the identity, ϕ, w are characterized by (5.2), and (5.5.ii) is immediate from (5.3.a) along with (5.2.iii). Finally, we denote by $\mathcal{B}$ and $\mathcal{D}$ the first-order linear differential operators, sending symmetric 2-tensors τ to 1-forms on Σ, or, respectively, 1-forms ξ on Σ to functions $\Sigma \to \mathbb{R}$, which are given by

$$(5.6) \qquad \text{a) } [(\mathcal{B}\tau)(v)]\rho = [d^\nabla \tau](\cdot, \cdot, v), \quad \text{ b) } 2[\mathcal{D}\xi]\rho = \xi \wedge \phi - d\xi$$

for any vector field v, where d^∇ is the Codazzi operator with (3.1.a). (Note that $[d^\nabla \tau](\cdot, \cdot, v)$ is a section of the bundle $[T^*\Sigma]^{\wedge 2}$, trivialized by ρ.) Using these $\phi, w, \mathcal{B}$ and $\mathcal{D}$, we also define a third-order linear differential operator $\mathcal{Z}$, sending each symmetric 2-tensor τ to the 1-form

$$(5.7) \qquad \mathcal{Z}\tau = 2d[\mathcal{D}(\mathcal{B}\tau)] + 4\mathcal{B}\tau - \tau(w, \cdot) + 3[\mathcal{D}(\mathcal{B}\tau)]\phi/2.$$

6. The vertical distribution of a type III SDNE manifold

Every type III SDNE manifold (M, g) carries a distinguished two-dimensional null distribution $\mathcal{V}$, which, in addition, is integrable and has totally geodesic leaves. Namely, W^+ acting on self-dual 2-forms, at any point x, has rank 2, and hence its kernel is one-dimensional. Thus, we may declare $\mathcal{V}_x$ to be the nullspace of some, or any, self-dual 2-form at x spanning $\mathrm{Ker}\, W_x^+$. That $\mathcal{V}$ has the properties just listed was shown in [**6**, Lemma 5.1].

We refer to $\mathcal{V}$ as the *vertical distribution* of (M, g), and say that g is a *type* III *SDNE Walker metric* if its vertical distribution $\mathcal{V}$ is parallel. Similarly, a type III SDNE metric g is called *strictly non-Walker* [**6**, Section 6] if the fundamental tensor of $\mathcal{V}$, which measures its deviation from being parallel [**6**, Section 24], is nonzero everywhere.

For g as above, being a Walker metric is equivalent to having the *Walker property* mentioned in the Introduction. Namely, the vertical distribution $\mathcal{V}$ of every type III SDNE manifold (M, g) is compatible with the orientation [**6**, Theorem 6.2(i)] and, if (M, g) admits any two-dimensional null parallel distribution compatible with the orientation, then $\mathcal{V}$ is such a distribution, that is, $\mathcal{V}$ itself must be parallel [**6**, Theorem 6.2(ii),(iv)].

Two constructions of type III SDNE manifolds are known. One, discovered by Díaz-Ramos, García-Río and Vázquez-Lorenzo [**8**, Theorem 3.1(ii.3)], always leads to Walker metrics. (See also the next section.) The other, described in [**6**, Theorem 22.1], gives rise to strictly non-Walker metrics. The resulting examples serve as *universal models*: as shown in [**8**, Theorem 3.1(ii.3)] and [**6**, Theorem 22.1], locally, at points in general position, up to isometries, every type III SDNE manifold arises from one of the two constructions just mentioned.

7. The structure theorem of Díaz-Ramos, García-Río and Vázquez-Lorenzo

In [**8**, Theorem 3.1(ii.3)], Díaz-Ramos, García-Río and Vázquez-Lorenzo described the local structure of all type III SDNE Walker metrics. With compatibility defined as in the lines preceding Remark 2.1, one can state their result as follows. (See also [**5**, p. 238].)

Theorem 7.1. *Let there be given a surface* Σ, *a torsionfree connection* ∇ *on* Σ *such that the Ricci tensor* ρ *of* ∇ *is skew-symmetric and nonzero everywhere, and a symmetric 2-tensor* τ *on* Σ. *Then, for a suitable orientation of the four-manifold* $\mathrm{M} = T^*\Sigma$, *the Riemann extension metric* $g = g^\nabla + 2\pi^*\tau$ *on* M, *with the neutral signature* $(--++)$, *is Ricci-flat and self-dual of Petrov type* III, *the distribution* $\mathcal{V} = \mathrm{Ker}\, d\pi$ *is* g-null, g-parallel, compatible *with the orientation, and constitutes the vertical distribution of* g, *cf. Section* 6, *while* g *and* $\mathcal{V}$ *satisfy the curvature condition* (4.3), *and the corresponding transversal connection on* Σ, *described in Lemma* 4.3(b), *coincides with our original* ∇.

Conversely, if (M, g) *is a neutral-signature oriented self-dual Einstein four-manifold of Petrov type* III *admitting a two-dimensional null parallel distribution* $\mathcal{V}$ *compatible with the orientation, then, for every* $x \in \mathrm{M}$, *there exist* Σ, ∇, τ *as above and a diffeomorphism of a neighborhood of* x *onto an open subset of* $T^*\Sigma$, *under which* g *corresponds to the metric* $g = g^\nabla + 2\pi^*\tau$, *and* $\mathcal{V}$ *to the vertical distribution* $\mathrm{Ker}\, d\pi$.

According to Theorem 7.1, every type III SDNE Walker metric g can be locally identified with a Riemann extension metric $g = g^\nabla + 2\pi^*\tau$ defined as at the beginning of Section 4 in the special case where Σ is a surface and the torsionfree connection ∇ on Σ has the property that its Ricci tensor ρ is skew-symmetric and nonzero at every point. By Lemma 4.2, g is Ricci-flat, while the formulae, appearing in (4.2), for the components of its curvature 4-tensor $\bar{R}$ in coordinates y^j, x^λ chosen for (4.2) may be rewritten as

$$(7.1) \qquad \bar{R}_{\lambda\mu..} = \bar{R}_{\lambda\cdot\mu\cdot} = 0, \quad \bar{R}_{jkl\lambda} = g_{l\lambda}\rho_{jk}, \quad \bar{R}_{jkls} = [g_{p\lambda}x^\lambda w^p + 2\mathcal{D}(\mathcal{B}\tau)]\rho_{jk}\rho_{ls},$$

where $\mathcal{B}$ and $\mathcal{D}$ are the operators defined by (5.6), and the index convention is the same as in (4.2). In fact, by (5.1.a), $R_{jkl}{}^s = \rho_{jk}\delta_l^s$, and so, on the right-hand side of the last equality in (4.2), after interchaning the indices p and s, we have $\Gamma_{jq}^p R_{lsk}{}^q - \Gamma_{kq}^p R_{lsj}{}^q + \Gamma_{lq}^p R_{jks}{}^q - \Gamma_{sq}^p R_{jkl}{}^q = (\Gamma_{jk}^p - \Gamma_{kj}^p)\rho_{ls} + (\Gamma_{ls}^p - \Gamma_{sl}^p)\rho_{jk} = 0$, as well as $R_{kjs}{}^p\tau_{pl} - R_{kjl}{}^p\tau_{ps} = (\tau_{sl} - \tau_{ls})\rho_{kj} = 0$. Similarly, $R_{lsj}{}^p{}_{,k} - R_{lsk}{}^p{}_{,j}$ equals $\rho_{ls,k}\delta_j^p - \rho_{ls,j}\delta_k^p$, and hence $w^p\rho_{jk}\rho_{ls}$, as $\rho_{ls,k} = \phi_k\rho_{ls}$ by (5.2.i), while $\phi_k\delta_j^p - \phi_j\delta_k^p = w^p\rho_{jk}$ in view of (5.2.ii) and (5.1.b) for $u = w$ and the 1-form β with $\beta_j = \delta_j^p$. Finally, since $(d^\nabla\tau)_{lsj} = (\mathcal{B}\tau)_j\rho_{ls}$ (cf. (5.6.a)) and $\rho_{ls,k} = \phi_k\rho_{ls}$ (see above), using (5.6.b), (2.5.a) and (2.5.c) we conclude that $(d^\nabla\tau)_{lsj,k} - (d^\nabla\tau)_{lsk,j} = 2\mathcal{D}(\mathcal{B}\tau)\rho_{jk}\rho_{ls}$.

Remark 7.2. Every type III SDNE Walker metric g, restricted to a suitable neighborhood of any given point of the underlying four-manifold, gives rise to a triple $(\Sigma, \nabla, [\tau])$ of invariants. Specifically, Σ is a surface (a local leaf space of the vertical distribution $\mathcal{V}$, cf. Section 6), ∇ is a torsionfree connection on Σ with everywhere-nonzero, skew-symmetric Ricci tensor (namely, the connection described in Theorem 7.1), and $[\tau]$ denotes a coset, in the vector space of all symmetric 2-tensors of class C^∞ on Σ, of the image of the Killing operator $\mathcal{L}$ for ∇, given by (3.1.b). (Here the coset is chosen so as to contain the 2-tensor τ appearing in Theorem 7.1.) Although τ itself is not an invariant of g, the coset $[\tau]$ is, as one sees using Lemma 4.6(ii) for $\Phi = \mathrm{Id}$, the local leaf space $\mathrm{S} = \Sigma$, and $\mathrm{D} = \nabla$, with t denoting the other choice of τ.

Conversely, by Theorems 7.1 and 14.1(c)), every triple $(\Sigma, \nabla, [\tau])$ with the properties just listed arises in this manner from some type III generic SDNE Walker metric g, namely, the Riemann extension $g = g^\nabla + 2\pi^*\tau$.

Finally, the original metric g, on a suitable neighborhood of the given point, is uniquely determined, up to an isometry, by the corresponding triple $(\Sigma, \nabla, [\tau])$. In fact, $g = g^\nabla + 2\pi^*\tau'$ for some τ' that lies in the coset $[\tau]$, while, for any two choices of such τ', the resulting metrics are isometric to each other (Lemma 4.1(b)).

8. Some natural tensor fields on a type III SDNE Walker manifold

In the next two lemmas (M, g) is assumed to be a type III SDNE Walker manifold. By Theorem 7.1, (M, g) may be identified, locally, with a Riemann extension $(T^*\Sigma, g^\nabla + 2\pi^*\tau)$ for a surface Σ with a torsionfree connection ∇, the Ricci tensor ρ of which is skew-symmetric and nonzero everywhere, and some symmetric 2-tensor τ on Σ. We will also choose, in M, local coordinates y^j, x^λ with (4.2) and (7.1).

Lemma 8.1. *For every type* III *SDNE Walker manifold* (M, g) *there exists a unique quintuple* $(\zeta, \eta, A, \gamma, v)$ *of local geometric invariants of* g *consisting of* 2-*forms* ζ, η, *a bundle morphism* $A : TM \to TM$, *a* 1-*form* γ, *and a vector field* v, *all defined globally on* M, *such that*

  (i) $2\bar{R} = \zeta \otimes \eta + \eta \otimes \zeta$, *where* $\bar{R}$ *is the curvature* 4-*tensor of* (M, g),
 (ii) $\overline{\nabla}\eta = 2\gamma \otimes \zeta$, *with* $\overline{\nabla}$ *denoting the Levi-Civita connection of* (M, g),
(iii) $\zeta = -2\pi^*\rho$, *the symbol* π *standing for the bundle projection* $T^*\Sigma \to \Sigma$,
 (iv) $\eta(u, \cdot) = g(Au, \cdot)$ *and* $g(v, u) = 4[\gamma(u) - \gamma(Au)]$ *for all vector fields* u.

In coordinates y^j, x^λ *chosen as above, with* Q *given by* (5.5.i), v *has the components*

$$(8.1) \qquad\qquad v^j = 0, \qquad v^\lambda = g^{j\lambda}(g_{k\mu}x^\mu Q_j^k + \xi_j),$$

where ξ *is a* 1-*form on* Σ *which may depend on the choice of the special coordinates.*

Proof. By (7.1), (i) and (iii) hold for the 2-forms ζ and η defined by $\zeta_{jk} = -2\rho_{jk}$, $\eta_{jk} = -[\mathcal{D}(\mathcal{B}\tau) + g_{p\lambda}x^\lambda w^p/2]\rho_{jk}$, $\eta_{\lambda j} = -\eta_{j\lambda} = g_{j\lambda}$, $\zeta_{\lambda j} = \zeta_{j\lambda} = \zeta_{\lambda\mu} = \eta_{\lambda\mu} = 0$. Both ζ and η are local geometric invariants of the metric: ζ by (iii) and Theorem 7.1, η in view of (i) and the fact that symmetric multiplication has no zero divisors.

We now establish (ii) for a 1-form γ with the components satisfying the conditions

$$(8.2) \qquad\qquad 8\gamma_\lambda = g_{k\lambda}w^k \quad \text{and} \quad 8\gamma_j \sim g_{k\mu}x^\mu(Q_j^k - \Gamma_{jl}^k w^l + \phi_j w^k/4),$$

the relation $\sim$ meaning, in the rest of the proof, that the two expressions differ by a function in Σ (which may itself depend both on some indices and on the choice of our coordinates); in other words, their difference is allowed to depend on the coordinates y^j, but not on x^λ. In fact, with the semicolons standing for $\overline{\nabla}$-covariant derivatives, using (4.2), we easily obtain $\eta_{\lambda\mu;\nu} = \eta_{\lambda\mu;j} = \eta_{\lambda j;\mu} = \eta_{\lambda j;k} = 0$ and $2\eta_{jk;\lambda} = -g_{s\lambda}w^s\rho_{jk}$. Next, $-\bar{\Gamma}_{jl}^\mu\eta_{k\mu} - \bar{\Gamma}_{jk}^\mu\eta_{\mu l} = g_{k\mu}\bar{\Gamma}_{jl}^\mu - g_{l\mu}\bar{\Gamma}_{jk}^\mu$ which, as a consequence of the formula for $g_{k\mu}\bar{\Gamma}_{jl}^\mu$ in (4.2), equals $2g_{s\lambda}x^\lambda R_{lkj}{}^s + 2(d^\nabla\tau)_{lkj}$ (note the numerous cancellations due to symmetry in k, l). Since $\rho_{kl,j} = \phi_j\rho_{kl}$, $R_{lkj}{}^s = \rho_{lk}\delta_j^s$ and $(d^\nabla\tau)_{lkj} = (\mathcal{B}\tau)_j\rho_{lk}$ (see (5.2.i), (5.1.a) and (5.6.a)), this gives, by (4.2), $2\eta_{kl;j} = -\gamma_j\rho_{kl}$ with γ_j as in (8.2), thus proving (ii) and (8.2).

Assertion (iv) is simply a definition of A and v, stating that they are obtained from η and $4(\gamma - A^*\gamma)$ by index raising. (Notation of (2.3).) Using (4.2) we thus get $A_j^k = -\delta_j^k$, $A_\lambda^j = 0$, $A_\lambda^\mu = \delta_\lambda^\mu$ and $A_j^\lambda \sim -g^{k\lambda}g_{s\mu}x^\mu(2\Gamma_{jk}^s + w^s\rho_{jk}/2)$. Consequently, $v^j = 4g^{j\lambda}(\gamma - A^*\gamma)_\lambda =$

$4g^{j\lambda}(\gamma_\lambda - A_\lambda^\mu \gamma_\mu) = 0$. Similarly, $v^\lambda = 4g^{j\lambda}(\gamma - A^*\gamma)_j = 4g^{j\lambda}(2\gamma_j - A_j^\mu \gamma_\mu)$. Now (8.2) and the equality $w^k \rho_{jk} = -\phi_j$ (see (5.2.ii)) yield (8.1).   □

Lemma 8.2. *Every type* III *SDNE Walker manifold* (M, g) *admits a globally defined section* θ *of the real line bundle* $[\mathcal{V}^*]^{\wedge 2}$, *where* $\mathcal{V}$ *denotes the vertical distribution, such that the restriction of* θ *to each leaf* N *of* $\mathcal{V}$ *is nonzero and parallel relative to the connection on* N *induced by the Levi-Civita connection of* g.

Proof. The 2-form ζ appearing in Lemma 8.1(iii) may be treated as a nowhere-zero section of the vector bundle $\mathcal{E}^{\wedge 2}$ over M, where $\mathcal{E}$ stands for the dual of the quotient bundle $(TM)/\mathcal{V}$. As $\mathcal{V}$ is a g-null subbundle of TM, the metric g constitutes a vector-bundle isomorphism $\mathcal{V} \to \mathcal{E}$, under which ζ corresponds to a trivializing section of $\mathcal{V}^{\wedge 2}$. Our θ is its dual trivializing section in $[\mathcal{V}^*]^{\wedge 2}$. That θ is parallel in the direction of $\mathcal{V}$ is immediate, since so are ζ, as one sees using (4.2), and g.   □

The local geometric invariants of type III SDNE Walker manifolds, described in this section, can be naturally generalized to arbitrary type III SDNE manifolds, with or without the Walker property. See [**6**, Lemma 5.1(c),(e) and Theorem 6.2(ii)].

9. Noncompactness of type III SDNE Walker manifolds

We begin with two lemmas. The first is obvious from (2.4.b) and (i) in Section 2.

Lemma 9.1. *If* $\mathcal{V}$ *is a* $\overline{\nabla}$-*parallel distribution on a manifold* M *endowed with a torsionfree connection* $\overline{\nabla}$, *and* v *is a section of* $\mathcal{V}$, *then* $\operatorname{div} v = \operatorname{div}^{\mathcal{V}} v$. *Here* div *is the* $\overline{\nabla}$-*divergence, given by* (2.4.b) *for* $\nabla = \overline{\nabla}$, *while the function* $\operatorname{div}^{\mathcal{V}} v : M \to \mathbb{R}$ *is defined so as to coincide, on each leaf* N *of* $\mathcal{V}$, *with the* D-*divergence of the restriction of* v *to* N, *where* D *denotes the connection on* N *induced by* $\overline{\nabla}$.

Lemma 9.2. *If* (M, g) *is a type* III *SDNE Walker manifold and* v *denotes the vector field appearing in Lemma* 8.1, *then* $\operatorname{div} v = \operatorname{div}^{\mathcal{V}} v = 10$, *with* $\operatorname{div}^{\mathcal{V}}$ *as in Lemma* 9.1 *for the Levi-Civita connection* $\overline{\nabla}$ *of* g, *and the vertical distribution* $\mathcal{V}$ *of* (M, g), *cf. Section* 6.

Proof. We use the notations and identifications described at the beginning of Section 8. The equality $\overline{\Gamma}_{\lambda\mu}^\cdot = 0$ in (4.2) states that x^λ are affine coordinates on each leaf $T_y^*\Sigma = \pi^{-1}(y)$. Thus, by (8.1) and (5.5.ii), $\operatorname{div}^{\mathcal{V}} v = \partial_\mu v^\mu = g^{j\mu} g_{k\mu} Q_j^k = Q_k^k = 10$, and our assertion is immediate from Lemma 9.1.   □

Lemma 9.2 leads to the following conclusion.

Theorem 9.3. *Suppose that* (M, g) *is a type* III *SDNE Walker manifold. Then*
  (a) M *is not compact,*
  (b) *the vertical distribution* $\mathcal{V}$ *has no compact leaves.*

Proof. The divergence formula is well-known to remain valid for any compact manifold with a torsionfree connection admitting a global parallel volume element. (Cf. [**7**, Remark 7.3].) Thus, compactness of M, or of some leaf of $\mathcal{V}$, would contradict Lemmas 9.2 and 8.2.   □

10. Left-invariant RSTS connections on a Lie group

Kowalski, Opozda and Vlášek [**13**] found a canonical coordinate form of RSTS connections that are also *locally homogeneous*. More general results later appeared in [**15**] and [**2**].

It is convenient for us to rephrase the result of [**13**] using left-invariant connections on a Lie group. This approach has the added benefit of providing a precise description of a local moduli space of the (nonflat) connections in question, which turns out to be a moduli *curve*, namely, the union of two subsets homeomorphic to $\mathbb{R}$, intersecting at one point. See Section 11.

We always identify the Lie algebra of a Lie group H with the space $\mathfrak{h}$ of left-invariant vector fields on H. If H is two-dimensional, non-Abelian, simply connected, and u, w is a basis of $\mathfrak{h}$ such that $[u, w] = 2u$, then there exists a function $f : H \to \mathbb{R}$ with

$$\text{(10.1)} \qquad d_u f = 0, \qquad d_w f = -2f, \qquad f > 0.$$

Such functions f are positive constant multiples of a specific Lie-group homomorphism from H into the multiplicative group $(0, \infty)$. In fact, by (2.5.b), the left-invariant 1-form sending u to 0 and w to -2 is closed, so that it equals $d \log f$ for some function $f > 0$. Left-invariance of $d \log f$ means in turn that left translations act on f via multiplications by constants, which characterizes nonzero multiples of homomorphisms $H \to (0, \infty)$.

Lemma 10.1. *The left-invariant connections ∇ with skew-symmetric Ricci tensor on any connected two-dimensional Lie group H are in a bijective correspondence with pairs (Ψ, f) formed by a Lie-algebra homomorphism $\Psi : \mathfrak{h} \to \mathfrak{sl}(\mathfrak{h})$ and a linear functional $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h}$ is the Lie algebra of H, consisting of left-invariant vector fields on H, and $\mathfrak{sl}(\mathfrak{h})$ stands for the Lie algebra of traceless vector-space endomorphisms of $\mathfrak{h}$.*

The correspondence is given by $\nabla_u v = [\Psi u]v + \lambda(u)v$ for $u, v \in \mathfrak{h}$, and ∇ has the Ricci tensor ρ with $\rho(u, v) = \lambda([u, v])$.

Proof. This is obvious from [**5**, Theorem 7.2] and [**5**, Lemma 4.1]. $\qquad\qquad\square$

Example 10.2. Let us fix a two-dimensional non-Abelian simply connected Lie group H along with a basis u, w of its Lie algebra $\mathfrak{h}$ such that $[u, w] = 2u$. Given real parameters a, b with $ab = 0$, we define a left-invariant torsionfree connection $\nabla = \nabla(a, b)$ on H by

$$\text{(10.2)} \qquad \begin{array}{ll} \nabla_u u = (3 + a)u - aw, & \nabla_u w = au + (3 - a)w, \\ \nabla_w u = (a - 2)u + (3 - a)w, & \nabla_w w = (a + b - 1)u + (2 - a)w. \end{array}$$

The Ricci tensor ρ of ∇ then is skew-symmetric and, for the recurrence 1-form ϕ of ∇,

$$\text{(10.3)} \qquad \text{a)} \quad \rho(u, w) = 6, \qquad \text{b)} \quad \phi(u) = -6, \quad \phi(w) = 0,$$

while w coincides with the vector field in (5.2.ii). Whenever $f : H \to \mathbb{R}$ satisfies (10.1),

$$\text{(10.4)} \qquad \text{i)} \quad f\rho \text{ is right-invariant}, \qquad \text{ii)} \quad d(f^s \phi) = 2(1 - s)f^s \rho \text{ for any } s \in \mathbb{R}.$$

There exists a function $\psi : H \to \mathbb{R}$ with $3d\psi = -f\phi$, and, for any such ψ,

(10.5) $v_1 = f^{-1}u$ and $v_2 = f^{-1}\psi u - w$ are right-invariant vector fields, while $[v_1, v_2] = 2v_1$.

Finally, if $(a, b) = (1, 0)$ and $\mathcal{Z}$ is the operator given by (5.7), we have

$$\text{(10.6)} \qquad \mathcal{Z}(\phi \otimes \phi) = 15\phi/2 \neq 0.$$

In fact, ∇ corresponds as in Lemma 10.1 to the pair (Ψ, λ) such that $\lambda(u) = 3$, $\lambda(w) = 0$, and the matrices representing Ψu and Ψw in the basis u, w are

$$(10.7) \qquad \mathfrak{B}_u = \begin{bmatrix} a & a \\ -a & -a \end{bmatrix}, \qquad \mathfrak{B}_w = \begin{bmatrix} a-2 & a+b-1 \\ 3-a & 2-a \end{bmatrix}.$$

We have $\mathfrak{B}_u \mathfrak{B}_w - \mathfrak{B}_w \mathfrak{B}_u = 2 \mathfrak{B}_u$. Hence Ψ is a Lie-algebra homomorphism, and so, according to Lemma 10.1, ∇ has skew-symmetric Ricci tensor with (10.3.a), while, evaluating $d_u[\rho(u,w)]$ and $d_w[\rho(u,w)]$ via the Leibniz rule and (10.2), we obtain (5.2.i,ii) for ϕ with (10.3.b). Next, (10.1) and (10.3.a) give $\rho(u, \cdot) = -3 \, d \log f$, so that (10.4.ii) follows from (5.2.iv) and the relation $\phi \wedge df = 2f\rho$, immediate from (5.1.b) and (10.3.b). As $\rho(u, \cdot) = -3 \, d \log f$, (2.6), (5.2.ii) and (10.4.ii) with $s = 1$ yield $\mathbb{L}_u(f\rho) = \mathbb{L}_w(f\rho) = 0$. Since the flows of left-invariant vector fields consist of right translations, this proves (10.4.i). For v_j as in (10.5), using (10.1) and (10.3.b) we easily get $\mathbb{L}_u v_j = \mathbb{L}_w v_j = 0$ for $j = 1, 2$, which implies (10.5), closedness of the 1-form $f\phi$ being obvious from (10.4.ii). Finally, by (10.2) with $(a, b) = (0, 1)$, the Leibniz rule and (10.3), $(\nabla_u \phi)(u) = 24$, $(\nabla_u \phi)(w) = 6$, $(\nabla_w \phi)(u) = -6$, $(\nabla_w \phi)(w) = 0$, so that $\nabla_w \phi = \phi$. Now (10.3.b) gives $[\nabla_u(\phi \otimes \phi)](w, \cdot) = 6\phi$ and $[\nabla_w(\phi \otimes \phi)](u, \cdot) = -12\phi$, which, combined with (5.6.a) and (10.3.a), yields $\mathcal{B}(\phi \otimes \phi) = 3\phi$. However, $\mathcal{D}\phi = -1$ by (5.6.b) and (5.2.iv). Therefore, (10.6) follows from (5.7) for $\tau = \phi \otimes \phi$ along with (5.2.iii).

Remark 10.3. If $(a, b) \neq (1, 0)$, the vector fields u and w are local geometric invariants of the connection $\nabla = \nabla(a, b)$ given by (10.2).

For w this is clear, also when $(a, b) = (1, 0)$, from (5.2.ii). On the other hand, (10.3.b) determines u uniquely up to its replacement by $u + \chi w$, where χ is any function. As $ab = 0$, the requirement that the equality $\nabla_u w = au + (3-a)w$ in (10.2) remain valid, even after u has been replaced by $u + \chi w$, easily gives $(a, b) = (1, 0)$ unless χ is identically zero.

Proposition 10.4. *If ∇ is a torsionfree connection on a surface Σ with everywhere-nonzero, skew-symmetric Ricci tensor, while (10.2) holds on a nonempty open set $\Sigma' \subset \Sigma$, for some constants a, b with $ab = 0$ such that $a + b \neq 1$, and some vector fields u, w defined on Σ', which are linearly independent at each point of Σ', then*

   (i) *w is the restriction to Σ' of the vector field w given by (5.2.ii),*
   (ii) *the vector field w with (5.2.ii) is nonzero everywhere in the closure of Σ'.*

Proof. Assertion (i) was established in Example 10.2. (Since ∇ is torsionfree, $[u, w] = 2u$.) If we now had $w \to 0$ on some sequence of points of Σ' converging in Σ, it would follow that $\nabla_w w \to 0$ as well, and so the last equality in (10.2) would give $(a + b - 1)u \to 0$, that is, $u \to 0$, contradicting (10.3.a). □

As shown by the next two examples, the assumption that $(0, 1) \neq (a, b) \neq (1, 0)$ (or, equivalently, $a + b \neq 1$) is essential for conclusion (ii) in Proposition 10.4. In Section 18 the same connections $\nabla(0, 1)$ and $\nabla(1, 0)$ are realized on Lorentzian quadric surfaces in a 3-space.

Example 10.5. Let y^1, y^2 be the Cartesian coordinates in $\Sigma = \mathbb{R}^2$. For the vector fields $u = (0, 1/y^1)$ on the open set $\Sigma' \subset \Sigma$ where $y^1 \neq 0$, and $w = (2y^1, 0)$ on Σ, we have $[u, w] = 2u$. Furthermore, the connection ∇ defined by (10.2) with $(a, b) = (0, 1)$ has a C^∞ extension from Σ' to Σ, since $\nabla_{(1,0)}(1, 0) = 0$, $\nabla_{(1,0)}(0, 1) = \nabla_{(0,1)}(1, 0) = (3y^1, 0)$ and $\nabla_{(0,1)}(0, 1) = (0, 3y^1)$, as one sees noting that $(1, 0) = w/(2y^1)$, $(0, 1) = y^1 u$, $d_u y^1 = 0$ and $d_w y^1 = 2y^1$. Our Σ, Σ' and ∇ thus satisfy the assumptions of Proposition 10.4 except for the

condition $a + b \neq 1$, and conclusion (ii) fails to hold: $w = 0$ on $\Sigma \smallsetminus \Sigma'$. (The Ricci tensor ρ is nonzero everywhere in Σ, since, by (10.3.a), $6 = \rho(u, w) = 2\rho((0, 1), (1, 0))$ on Σ'.)

Both vector fields u, w, and hence also the connection ∇, are easily seen to be invariant under the group H of affine transformations of $\mathbb{R}^2$ having a diagonal linear part of determinant 1 and a translational part parallel to the y^2 axis $\Sigma \smallsetminus \Sigma'$.

Example 10.6. Let Π be a two-dimensional real vector space with a fixed area form Ω. Thus, Ω is an element of $[\Pi^*]^{\wedge 2} \smallsetminus \{0\}$, treated as a constant 2-form on Π. Denoting by w the radial (identity) vector field on Π and by c a fixed nonzero real constant, we define a connection ∇ on Π by requiring that, for all vector fields u and all constant vector fields v on Π,

$$(10.8) \qquad \nabla_u v \; = \; 2c[\Omega(w, u)v \, + \, \Omega(w, v)u] \, - \, c^2 \Omega(w, u)\Omega(w, v)w.$$

Using u, v which are both constant, one sees that ∇ is torsionfree and its pullback under any linear isomorphism $A : \Pi \to \Pi$ is an analogous connection corresponding, instead of c, to $c \det A$. Thus, ∇ is invariant under the action of the unimodular group $\mathrm{SL}(\Pi)$, and, although ∇ varies with the parameter c, its diffeomorphic equivalence class is independent of c.

Next, (iii) in Section 2 easily implies both that $w = \Omega(w, v')v - \Omega(w, v)v'$ for constant vector fields v, v' with $\Omega(v, v') = 1$, and that, as a result,

$$(10.9) \qquad \nabla_u w \; = \; u \, + \, 2c\Omega(w, u)w \qquad \text{for all vector fields } u,$$

even if one replaces c^2 in (10.8) with just any constant c'. Let us now fix a nonzero constant vector field v on Π and set $u = [c\Omega(w, v)]^{-1}v$ on the open subset $\Sigma' = \Pi \smallsetminus \mathbb{R}v$. Then (10.8) and (10.9) yield (10.2) for $(a, b) = (1, 0)$ and our u, w. Again, the assumptions of Proposition 10.4 hold in this case, with $\Sigma = \Pi$, except for $a + b \neq 1$, and conclusion (iii) fails, as $w = 0$ at 0. (The Ricci tensor ρ is nonzero everywhere in Σ, since, by (10.3.a), $c\rho = 6\Omega$.)

Lemma 10.7. *If Π is a two-dimensional real vector space with a fixed area form Ω, and an RSTS connection ∇ on a nonempty connected open set $U \subset \Pi$ is invariant under the infinitesimal action of $\mathrm{SL}(\Pi)$, then ∇ satisfies (10.8) for some $c \in \mathbb{R}$, all vector fields u, and all constant vector fields v on U.*

Proof. Let $\mathrm{H} \subset \mathrm{SL}(\Pi)$ be the isotropy subgroup of a fixed point $y \in U \smallsetminus \{0\}$. Thus, $\mathbb{R}y$ is the only line (one-dimensional vector subspace) in $T_y U = \Pi$ with the property of H-*invariance*, here meaning invariance under the infinitesimal action of H.

Multiples of $\Omega(y, \cdot) \otimes \Omega(y, \cdot)$ are, in turn, the only H-invariant symmetric 2-tensors τ at y. In fact, such τ, if nonzero, must be of rank 1, so that its nullspace, being an H-invariant line, must coincide with $\mathbb{R}y$, the nullspace of $\Omega(y, \cdot) \otimes \Omega(y, \cdot)$. (The rank of τ cannot be 2, or else τ would be a pseudo-Euclidean inner product, and so $\mathbb{R}y$ would give rise to a second H-invariant line: the τ-orthogonal complement of $\mathbb{R}y$, if $\mathbb{R}y$ is not τ-null, or the other τ-null line, if $\mathbb{R}y$ is τ-null.)

Let D be the restriction to U of the standard flat connection on Π. The difference $\Xi = \nabla - \mathrm{D}$ is an $\mathrm{SL}(\Pi)$-invariant section of $[T^*U]^{\odot 2} \otimes TU$, and its value at y is an H-invariant symmetric bilinear mapping $\Xi_y : \Pi \times \Pi \to \Pi$. Since the H-invariant symmetric 2-tensor $\tau = \Omega(y, \Xi_y(\cdot, \cdot))$ must equal $-c\Omega(y, \cdot) \otimes \Omega(y, \cdot)/4$ for some $c \in \mathbb{R}$, so that $\Xi_y(u, v) - 2c[\Omega(y, u)v + \Omega(y, v)u]$ lies, for all $u, v \in \Pi$, in $\mathrm{Ker}\,\Omega(y, \cdot) = \mathbb{R}y$. Therefore, if u, v are *constant* vector fields, $\nabla_u v = \Xi(u, v)$ is given by the formula obtained from (10.8) by replacing the coefficient c^2 with a constant c' unrelated to c. (The values of Ξ at points other than y are the images of Ξ_y under the infinitesimal action of $\mathrm{SL}(\Pi)$.) Using (2.1), (10.9) (still valid

in this case), and (iii) in Section 2 we get $R(u,v)v' = -2\Omega(u,v)[3cv' + 2(c^2 - c')\Omega(w,v')w]$
for the curvature tensor R of ∇ and all constant vector fields u, v, v'. Thus, the Ricci tensor
of ∇ is skew-symmetric if and only if $c' = c^2$, which completes the proof. $\square$

Remark 10.8. We will use the following well-known fact. Let e_j, $j = 1, \ldots, n$, be vector fields
on an n-dimensional manifold Σ, trivializing the tangent bundle $T\Sigma$ and spanning an n-
dimensional Lie algebra. (Thus, the Lie brackets $[e_j, e_k]$ are constant-coefficient combinations
of $e_1, \ldots, e_n$.) Then, locally, Σ may be diffeomorphically identified with a Lie group so that
$e_1, \ldots, e_n$ correspond to left-invariant vector fields. See, for instance, [**5**, Appendix B].

11. The moduli curve of locally homogeneous RSTS connections

Let ∇ be a torsionfree connection on a surface Σ such that the Ricci tensor ρ of ∇ is skew-
symmetric and nonzero everywhere, and let w be the vector field with (5.2.ii). As in Section 2,
we denote by $\mathfrak{a}_y$ the Lie algebra of germs, at $y \in \Sigma$, of all infinitesimal affine transformations
of ∇. If $w_y \neq 0$, then $\dim \mathfrak{n}_y \leq 1$ for $\mathfrak{n}_y = \{v \in \mathfrak{a}_y : v_y = 0\}$, that is,

(11.1)   the isotropy subalgebra $\mathfrak{n}_y$ of $\mathfrak{a}_y$ is at most one-dimensional, and so $\dim \mathfrak{a}_y \leq 3$.

Namely, the differentials at y of affine transformations in Σ keeping y fixed lie in the one-
dimensional group of linear automorphism of $T_y\Sigma$ that preserve both the area form ρ_y and
the vector $w_y \neq 0$, so that we get (11.1). By (11.1), for $y \in \Sigma$ with $w_y \neq 0$,

(11.2)   the pair $(\dim \mathfrak{a}_y, \dim \mathfrak{n}_y)$ is one of $(3,1), (2,1), (2,0), (1,1), (1,0), (0,0)$.

In addition, let $v_y \in T_y\Sigma$ be a vector naturally distinguished by ∇. It follows that

(11.3)   if v_y and w_y are linearly independent, then $\mathfrak{n}_y = \{0\}$ and $\dim \mathfrak{a}_y \leq 2$.

In fact, the flows of elements of $\mathfrak{n}_y$ keep the basis v_y, w_y of $T_y\Sigma$ fixed, while, in general,

(11.4)   an affine transformation F between two manifolds with torsionfree connec-
              tions is uniquely determined by its value and differential at any given point,

since, in geodesic coordinates, F appears as a linear operator.

Remark 11.1. Let ∇ be a torsionfree connection on a surface Σ such that the Ricci tensor ρ
of ∇ is skew-symmetric and nonzero at every point $y \in \Sigma$

(a) The inequality $\dim \mathfrak{a}_y \leq 3$ in (11.1) remains valid, by (5.4), also when $w_y = 0$.
(b) Since $\dim \mathrm{SL}(\Pi) = 3$, (a) implies that the connections ∇ described in Example 10.6
    have $\dim \mathfrak{a}_y = 3$ at each point y, while $w_y = 0$ if $y = 0$.
(c) Conversely, if $\dim \mathfrak{a}_z = 3$ and $w_z = 0$ at a point $z \in \Sigma$, then the restriction of ∇
    to some neighborhood of z is diffeomorphically equivalent to one of the connections in
    Example 10.6.

To verify (c), we first note that $(\nabla w)_z$ cannot have two distinct real eigenvalues: if it did,
the same would be true of nearby points y, including, in view of (5.4), one with $w_y \neq 0$ and
$\dim \mathfrak{a}_y = 3$. The condition $\dim \mathfrak{a}_y = 3$ would now contradict (11.3) for v_y chosen to be an
eigenvector of $(\nabla w)_y$ such that $\rho(v_y, w_y) = 1$. Consequently, $(\nabla w)_z$ must be a linear automor-
phism of $T_z\Sigma$, for otherwise, according to (5.3.a) and (2.4.b), $(\nabla w)_z$ would have the distinct
real eigenvalues 0 and 2. Since the matrix $[(\partial_j w^k)(z)]$ of the components of $(\nabla w)_z$ in any
local coordinates is nonsingular, the inverse mapping theorem implies that z is an isolated
zero of w. Thus, the flows of all elements of $\mathfrak{a}_y$ keep z fixed, and so the isotropy subalgebra

$\mathfrak{n}_y = \mathfrak{a}_y$ is three-dimensional. The Lie algebra $\mathfrak{n}_y = \mathfrak{a}_y$ treated as acting in $T_z\Sigma$ preserves the area form ρ_z. Being three-dimensional, it must therefore coincide with $\mathfrak{sl}(T_z\Sigma)$, and (c) is immediate from Lemma 10.7 along with the line following (11.4).

Lemma 11.2. *Under the same assumptions as in Remark* 11.1, *let* ϕ *and* w *be characterized by* (5.2). *If* $y \in \Sigma$ *is a point with* $\dim \mathfrak{a}_y = 3$ *and* $w_y \neq 0$, *then* y *has a connected neighborhood* U *such that* ∇ *restricted to* U *is locally homogeneous, and the only symmetric* 2-*tensors on* U, *naturally distinguished by* ∇, *are constant multiples of* $\phi \otimes \phi$.

Proof. Since $w_y \neq 0$, (11.1) implies local homogeneity of ∇ on some connected neighborhood U of y. If σ now is a symmetric 2-tensor with the stated properties, we must have $\sigma(w, w) = 0$, for otherwise (11.3) applied to the vector field v with $\sigma(w, \cdot) = \rho(v, \cdot)$ would contradict the assumption that $\dim \mathfrak{a}_y = 3$. Thus, if σ were nondegenerate at y, it would have the Lorentzian signature $(-+)$, again leading to a contradiction with (11.3), this time for v chosen so that $\sigma(v, v) = 0$ and $\rho(v, w) = 1$. Therefore, $\operatorname{rank}\sigma$, obviously constant on U, equals 1 or 0. If $\operatorname{rank}\sigma = 1$, we have $\sigma = \pm\alpha \otimes \alpha$ for some 1-form α without zeros, which is a constant multiple of ϕ due to the relation $\sigma(w, w) = 0$, (5.2.iii) and local homogeneity of ∇ on U. $\square$

Remark 11.3. For a torsionfree connection ∇ with everywhere-nonzero, skew-symmetric Ricci tensor ρ on a surface Σ, the vector field w given by (5.2.ii), and a point $y \in \Sigma$, we have $\mathfrak{n}_y = \{0\}$ and $\dim \mathfrak{a}_y \leq 2$ whenever $d_w\psi \neq 0$ at y for some function ψ which is naturally distinguished by ∇, and defined on a neighborhood of y. (This is clear from (11.3) for the vector field v characterized by $d\psi = \rho(\cdot, v)$.)

Theorem 11.4. *The assignment* $(a, b) \mapsto \nabla$, *with* $\nabla = \nabla(a, b)$ *defined as in Example* 10.2, *establishes a bijective correspondence between*

(i) *the union* $(\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$ *of the coordinate axes in the* (a, b)-*plane* $\mathbb{R}^2$, *and*
(ii) *the set of local-equivalence classes of locally homogeneous nonflat torsionfree surface connections with skew-symmetric Ricci tensor.*

The degree of mobility of $\nabla(a, b)$, *defined as in Section* 2, *equals* 3 *for* $(a, b) = (1, 0)$, *and* 2 *when* $(a, b) \neq (1, 0)$.

Proof. That $\nabla = \nabla(a, b)$ all have the properties listed in (ii) is immediate from Example 10.2. The final clause about the degrees of mobility $\dim \mathfrak{a}_y$ is in turn an obvious consequence either of Remark 10.3 combined with (11.3), for $(a, b) \neq (1, 0)$, or of Remark 11.1(b), when $(a, b) = (1, 0)$. Our assertion will thus follow if we show that, for any given nonflat locally homogeneous RSTS connection ∇,

(iii) ∇ is locally equivalent to $\nabla(a, b)$ for some (a, b) with $ab = 0$,
(iv) the pair (a, b) in (iii) is uniquely determined by ∇.

To prove (iii) – (iv), we first note that the degree of mobility $\dim \mathfrak{a}_y$ is the same for all $y \in \Sigma$ and, by Remark 11.1(a), equals 2 or 3. If $\dim \mathfrak{a}_y = 2$, then ∇ is locally equivalent to a left-invariant connection on a connected Lie group H, and so ∇ has, locally, the form appearing in Lemma 10.1, with suitable Ψ and λ, while H is not Abelian (for otherwise ∇ would be Ricci-flat by Lemma 10.1, and hence flat by (5.1.a)). Choosing a basis u, w of the Lie algebra $\mathfrak{h}$ with $[u, w] = 2u$, we get $\lambda(u) \neq 0$, as ∇ is not Ricci-flat. Thus, rescaling u and adding to w a multiple of u, we may also assume that $\lambda(u) = 3$ and $\lambda(w) = 0$. If $\mathfrak{B}_u, \mathfrak{B}_w$ are the matrices representing Ψu and Ψw in the basis u, w, the equality $(\Psi u)w - (\Psi w)u = 2u - 3w$,

meaning that ∇ is torsionfree, amounts to

$$\mathfrak{B}_u \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \mathfrak{B}_w \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \text{and hence} \quad \mathfrak{B}_u = \begin{bmatrix} a & c \\ -s & -a \end{bmatrix}, \quad \mathfrak{B}_w = \begin{bmatrix} c-2 & a+b-1 \\ 3-a & 2-c \end{bmatrix}$$

for some $a, b, c, s \in \mathbb{R}$. As Ψ is a Lie-algebra homomorphism (see Lemma 10.1),

(11.5) i) $(a+b-1)s = (a-3)c+2a$, ii) $(a+b-1)a = (c-1)c$, iii) $(a-3)a = (c-3)s$,

or, in other words, $\mathfrak{B}_u \mathfrak{B}_w - \mathfrak{B}_w \mathfrak{B}_u = 2\mathfrak{B}_u$. From (11.5) it follows that $c = s = a$ and $ab = 0$ (which yields (10.7), and hence (10.2), thus proving (iii) when $\dim \mathfrak{a}_y = 2$). Namely, (11.5.i) and (11.5.ii) give $[(a-3)c+2a]a = (c-1)cs$, so that, by (11.5.iii), $(c-3)cs + 2a^2 = (c-1)cs$, and, consequently, $cs = a^2$. Again using (11.5.i) and (11.5.ii), we thus obtain $[(a-3)c+2a]c = (a+b-1)cs = (a+b-1)a^2 = (c-1)ca$, that is, $(a-c)c = 0$. There are now two cases: $c = 0$, and $a = c$. If $c = 0$, (11.5.ii) implies that $a + b = 1$ or $a = 0$, so that $a = 0$ by (11.5.i) with $c = 0$, and $s = 0$ from (11.5.iii) with $c = 0$. Hence $c = s = a$ and $ab = 0$, as required. If $a = c$, (11.5.ii) yields $ab = 0$, and either $a = c \neq 3$ (which, by (11.5.iii), gives $a = s$), or $a = c = 3$ (and, as the equality $ab = 0$ now reads $b = 0$, (11.5.i) becomes $6 = (a-3)c + 2a = (a+b-1)s = (a-1)s = 2s$, so that, again, $c = s = a$).

Next, let $\dim \mathfrak{a}_y = 3$ at every point y. Then the tangent bundle of the underlying surface Σ is, locally, trivialized by w with (5.2.ii) and some vector field u with (10.3) such that

$$(11.6) \qquad \begin{aligned} \nabla_u u &= 4u - \varepsilon w, & \nabla_u w &= u + 2w, \\ \nabla_w u &= -u + 2w, & \nabla_w w &= w \end{aligned}$$

for some function ε. In fact, $w \neq 0$ everywhere due to (5.4) and local homogeneity of ∇, so that, by (5.2), we may, locally, choose u with (10.3). Such u is unique up to being replaced by $u + \psi w$, for an arbitrary function ψ. We may further require that $[u, w] = 2u$, and then

(11.7) u becomes unique up to replacement by $u + \psi w$ for a function ψ with $d_w \psi = -2\psi$.

(Namely, (10.3), (5.2.iv) and (2.5.b) yield $12 = 2\rho(u, w) = (d\phi)(u, w) = -\phi([u, w])$, and so $[u, w] - 2u$ equals a function times w, which, locally, gives $[u, w] = 2u$ provided that instead of u one uses $u + \psi w$, for suitable ψ.) On the other hand, as $w \neq 0$ everywhere, we have $\nabla_w w = \kappa w$ for some function $\kappa : \Sigma \to \mathbb{R}$ (or else, setting $v = \nabla_w w$ in (11.3), we would get $\dim \mathfrak{a}_y \leq 2$ at some point y). Now, by (5.3.a) and (2.4.b), the functions κ and $2 - \kappa$ form the eigenvalues of $\nabla w : T\Sigma \to T\Sigma$ at every point of Σ, which implies that $\kappa = 1$ everywhere, since otherwise (11.3), applied to v defined by the conditions $\nabla_v w = (2 - \kappa)v$ and $\rho(v, w) = 1$, would again give $\dim \mathfrak{a}_y \leq 2$ at some y. Consequently, $\nabla_w w = w$ and $\nabla_u w = u + \chi w$ for some function χ. Thus, by (10.3.b), $\nabla w = \mathrm{Id} - \chi \phi \otimes w/6$, which shows that χ is naturally distinguished by ∇, and so $d_w \chi = 0$ everywhere (or else Remark 11.3 would give $\dim \mathfrak{a}_y \leq 2$ at some y). From (10.3.a), (5.1.a) and (2.1) we now get $6w = \rho(u, w)w = R(u, w)w = \nabla_w(u + \chi w) - \nabla_u w + 2\nabla_u w = -[u, w] + \chi w + 2\nabla_u w = 3\chi w$, and hence $\chi = 2$. Therefore, $\nabla_u w = u + 2w$. As $\rho(u, w) = 6$ is constant, cf. (10.3.a), differentiation by parts gives $\rho(\nabla_u u, w) = -[\nabla_u \rho](u, w) - \rho(u, \nabla_u w) = 4\rho(u, w)$. (Note that $\nabla_u \rho = \phi(u)\rho = -6\rho$ by (5.2.i) and (10.3.b).) Hence $\nabla_u u = 4u - \varepsilon w$ for some function ε. Combined with the equality $[u, w] = 2u$, this proves (11.6).

Also, as in the last paragraph, $6u = \rho(u, w)u = R(u, w)u = \nabla_w(4u - \varepsilon w) - \nabla_u(2w - u) + 2\nabla_u u$, so that (11.6) yields $d_w \varepsilon = 4(1 - \varepsilon)$. However, ε depends on the choice of u, and we are free to modify u as in (11.7). Some such modification gives $\varepsilon = 1$ (and so (11.6) becomes (10.2) with $(a, b) = (1, 0)$, proving (iii) also in the case $\dim \mathfrak{a}_y = 3$). Namely, the use of $u + \psi w$ instead of

u, with $d_w\psi = -2\psi$, causes ε to be replaced by $\varepsilon + \psi^2 - d_u\psi$, while $\varepsilon + \psi^2 - d_u\psi = 1$ if and only if ψ is a solution of the system $d_u\psi = \psi^2 + \varepsilon - 1$, $d_w\psi = -2\psi$. As $d_w\varepsilon = 4(1 - \varepsilon)$, this system is completely integrable. In fact, it implies its own integrability conditions, and so the graphs of its solutions are, locally, the integral manifolds of a distribution on $\Sigma \times \mathbb{R}$, which happens to be integrable.

Finally, to obtain (iv), note that, according to the final clause of the theorem, the case $(a, b) = (1, 0)$ is uniquely distinguished by the value of the degree of mobility, while, if $(a, b) \neq (1, 0)$, the invariant character of both u and w, established in Remark 10.3, allows us to treat the last equality in (10.2) as an explicit geometric definition of a and b. $\qquad\square$

12. The Killing equation for an RSTS connection

If the Ricci tensor ρ of a torsionfree connection ∇ on a surface Σ is skew-symmetric and nonzero at every point, (5.1.a) and (5.6.a) allow us to rewrite (3.2) as

$$(12.1) \qquad \nabla\nabla\xi = (\mathcal{B}\tau - \xi) \otimes \rho + \nabla\tau \quad \text{for any 1-form } \xi \text{ on } \Sigma, \text{ with } \tau = \mathcal{L}\xi.$$

Lemma 12.1. *Let ∇ be a torsionfree connection with everywhere-nonzero, skew-symmetric Ricci tensor ρ on a surface Σ. For any 1-form ξ on Σ, setting $\tau = \mathcal{L}\xi$, we then have*

$$(12.2) \qquad \text{a) } Q^*\xi = \mathcal{Z}\tau, \qquad \text{b) } \nabla\xi = \tau + [\xi(w) - 2\mathcal{D}(\mathcal{B}\tau)]\rho/4,$$

with w given by (5.2.ii), $\mathcal{Z}, \mathcal{B}, \mathcal{D}$ as in (5.6) – (5.7), and Q^* standing for the dual of the morphism Q in (5.5.i), cf. (2.3).

Proof. Being skew-symmetric, $\nabla\xi - \mathcal{L}\xi = \nabla\xi - \tau$ equals $\psi\rho$ for some function ψ. Now

$$(12.3) \qquad \text{i) } d\psi = \mathcal{B}\tau - \xi - \psi\phi, \quad \text{ii) } d\xi = [\xi(w) - 2\psi - 2\mathcal{D}(\mathcal{B}\tau)]\rho.$$

In fact, (12.1) yields (12.3.i) since, by (5.2.i), $\nabla\nabla\xi = \nabla(\psi\rho + \tau) = (d\psi + \psi\phi) \otimes \rho + \nabla\tau$. Next, (12.3.i) gives $d(\mathcal{B}\tau - \xi - \psi\phi) = 0$ and so, again from (12.3.i), $d\xi = d(\mathcal{B}\tau) + \phi \wedge d\psi - \psi d\phi = d(\mathcal{B}\tau) + \phi \wedge \mathcal{B}\tau + \xi \wedge \phi - \psi d\phi$. Using (5.2.iv), (5.6.b) and the relation $\xi \wedge \phi = \xi(w)\rho$, immediate from (5.1.b) with $\beta = \xi$, $u = w$ and (5.2.ii), we thus get (12.3.ii).

However, (2.5.c) and the equality $\nabla\xi = \psi\rho + \tau$ give $d\xi = 2\psi\rho$. Equating this expression for $d\xi$ with (12.3.ii), we obtain $4\psi = \xi(w) - 2\mathcal{D}(\mathcal{B}\tau)$, which, as $\nabla\xi = \psi\rho + \tau$, implies (12.2.b). In view of (12.2.b) and (5.2.ii), for any vector field v we have

$$(12.4) \qquad 4(\nabla_v\xi)(w) = 4\tau(w, v) + [2\mathcal{D}(\mathcal{B}\tau) - \xi(w)]\phi(v).$$

Applying ∇_v to (12.2.b), we see that $4\nabla_v(\nabla\xi - \tau)$ equals $\xi(\nabla_v w) - 2d_v[\mathcal{D}(\mathcal{B}\tau)] + \tau(w, v) - 3[2\mathcal{D}(\mathcal{B}\tau) - \xi(w)]\phi(v)/4$ times ρ, due to the Leibniz rule, (12.4) and the relation $\nabla_v\rho = \phi(v)\rho$, cf. (5.2.i). Since, by (12.1), $4\nabla_v(\nabla\xi - \tau) = 4[(\mathcal{B}\tau - \xi)(v)]\rho$, (12.2.a) follows. $\qquad\square$

Lemma 12.1 immediately leads to the following conclusion about 1-forms $\xi \in \operatorname{Ker}\mathcal{L}$, that is, C^∞ solutions ξ to the *Killing equation* $\mathcal{L}\xi = 0$, where $\mathcal{L}$ is the Killing operator with (3.1.b).

Theorem 12.2. *For the Killing operator $\mathcal{L}$ of a surface Σ with a torsionfree connection ∇ such that the Ricci tensor ρ of ∇ is skew-symmetric and nonzero everywhere,*

(i) $\dim \operatorname{Ker}\mathcal{L} \leq 1$,

(ii) *each 1-form $\xi \in \operatorname{Ker}\mathcal{L}$ is either identically zero, or nonzero at every point.*

Proof. Equality (12.2.b) with $\tau = 0$ states that any $\xi \in \operatorname{Ker} \mathcal{L}$, restricted to any geodesic, satisfies a first-order linear homogeneous ordinary differential equation, which proves (ii). If we now had $\dim \operatorname{Ker} \mathcal{L} \geq 2$, formula (12.2.a) with $\tau = 0$ would, by (ii), imply that $Q = 0$ at every point, contradicting (5.5.ii). $\square$

We say that a torsionfree connection ∇ on a surface Σ with skew-symmetric Ricci tensor ρ is *generic* if $\rho \neq 0$ at every point and Q defined by (5.5.i) is an isomorphism $T\Sigma \to T\Sigma$. For such a generic connection ∇, formula (12.2.a) leads to a complete description of both the kernel and the image of the Killing operator $\mathcal{L}$. In fact, by (12.2.a),

(12.5)   if ∇ is generic, $\operatorname{Ker} \mathcal{L} = \{0\}$.

About the image of $\mathcal{L}$, see Section 14. Here we just note that, for $\nabla(a, b)$ as in Example 10.2,

(12.6) $\nabla(a, b)$ is generic except when $(a, b) = (-9, 0)$ or $(a, b) = (0, -15)$.

Namely, by (10.2) and (10.3.b), at each point y, in the basis u_y, w_y of the tangent plane,

(12.7) Q is represented by the matrix $\begin{bmatrix} a+4 & a+b-1 \\ -a-3/2 & 6-a \end{bmatrix}$.

Thus, $2 \det Q = 5a + 3b + 45$, and so (12.6) follows since $ab = 0$.

Example 12.3. For the non-generic locally homogeneous connection $\nabla(-9, 0)$ defined in Example 10.2 with $(a, b) = (-9, 0)$, we have $\dim \operatorname{Ker} \mathcal{L} = 1$. In fact, choosing f with (10.1), and then setting $\xi(u) = 3f^2$, $\xi(w) = 2f^2$, we define a nonzero 1-form ξ such that $\mathcal{L}\xi = 0$. By Theorem 12.2(i), ξ spans $\operatorname{Ker} \mathcal{L}$.

Example 12.4. The remaining non-generic locally homogeneous connection $\nabla(0, -15)$ in (12.6) has $\operatorname{Ker} \mathcal{L} = \{0\}$. To see this, note that a nonzero 1-form ξ with $\mathcal{L}\xi = 0$, if it existed, would give rise to the line subbundle $\operatorname{Ker} \xi$ in $T\Sigma$ (cf. Theorem 12.2(ii)), and, by (12.2.a) with $\tau = 0$, the image of Q would be contained in $\operatorname{Ker} \xi$. From (12.7) with $(a, b) = (0, -15)$ it would now follow that $8u - 3w$ spans $\operatorname{Ker} \xi$, that is, $\xi(u) = 3\psi$ and $\xi(w) = 8\psi$ for some function ψ, not identically equal to 0. The relation $\mathcal{L}\xi = 0$ would now yield $0 = [\nabla_u \xi](u) = d_u[\xi(u)] - \xi(\nabla_u u) = 3(d_u \psi - 3\psi)$ and, similarly, $0 = [\nabla_w \xi](w) = 8(d_w \psi + 4\psi)$. The resulting equalities $d_u \psi = 3\psi$, $d_w \psi = -4\psi$ would in turn give $6\psi = 2d_u \psi = d_{[u,w]} \psi = d_u d_w \psi - d_w d_u \psi = 0$, contradicting our earlier conclusion that $\psi \neq 0$ somewhere.

The following lemma will be needed in Section 16. As usual, $\mathcal{L}$ denotes the Killing operator.

Lemma 12.5. *Let ∇ be one of the connections $\nabla(a, b)$ described in Example 10.2. If a left-invariant symmetric 2-tensor on the underlying two-dimensional Lie group H is the $\mathcal{L}$-image of some 1-form, then it is also the $\mathcal{L}$-image of some left-invariant 1-form.*

Proof. With $\mathbb{L}_j$ denoting the Lie derivatives with respect to the right-invariant vector fields v_j, $j = 1, 2$, chosen as in (10.5), the left-invariance of the symmetric 2-tensor $\mathcal{L}\beta$, for a given 1-form β, means that $\mathbb{L}_j \mathcal{L}\beta = 0$, since the flows of right-invariant vector fields on H consist of left translations. The last fact also implies that both $\mathbb{L}_j$ commute with $\mathcal{L}$. Left-invariance of $\mathcal{L}\beta$ thus gives $\mathbb{L}_j \beta \in \operatorname{Ker} \mathcal{L}$. If $(a, b) \neq (-9, 0)$, then, according to (12.5), (12.6) and Example 12.4, $\operatorname{Ker} \mathcal{L} = \{0\}$, so that β is left-invariant, which yields our assertion. On the other hand, if $(a, b) = (-9, 0)$, we may use the basis α, ϕ of left-invariant 1-forms given by $\alpha(u) = 3$, $\alpha(w) = 2$ (for u, w as in Example 10.2) and (5.2.i), and fix a function f with (10.1). The 1-form $\xi = f^2 \alpha$ then spans $\operatorname{Ker} \mathcal{L}$ (see Example 12.3), while $\beta = \mu\alpha + \chi\phi$ for some

functions χ and μ. Since $\mathbb{L}_j\beta \in \mathrm{Ker}\,\mathcal{L}$, there exist $c_j \in \mathbb{R}$ such that $\mathbb{L}_j\beta = 4c_j\xi = 4c_j f^2\alpha$. At the same time, $\mathbb{L}_j\beta = (\mathbb{L}_j\mu)\alpha + (\mathbb{L}_j\chi)\phi$. The resulting system of equations, rewritten with the aid of (10.5), states that χ is constant and $d_u\mu = 4c_1 f^3$, $d_w\mu = 4c_1 f^2\psi - 4c_2 f^2$. From (10.1) and (10.3.b) with $3d\psi = -f\phi$ we obtain $0 = d_u d_w\mu - d_w d_u\mu - 2d_u\mu = 24c_1 f^3$, that is, $c_1 = 0$. The equations imposed on μ, combined with (10.1), now give $\mu = c_2 f^2 + c'$ for some $c' \in \mathbb{R}$. Thus, $\beta = \mu\alpha + \chi\phi = c_2\xi + c'\alpha + \chi\phi$, and β has the same $\mathcal{L}$-image as the left-invariant 1-form $c'\alpha + \chi\phi$. $\qquad\square$

13. Degree of mobility for type III SDNE Walker manifolds

Let (M, g) be a type III SDNE Walker manifold. Since our discussion is local, we may use Theorem 7.1 to identify M with $T^*\Sigma$ for some surface Σ with a torsionfree connection ∇ such that the Ricci tensor ρ of ∇ which is skew-symmetric and nonzero at each point. At any $x \in \mathrm{M} = T^*\Sigma$, the bundle projection $\pi : T^*\Sigma \to \Sigma$ induces a Lie-algebra homomorphism

$$(13.1) \qquad\qquad \pi_x : \mathfrak{i}_x \to \mathfrak{a}_y, \quad \text{where } y = \pi(x),$$

$\mathfrak{a}_y$ and $\mathfrak{i}_x$ being as in Section 2 for ∇ (on Σ) and g (on M). Thus, if $\delta = \dim \mathrm{Ker}\,\pi_x$, and $(\mathrm{Ker}\,\mathcal{L})_y$ is the space of germs at y of 1-forms ξ with $\mathcal{L}\xi = 0$ on a neighborhood of y in Σ,

$$(13.2) \qquad \text{a) } \dim \mathfrak{i}_x = \delta + \mathrm{rank}\,\pi_x \leq \delta + \dim \mathfrak{a}_y, \qquad \text{b) } \delta = \dim(\mathrm{Ker}\,\mathcal{L})_y \leq 1,$$

relation (13.2.b) being immediate from Lemma 4.4 and Theorem 12.2(i), as $\mathrm{Ker}\,\pi_x$ consists of germs, at x, of those Killing fields for g which are vertical (tangent to $\mathcal{V} = \mathrm{Ker}\,d\pi$). Thus,

$$(13.3) \qquad \text{if } \dim \mathfrak{i}_x \geq 3, \text{ the pair } (\dim \mathfrak{a}_y, \delta) \text{ is one of } (3,0) \text{ and } (2,1).$$

Namely, according to Remark 11.1(a), we just need to show that $(\dim \mathfrak{a}_y, \delta)$ cannot equal $(3, 1)$. If it did, however, we would be free to assume that $w_y \neq 0$ (replacing y with a point arbitrarily close to it, cf. (5.4)). Then ∇ would be locally homogeneous at y as a consequence of (11.1), and the final clause of Theorem 11.4 would imply that ∇ is locally equivalent to the connection $\nabla(1, 0)$ of Example 10.2, contradicting in turn relations (12.6) and (12.5) (as $\delta = \dim(\mathrm{Ker}\,\mathcal{L})_y = 1$).

Theorem 13.1. *Let (M, g) be a neutral-signature oriented Ricci-flat self-dual Walker four-manifold of Petrov type* III. *Then its degree of mobility* $\dim \mathfrak{i}_x$, *defined as in Section 2, does not exceed* 3 *at any point* $x \in \mathrm{M}$. *In particular,* (M, g) *cannot be locally homogeneous.*

In fact, if $\dim \mathfrak{i}_x \geq 3$, (13.3) gives $\delta + \dim \mathfrak{a}_y = 3$, and so, by (13.2.a), $\dim \mathfrak{i}_x = 3$.

14. The image of the Killing operator

Let a torsionfree connection ∇ with skew-symmetric Ricci tensor ρ on a surface Σ be generic, as defined in the lines preceding formula (12.5). We use the symbol $\mathcal{P}$ for the fourth-order linear differential operator sending any symmetric 2-tensor τ on Σ to the symmetric 2-tensor

$$(14.1) \qquad\qquad \mathcal{P}\tau = \tau - \mathcal{L}[(Q^*)^{-1}\mathcal{Z}\tau].$$

Here $\mathcal{L}, \mathcal{Z}$ and Q are given by (3.1.b), (5.7) and (5.5), while $(Q^*)^{-1} : T\Sigma \to T\Sigma$ denotes the inverse of the dual of Q, defined as in (2.3), so that $(Q^*)^{-1}\mathcal{Z}\tau$ is the 1-form ξ with $Q^*\xi = \mathcal{Z}\tau$, in the notation of (12.2.a). Finally, we let $\mathcal{S}_1$ (or, $\mathcal{S}_2$) stand for the space of all 1-forms (or, symmetric 2-tensors) of class C^∞ on Σ.

Theorem 14.1. *Suppose that* ∇ *is a generic torsionfree connection with skew-symmetric Ricci tensor on a surface* Σ, *while* $\mathcal{L}, \mathcal{Z}$ *and* $\mathcal{P}$ *are given by* (3.1.b), (5.7) *and* (14.1).

(a) $\mathcal{L}(\mathcal{S}_1) = \mathrm{Ker}\,\mathcal{P}$, *that is, the image of the Killing operator* $\mathcal{L} : \mathcal{S}_1 \to \mathcal{S}_2$ *is at the same time the kernel of the fourth-order operator* $\mathcal{P} : \mathcal{S}_2 \to \mathcal{S}_2$.

(b) *We have a direct-sum decomposition* $\mathcal{S}_2 = \mathcal{L}(\mathcal{S}_1) \oplus \mathrm{Ker}\,(\mathcal{P} - \mathrm{Id})$, *for which* $\mathcal{P} : \mathcal{S}_2 \to \mathcal{S}_2$ *serves as the projection onto the second summand.*

(c) *The image of* $\mathcal{P} : \mathcal{S}_2 \to \mathcal{S}_2$ *coincides both with* $\mathrm{Ker}\,(\mathcal{P} - \mathrm{Id})$ *and with the space of all* C^∞ *solutions* $\tau \in \mathcal{S}_2$ *to the third-order linear differential equation* $\mathcal{Z}\tau = 0$.

Proof. If $\tau = \mathcal{L}\xi$, (12.2.a) gives $\mathcal{P}\tau = 0$, while, if $\mathcal{P}\tau = 0$, then $\tau \in \mathcal{L}(\mathcal{S}_1)$ by (14.1), which yields (a). Next, (a) and (14.1) imply that $\mathcal{P}^2 = \mathcal{P}$. Thus, $\mathcal{P}$ is a projection onto its image, and (b) follows from (a). Finally, by (b), $\mathrm{Ker}\,(\mathcal{P} - \mathrm{Id})$ is the image of $\mathcal{P}$, while $\tau \in \mathcal{S}_2$ lies in $\mathrm{Ker}\,(\mathcal{P} - \mathrm{Id})$ if and only if $\tau = \mathcal{P}\tau = \tau - \mathcal{L}[(Q^*)^{-1}\mathcal{Z}\tau]$, which amounts to $\mathcal{L}[(Q^*)^{-1}\mathcal{Z}\tau] = 0$. Since $\mathrm{Ker}\,\mathcal{L} = \{0\}$ (see (12.5)), this is equivalent to $\mathcal{Z}\tau = 0$, as required in (c). $\qquad\square$

15. Type III SDNE generic Walker metrics

Generic RSTS connections were defined in the lines preceding formula (12.5). We will now refer to a type III SDNE Walker manifold (M, g) as *generic* if so is, at every point $x \in \mathrm{M}$, the RSTS connection ∇ associated, in the sense of Theorem 7.1, with the restriction of g to a neighborhood of x. The connection ∇ is a part of the triple $(\Sigma, \nabla, [\tau])$ of local invariants of g, introduced in Remark 7.2. In the generic case, however, the coset $[\tau]$ contains a distinguished element σ given by $\sigma = \mathcal{P}\tau$, with the operator $\mathcal{P}$ determined by ∇ via (14.1). That σ is independent of the choice of τ in the coset is immediate from Theorem 14.1(a). In view of Theorem 14.1(c), $\mathcal{Z}\sigma = 0$.

Thus, by Theorem 14.1(b), if a type III SDNE Walker metric g is generic, the invariant σ described above constitutes a canonical choice of the 2-tensor τ appearing in Theorem 7.1. The two final paragraphs of Remark 7.2 then remain valid also after $[\tau]$ has been replaced by a 2-tensor σ on Σ, subject only to the condition $\mathcal{Z}\sigma = 0$ (cf. Theorem 14.1(c)).

16. The case of maximum mobility

According to Theorem 13.1, the degree of mobility of a type III SDNE Walker manifold cannot exceed 3 at any point. This section provides a complete characterization of the case where it is 3. We begin by constructing, in Examples 16.1 and 16.2, a single manifold and, respectively, a one-parameter family of type III SDNE Walker manifolds having the degree of mobility equal to 3. These mutually non-isometric manifolds are shown, in Theorem 16.3, to represent all local isometry classes of type III SDNE Walker metrics with maximum mobility.

Example 16.1. Let ∇ be one of the connections, described in Example 10.6, on a two-dimensional real vector space Π. (They are all diffeomorphically equivalent.) Using the first part of Theorem 7.1, with $\tau = 0$, we see that $g = g^\nabla$ then is a type III SDNE Walker metric on $\mathrm{M} = T^*\Pi$. Invariance of g under the cotangent action of $\mathrm{SL}(\Pi)$ implies, in view of Remark 11.1(a), that the degree of mobility of g equals 3 at every point.

Example 16.2. Given the connection $\nabla = \nabla(-9, 0)$ defined as in Example 10.2, with $(a, b) = (-9, 0)$, on a two-dimensional non-Abelian simply connected Lie group H, and any left-invariant symmetric 2-tensor τ on H, the Riemann extension $g = g^\nabla + 2\pi^*\tau$ is, according to

the first part of Theorem 7.1, a type III SDNE Walker metric on $\mathrm{M} = T^*\mathrm{H}$, invariant under the cotangent left action of H, the orbits of which are obviously transverse to the fibres of $\mathrm{M} = T^*\mathrm{H}$. In view of Example 12.3 and Lemma 4.4, (M, g) also admits a Killing vector field tangent to the fibres, which is nonzero everywhere (Theorem 12.2(ii)), and, so by Theorem 13.1, its degree of mobility is 3 at every point.

Even though the tensors τ used here form a three-dimensional space, the construction gives rise only to one-parameter family of nonequivalent metrics. In fact, the Killing operator $\mathcal{L}$ restricted to the space of left-invariant 1-forms α is injective (Example 12.3), while the metrics $g = g^{\nabla} + 2\pi^*\tau$ and $g' = g^{\nabla} + 2\pi^*\tau'$ corresponding to the tensors τ and $\tau' = \tau + \mathcal{L}\alpha$ are isometric to each other in view of Lemma 4.1(b).

Theorem 16.3. *If* (M, g) *is a neutral-signature oriented Ricci-flat self-dual Walker four-manifold of Petrov type* III, *and the degree of mobility of* g *at a point* x *equals* 3, *then* x *has a connected neighborhood isometric to an open submanifold of one of the manifolds described in Examples* 16.1 *and* 16.2.

Proof. We use the same assumptions and identifications as in the lines preceding (13.1), so that $\mathrm{M} = T^*\Sigma$ for a surface Σ carrying a torsionfree connection ∇ with everywhere-nonzero, skew-symmetric Ricci tensor ρ. We also fix a point $x \in \mathrm{M}$ at which $\dim \mathfrak{i}_x = 3$, and set $y = \pi(x)$, where $\pi : T^*\Sigma \to \Sigma$ is the bundle projection. By (13.3), the pair $(\dim \mathfrak{a}_y, \delta)$ equals $(3, 0)$ or $(2, 1)$.

If $(\dim \mathfrak{a}_y, \delta) = (3, 0)$, then the conclusion of Remark 11.1(c) holds. In fact, when $w_y = 0$, this is explicitly stated in Remark 11.1(c), while, in the case $w_y \neq 0$, Lemma 11.2 implies local homogeneity of ∇ at y, and our claim follows from the final clause of Theorem 11.4. Thus, ∇ is generic, as one sees using (12.6) with $(a, b) = (1, 0)$ if $w_y \neq 0$, and noting that the same is true when $w_y = 0$ since points y with $w_y \neq 0$ form a dense set (see (5.4)), and so, by (12.7), $\det Q$ is constant, namely, equal to 25. The invariant σ introduced in Section 15 may thus be treated, locally, as a symmetric 2-tensor on an open connected subset of Π, invariant under the infinitesimal action of $\mathrm{SL}(\Pi)$ (notation of Example 10.6). Hence, by Lemma 11.2, σ is a constant multiple of $\phi \otimes \phi$. However, as $\mathcal{Z}\sigma = 0$, cf. Section 15, (10.6) now gives $\sigma = 0$, and our claim follows in this case since g and $g^{\nabla} + 2\pi^*\sigma$ are locally isometric (Section 15).

Now let $(\dim \mathfrak{a}_y, \delta) = (2, 1)$. By (13.2.b), this is also the case if y is replaced with any point of some connected neighborhood U of y in Σ. We fix $\xi \in \mathrm{Ker}\,\mathcal{L}$, defined on U, so that $\xi \neq 0$ everywhere in U (cf. (13.2.b) and Theorem 12.2(ii)). On any open subset of U on which $\xi(w) = 0$, (12.2.b) and (12.1) with $\tau = 0$ give $\nabla\xi = 0$ and $\xi \otimes \rho = -\nabla\nabla\xi = 0$. Hence that such a subset must be empty, and so $\xi(w) \neq 0$ at all points of a dense open subset U' of U. Applying (11.3) to v given by $\rho(v, \cdot) = \xi$ we see that ∇ restricted to U' is locally homogeneous. As $\dim (\mathrm{Ker}\,\mathcal{L})_y = \delta = 1$, (12.5), (12.6) and Example 12.4 show that ∇ represents, on U', the point $(a, b) = (-9, 0)$ of the moduli curve in Theorem 11.4(i). Therefore, on U', (12.7) with $(a, b) = (-9, 0)$ yields $Qw = 15w - 10u$, and, by (10.3.a), $\rho(w, Qw) = 60$, which, due to denseness of U' in U, holds on U as well. Consequently, w and Qw are linearly independent at each point of U, so that the same is true of w and u, where u has now been extended to U via the formula $10u = 15w - Qw$. By continuity, we have (10.2) everywhere in U, and hence $[u, w] = 2u$. This allows us to treat U, locally, as an open set in

a two-dimensional non-Abelian simply connected Lie group H, while u, w then become left-invariant vector fields on H, and elements of $\mathfrak{a}_y$ are the germs at y of right-invariant vector fields on H (the flows of which consist of left translations).

Making U smaller, if necessary, and using Theorem 7.1, we may assume that $g = g^\nabla + 2\pi^*\tau$ for some symmetric 2-tensor τ on U. Since (13.1) is surjective (by (13.2.b)), for every left translation F close to the identity Lemma 4.6(ii) yields $F^*\tau = \tau + \mathcal{L}\beta$, where β is a 1-form depending on F. (Local isometries of (M, g) leave the vertical distribution $\mathcal{V}$ invariant.) Infinitesimally, this gives $\mathbb{L}_v\tau = \mathcal{L}\beta_v$ for every right-invariant vector field v, with a 1-form β_v that depends linearly on v. Let us now choose a basis v_1, v_2 of right-invariant vector fields such that $[v_1, v_2] = 2v_1$ and write $\mathbb{L}_j, \beta_j$ instead of $\mathbb{L}_v$ and β_v for $v = v_j$. As $\mathbb{L}_j$ commute with $\mathcal{L}$, for the 1-form $\xi' = \mathbb{L}_1\beta_2 - \mathbb{L}_2\beta_1 - 2\beta_1$ we have $\mathcal{L}\xi' = \mathbb{L}_1\mathbb{L}_2\tau - \mathbb{L}_2\mathbb{L}_1\tau - 2\mathbb{L}_1\tau = 0$. On the other hand, using the basis α, ϕ of left-invariant 1-forms given by $\alpha(u) = 3$, $\alpha(w) = 2$ and (5.2.i), we get $\beta_j = \Delta(v_j)\alpha + \Xi(v_j)\phi$ with some 1-forms Δ and Ξ. The equality $\xi' = \mathbb{L}_1\beta_2 - \mathbb{L}_2\beta_1 - 2\beta_1$ now reads $\xi' = (d\Delta)_{12}\alpha + (d\Xi)_{12}\phi$ (cf. (2.5.b)), where, for any 2-form ζ, we use the subscript convention $\zeta_{12} = \zeta(v_1, v_2)$. Also, as $\mathcal{L}\xi' = 0$, if we fix a function f with (10.1), ξ' equals a constant times $\xi = f^2\alpha$ (see Example 12.3). Thus, $(d\Xi)_{12} = 0$ and $(d\Delta)_{12} = 4cf^3\rho_{12}$ for some $c \in \mathbb{R}$. (By (10.4.i), $f\rho_{12}$ is constant.) Hence $d\Xi = 0$ and $d\Delta = 4cf^3\rho = -cd(f^3\phi)$, cf. (10.4.ii), so that $\Xi = d\chi$ and $\Delta = d\mu - cf^3\phi$ for some functions χ and μ. We may in addition assume that, for a suitable function ψ,

$$(16.1) \qquad \text{i)} \quad \mathbb{L}_j[\tau - \mathcal{L}(\mu\alpha + \chi\phi)] = -c\mathcal{L}[f\phi(v_j)\xi], \quad \text{ii)} \quad \mathcal{L}[f\phi(v_j)\xi] = -\mathbb{L}_j\mathcal{L}(\psi\xi).$$

Namely, (16.1.i) follows in any case since $\mathbb{L}_j\tau = \mathcal{L}\beta_j$ and $\mathcal{L}\beta_j = \mathcal{L}[\Delta(v_j)\alpha + \Xi(v_j)\phi] = \mathcal{L}[(\mathbb{L}_j\mu)\alpha + (\mathbb{L}_j\chi)\phi] - c\mathcal{L}[f^3\phi(v_j)\alpha]$, while $\xi = f^2\alpha$, and $\mathbb{L}_j$ commute with $\mathcal{L}$. To obtain (16.1.ii), instead of letting the right-invariant fields v_j with $[v_1, v_2] = 2v_1$ be otherwise arbitrary, we choose them as in (10.5). Then $f\phi(v_1) = -6$, $f\phi(v_2) = -6\psi$ (cf. (10.3.b)), and so $\mathcal{L}[f\phi(v_1)\xi] = 0$, as $\xi \in \mathrm{Ker}\,\mathcal{L}$, while $\mathcal{L}[f\phi(v_2)\xi] = -6\mathcal{L}(\psi\xi)$. However, $\mathbb{L}_1(f^2\psi) = 2f^2$ and $\mathbb{L}_2(f^2\psi) = 6f^2\psi$ by (10.5), (10.1) and (10.3.b) with $3d\psi = -f\phi$. Thus, $\mathbb{L}_j(\psi\xi) = \mathbb{L}_j(f^2\psi\alpha) = [\mathbb{L}_j(f^2\psi)]\alpha$ and $\mathbb{L}_j\mathcal{L}(\psi\xi) = \mathcal{L}\mathbb{L}_j(\psi\xi)$, so that $\mathbb{L}_1\mathcal{L}(\psi\xi) = 2\mathcal{L}(f^2\alpha) = 2\mathcal{L}\xi = 0 = -\mathcal{L}[f\phi(v_1)\xi]$, and $\mathbb{L}_2\mathcal{L}(\psi\xi) = 6\mathcal{L}(f^2\psi\alpha) = 6\mathcal{L}(\psi\xi) = -\mathcal{L}[f\phi(v_2)\xi]$, as required.

Setting $\tau' = \tau - \mathcal{L}(\mu\alpha + \chi\phi + c\psi\xi)$ we see that, by (16.1), $\mathbb{L}_j\tau' = 0$, and so τ' is left-invariant. Therefore, in view of Lemma 4.1(b), the restriction of g to some neighborhood of x is isometric to one of the metrics of Example 16.2. $\qquad\square$

The single manifold of Example 16.1 and the one-parameter family of Example 16.2 together form a collection of type III SDNE Walker manifolds that are mutually non-isometric, even locally. More precisely, an open submanifold in one of them is never isometric to an open submanifold of another.

In fact, the manifolds of Example 16.1 differs from those in Example 16.2 by the value of the local invariant $(\dim \mathfrak{a}_y, \delta)$ (see the proof of Theorem 16.3). Thus, we may restrict our discussion to the latter manifolds, assuming that τ and τ' are left-invariant symmetric 2-tensors on H, while the restrictions of $g^\nabla + 2\pi^*\tau$ and $g^\nabla + 2\pi^*\tau'$ to some open submanifolds are isometric. Since the vertical distribution is a local invariant of the metric (Section 6), and so is the transversal connection ∇ (cf. Lemma 4.3(b) and Theorem 7.1), applying Lemma 4.6(ii) we conclude that the left-invariant symmetric 2-tensor $\tau - \tau'$ is the $\mathcal{L}$-image of some 1-form on H. We used here the fact that the left translations in H are the only diffeomorphisms F

between open submanifolds of H, satisfying the condition $F^*\nabla = \nabla$, which itself easily follows from (11.4) and Remark 10.3.

As a consequence of Lemma 12.5, the two metrics represent the same element of the one-parameter family in Example 16.2.

17. Non-generic RSTS connections and type III SDNE Walker metrics

We refer to an RSTS connection ∇ as *special* if, at every point, the Ricci tensor ρ is nonzero and the bundle morphism Q given by (5.5.i) is noninjective. This is the extreme opposite of the case where ∇ is generic, defined in the lines preceding (12.5). For an RSTS connection with $\rho \neq 0$ at every point of the underlying surface Σ, being generic *or* special is a general-position requirement: Σ obviously contains a dense open subset U such that the restriction of ∇ to each connected component of U is either generic or special.

According to (5.5.ii), if an RSTS connection ∇ is special, Q and Q^* have, at each point, the eigenvalues 0 and 10, so that

$$(17.1) \qquad (Q - 10)Q = 0, \qquad (Q^* - 10)Q^* = 0.$$

The next result may be viewed as a counterpart of Theorem 14.1 for RSTS connections which, this time, are assumed special rather than generic. As before, $\mathcal{S}_1$ (or, $\mathcal{S}_2$) is the space of all 1-forms (or, symmetric 2-tensors) of class C^∞ on the surface Σ in question. In view of (17.1), $T^*\Sigma = \mathrm{Ker}\,(Q^* - 10) \oplus \mathrm{Ker}\,Q^*$, so that $\mathcal{S}_1 = \mathcal{S}^+ \oplus \mathcal{S}^0$, where $\mathcal{S}^+$ (or, $\mathcal{S}^0$) is the space of all C^∞ sections of the line bundle $\mathrm{Ker}\,(Q^* - 10)$ (or, $\mathrm{Ker}\,Q^*$). We also define a fourth-order linear differential operator $\mathcal{W} : \mathcal{S}_2 \to \mathcal{S}_2$ by $10\mathcal{W} = \mathcal{L}\mathcal{Z}$, with $\mathcal{L}$ and $\mathcal{Z}$ as in (3.1.b) and (5.7).

Theorem 17.1. *For any special RSTS connection,*
  (a) $\mathcal{W}^3 = \mathcal{W}^2$, *that is,* $(\mathcal{W} - \mathrm{Id})\mathcal{W}^2 = 0$, *and* $\mathcal{S}_2 = \mathrm{Ker}\,\mathcal{W}^2 \oplus \mathrm{Ker}\,(\mathcal{W} - \mathrm{Id})$,
  (b) $\mathcal{L}(\mathcal{S}_1) = \mathcal{L}(\mathcal{S}^+) \oplus \mathcal{L}(\mathcal{S}^0)$ *and* $\mathcal{L}(\mathcal{S}^+) = \mathrm{Ker}\,(\mathcal{W} - \mathrm{Id})$,
  (c) $\mathcal{L}(\mathcal{S}^0) \subset \mathrm{Ker}\,\mathcal{Z} \subset \mathrm{Ker}\,\mathcal{W} \subset \mathrm{Ker}\,\mathcal{W}^2 = \mathrm{Ker}\,Q^*\mathcal{Z}$.

Proof. By (12.2.a) and (17.1), $\mathcal{Z}\mathcal{L} = Q^*$ and $(Q^*)^2 = 10Q^*$, which yields $\mathcal{W}^3 = \mathcal{W}^2$, and hence (a). (Explicitly, $\tau = \mathcal{W}^2\tau - (\mathcal{W} + \mathrm{Id})(\mathcal{W} - \mathrm{Id})\tau$, and $(\mathcal{W} + \mathrm{Id})(\mathcal{W} - \mathrm{Id})\tau \in \mathrm{Ker}\,\mathcal{W}^2$, $\mathcal{W}^2\tau \in \mathrm{Ker}\,(\mathcal{W} - \mathrm{Id})$ for any $\tau \in \mathcal{S}_2$.) The relation $\mathcal{L}(\mathcal{S}^+) \cap \mathcal{L}(\mathcal{S}^0) = \{0\}$, that is, the first part of (b), is clear since $\mathcal{Z}\mathcal{L} = Q^*$ is zero on $\mathcal{S}^0$, and injective on $\mathcal{S}^+$. Next, $10\mathcal{W}\mathcal{L}\alpha = \mathcal{L}\mathcal{Z}\mathcal{L}\alpha = \mathcal{L}Q^*\alpha = 10\mathcal{L}\alpha$ whenever $\alpha \in \mathcal{S}^+$, so that $\mathcal{L}(\mathcal{S}^+) \subset \mathrm{Ker}\,(\mathcal{W} - \mathrm{Id})$. On the other hand, if $\mathcal{W}\tau = \tau$, setting $\alpha = \mathcal{Z}\tau$ we obtain $Q^*\alpha = \mathcal{Z}\mathcal{L}\alpha = \mathcal{Z}\mathcal{L}\mathcal{Z}\tau = 10\mathcal{Z}\mathcal{W}\tau = 10\mathcal{Z}\tau = 10\alpha$, and so $\alpha \in \mathcal{S}^+$, while $\mathcal{L}\alpha = \mathcal{L}\mathcal{Z}\tau = 10\mathcal{W}\tau = 10\tau$. Consequently, $\mathrm{Ker}\,(\mathcal{W} - \mathrm{Id}) \subset \mathcal{L}(\mathcal{S}^+)$, which proves (b). The first two inclusions in (c) are immediate as $\mathcal{Z}\mathcal{L} = Q^*$ and $10\mathcal{W} = \mathcal{L}\mathcal{Z}$, and the third one is obvious. Finally, $100\mathcal{W}^2 = \mathcal{L}\mathcal{Z}\mathcal{L}\mathcal{Z} = \mathcal{L}Q^*\mathcal{Z}$. Thus, $\mathrm{Ker}\,Q^*\mathcal{Z} \subset \mathrm{Ker}\,\mathcal{W}^2$. The opposite inclusion follows since, if $\mathcal{L}Q^*\mathcal{Z}\tau = 0$, then $0 = \mathcal{Z}\mathcal{L}Q^*\mathcal{Z}\tau = (Q^*)^2\mathcal{Z}\tau = 10Q^*\mathcal{Z}\tau$. $\square$

By analogy with Section 15, we will also say that a type III SDNE Walker manifold (M, g) is *special* if so is, at every point x of M, the RSTS connection ∇ determined, as in Theorem 7.1, by the restriction of g to a neighborhood of x.

The assumption that (M, g) is special allows us, as in Section 15, to replace $(\Sigma, \nabla, [\tau])$ by a more tangible triple (Σ, ∇, σ) of local invariants. Here σ is the $\mathrm{Ker}\,\mathcal{W}^2$ component of τ relative to the decomposition $\mathcal{S}_2 = \mathrm{Ker}\,\mathcal{W}^2 \oplus \mathrm{Ker}\,(\mathcal{W} - \mathrm{Id})$ in Theorem 17.1(a), so that, due to the second equality in Theorem 17.1(b), the coset $[\sigma] = [\tau]$ remains unchanged, and $Q^*\mathcal{Z}\sigma = 0$, cf. Theorem 17.1(c).

In contrast with the situation discussed in Section 15, σ does not constitute a unique, canonical choice of a representative from the coset $[\tau]$. In fact, according to Theorem 17.1(c), by replacing τ with σ we have merely reduced the freedom of choosing τ, which originally ranged over a coset of the subspace $\mathcal{L}(\mathcal{S}_1)$ in the space $\mathcal{S}_2$, to the freedom of selecting σ out of a fixed coset of the subspace $\mathcal{L}(\mathcal{S}^0)$ in the space $\operatorname{Ker} Q^* \mathcal{Z}$.

The remainder of this section is devoted to a description of the local structure of an arbitrary RSTS connection with $\dim \operatorname{Ker} \mathcal{L} = 1$, where $\mathcal{L}$ is the Killing operator. By (12.2.a), all such connections are special. We begin with some examples.

Let Σ be a surface with fixed vector fields v, w that trivialize the tangent bundle $T\Sigma$ and functions $\psi, \chi : \Sigma \to \mathbb{R}$, satisfying the conditions

$$(17.2) \qquad [v, w] = 6v + 2\psi w, \quad d_v \psi = 4\psi^2, \quad d_w \psi = -4\psi, \quad d_v \chi = 4, \quad \psi \neq 0 \text{ everywhere.}$$

Such v, w, ψ, χ are in a bijective correspondence with quadruples u, w, ψ, χ in which u, w again trivialize the tangent bundle, while $[u, w] = 6u$, $d_u \psi = 0$, $d_w \psi = -4\psi$, $d_{u-\psi w} \chi = 4$ and $\psi \neq 0$ everywhere. (The correspondence is given by $u = v + \psi w$.) Locally, the triples u, w, ψ within the latter quadruples arise precisely when u, w constitute a suitably chosen basis of left-invariant vector fields on a two-dimensional non-Abelian Lie group H and ψ is a nonzero constant multiple of a specific Lie-group homomorphism from H into the multiplicative group $(0, \infty)$. (Cf. Remark 10.8 and (10.1).)

For Σ and v, w, ψ, χ as above, we define a torsionfree connection ∇ on Σ by

$$(17.3) \qquad \nabla_v v = 5\psi v, \quad \nabla_w v = \psi w, \quad \nabla_v w = 6v + 3\psi w, \quad \nabla_w w = 15\chi v - 4w.$$

One easily verifies that the Ricci tensor ρ of ∇ is skew-symmetric, $\rho(v, w) = 4\psi$, and w coincides with the vector field characterized by (5.2.ii). Furthermore, $\dim \operatorname{Ker} \mathcal{L} = 1$ due to Theorem 12.2(i) and the fact that $\mathcal{L}\xi = 0$ for the nonzero 1-form ξ with $\xi(v) = 0$ and $\xi(w) = 4\psi$. In addition, $60\psi\chi = \rho(\nabla_w w, w)$ is an affine invariant and $d_v(\psi\chi) = 4(\psi\chi + 1)\psi$. Thus, ∇ is locally homogeneous if and only if $\psi\chi$ is constant, namely, equal to -1. (The 'if' part is immediate from Example 10.2, since, setting $u = 3(w - \chi v)/2$, we may then rewrite (17.3) as (10.2) with $(a, b) = (-9, 0)$.)

Theorem 17.2. *Every RSTS connection with* $\dim \operatorname{Ker} \mathcal{L} = 1$ *is, locally, at points in general position, given by* (17.3) *for some quadruple* v, w, ψ, χ *as above, with* (17.2).

Proof. Let us define w by (5.2.ii) and ψ by $4\psi = \xi(w)$, where ξ is a fixed nontrivial 1-form with $\mathcal{L}\xi = 0$. Thus, $\psi \neq 0$ at all points of a dense open subset: in fact, if we had $\psi = 0$ on some nonempty open set U, (12.2.b) with $\tau = 0$ would give $\nabla\xi = 0$ on U, and so (12.1) with $\tau = 0$ would imply that $\xi = 0$ on U, contradicting Theorem 12.2(ii).

From now on we assume that $\psi \neq 0$ everywhere. Using (12.2.b) with $\tau = 0$ we obtain $\nabla\xi = \psi\rho$, and so $\nabla\nabla\xi = (d\psi + \psi\phi) \otimes \rho$ (cf. (5.2.i)), while (12.1) with $\tau = 0$ yields $\nabla\nabla\xi = -\xi \otimes \rho$. Hence $d\psi = -\xi - \psi\phi$.

For the vector field v characterized by $\xi = \rho(v, \cdot)$, we see that $\xi(v) = 0$ and $\phi(v) = \rho(w, v) = -\xi(w) = -4\psi$. Since $d\psi = -\xi - \psi\phi$, (5.2.iii) now gives $d_v \psi = 4\psi^2$, $d_w \psi = -4\psi$, as required in (17.2). From the relation $\nabla\xi = \psi\rho$ and the Leibniz rule, $\rho(\nabla_u v, \cdot) = \nabla_u \xi - \phi(u)\xi = \psi\rho(u, \cdot) - \phi(u)\rho(v, \cdot)$ for any vector field u, so that $\nabla_u v = \psi u - \phi(u)v$. In particular, setting $u = v$ or $u = w$, we obtain the first two equalities in (17.3).

As $\mathcal{L}\xi = 0$, (12.2.a) shows that $Q^*\xi = 0$. Thus, by (5.5.i), $0 = \xi(Qw) = \xi(4w + \nabla_w w) = 16\psi + \xi(\nabla_w w)$ and $0 = \xi(Qv) = \xi(4v + \nabla_v w - 3\psi w) = \xi(\nabla_v w) - 12\psi^2$. We used here the fact that

$\xi(v) = 0$, which now also implies the last equality in (17.3), for some function χ, as well as the relation $\nabla_v w = \mu v + 3\psi w$ for some function μ. At the same time, by (5.2.iv) and (2.5.b), $8\psi = 2\xi(w) = 2\rho(v, w) = (d\phi)(u, w) = -d_w[\phi(v)] - \phi(\nabla_v w - \nabla_w v) = 4d_w\psi - \mu\phi(v) = (-16 + 4\mu)\psi$, so that $\mu = 6$. We thus have (17.3), as well as (17.2) except for the relation $d_v\chi = 4$. To establish it, we use (5.1.a) and (2.1), obtaining $4\psi w = \rho(v, w)w = R(v, w)w = 4\psi w + 15(4 - d_v\chi)v$, which completes the proof. $\qquad\square$

18. RSTS connections associated with a Lorentzian 3-space

Let Π be a two-dimensional real vector space with a fixed area form Ω (cf. Example 10.6). The unimodular group $\mathrm{SL}(\Pi)$ acts, by conjugation, on the three-dimensional vector space V of all traceless endomorphisms of Π, and its action preserves the Lorentzian $(-++)$ inner product $\langle\,,\rangle$ in V characterized by $\langle A, A\rangle = -\det A$ for $A \in$ V. In other words, V is the Lie algebra of $\mathrm{SL}(\Pi)$, the action amounts to the adjoint representation, and $\langle\,,\rangle$ is, up to a factor, the Killing form of $\mathrm{SL}(\Pi)$. The action of $\mathrm{SL}(\Pi)$ is not effective: its kernel is the center $\mathbf{Z}_2 = \{\mathrm{Id}, -\mathrm{Id}\}$ of $\mathrm{SL}(\Pi)$, and $\mathrm{SL}(\Pi)/\mathbf{Z}_2$ acting on V is nothing else than the identity component $\mathrm{SO}^\uparrow(\mathrm{V})$ of the Lorentz group of $(\mathrm{V}, \langle\,,\rangle)$.

The $\mathrm{SL}(\Pi)$-equivariant quadratic mapping $\Phi : \Pi \to \mathrm{V}$ defined by $\Phi(y) = \Omega(y, \cdot) \otimes y$ sends $\Pi \smallsetminus \{0\}$ onto the *future null cone* Σ in V, which is a specific connected component Σ of the set of nonzero $\langle\,,\rangle$-null vectors.

Proposition 18.1. *Under the above hypotheses, the future null cone Σ in V admits a one-parameter family of torsionfree connections invariant under the transitive action of $\mathrm{SO}^\uparrow(\mathrm{V})$ on Σ and having nonzero, skew-symmetric Ricci tensor. All these connections represent the point $(a, b) = (1, 0)$ of the moduli curve in Theorem 11.4(i).*

Proof. Since $\Phi : \Pi \smallsetminus \{0\} \to \Sigma$ is a two-fold covering, we may choose the connections in question to be the Φ-images of the $\mathrm{SL}(\Pi)$-invariant connections ∇ on Π described in Example 10.6. (The deck transformation $-\mathrm{Id} \in \mathrm{SL}(\Pi)$ leaves any such ∇ invariant.) $\qquad\square$

As before, let Σ be a future null cone in a 3-space V endowed with a Lorentzian $(-++)$ inner product $\langle\,,\rangle$, and let Y be the sheet, adjacent to Σ, of the two-sheeted hyperboloid formed by all $\langle\,,\rangle$-unit timelike vectors in V. We refer to Y as the *hyperbolic plane*, since $\langle\,,\rangle$ induces on Y a Riemannian metric of constant curvature -1. Similarly, $\langle\,,\rangle$ induces a Lorentzian $(-+)$ metric of constant curvature 1 on the one-sheeted hyperboloid S of all $\langle\,,\rangle$-unit spacelike vectors in V.

The unit tangent bundle $T^1\mathrm{Y}$ of Y may be identified with the submanifold of $\mathrm{Y} \times \mathrm{S}$ consisting of all $\langle\,,\rangle$-orthogonal pairs $(p, q) \in \mathrm{Y} \times \mathrm{S}$. The formula $F(p, q) = p + q$ defines a mapping $F : T^1\mathrm{Y} \to \Sigma$ which is a fibration, as the connected Lorentz group $\mathrm{SO}^\uparrow(\mathrm{V})$ acts transitively on both $T^1\mathrm{Y}$ and Σ, while F is obviously $\mathrm{SO}^\uparrow(\mathrm{V})$-equivariant. The fibres of F are the leaves of the *horocycle foliation* on $T^1\mathrm{Y}$, called so because they are easily verified to be the natural lifts to $T^1\mathrm{Y}$ of oriented horocycles in Y.

Remark 18.2. The horocycle foliation descends from $T^1\mathrm{Y}$ to the unit tangent bundle $T^1\Sigma$ of any closed orientable surface Σ of genus greater than 1 endowed with a hyperbolic metric. This is due to its invariance under the action on $T^1\mathrm{Y}$ of the group $\mathrm{SO}^\uparrow(\mathrm{V})$ of all orientation-preserving isometries of the hyperbolic plane Y. The invariance follows in turn from the $\mathrm{SO}^\uparrow(\mathrm{V})$-equivariance of F, mentioned above.

Remark 18.3. If Λ is a $\langle\,,\rangle$-null one-dimensional subspace of our Lorentzian 3-space V, then $\langle\,,\rangle$ restricted to the plane $\Lambda^\perp$ is positive semidefinite and degenerate. Thus, the set $\Lambda^\perp \cap$ S of all $\langle\,,\rangle$-unit spacelike vectors in $\Lambda^\perp$ is the union of two parallel lines, cosets of Λ, each of which is a null geodesic in the one-sheeted hyperboloid S with its submanifold metric. Consequently, as $\mathrm{SO}^\uparrow(\mathrm{V})$ acts on S transitively, S carries two foliations, the leaves of which are maximal null geodesics in S and, simultaneously, straight lines in V, in such a way that each leaf of one foliation is disjoint with (and, as a line, parallel to) exactly one leaf of the other foliation.

Proposition 18.4. *For* $\mathrm{V},\langle\,,\rangle$ *and the one-sheeted hyperboloid* $\mathrm{S} \subset \mathrm{V}$ *as above, let* $\Lambda^\pm$ *be the two parallel lines forming the set* $\Lambda^\perp \cap$ S, *where* Λ *is a fixed* $\langle\,,\rangle$-*null one-dimensional subspace of* V. *Then the surface* $\mathrm{S}' = \mathrm{S} \smallsetminus \Lambda^-$ *admits a torsionfree connection* ∇ *invariant under the action of the two-dimensional Lie group* $\mathrm{H} = \{C \in \mathrm{SO}^\uparrow(\mathrm{V}) : C(\Lambda) = \Lambda\}$ *and having everywhere-nonzero, skew-symmetric Ricci tensor. Furthermore, the restriction of* ∇ *to the open subset* $\mathrm{S}' \smallsetminus \Lambda^+ = \mathrm{S} \smallsetminus \Lambda^\perp$ *is locally homogeneous and represents the point* $(a, b) = (0, 1)$ *on the moduli curve of Theorem* 11.4(i).

Proof. Define vector fields $v_\pm$ on the open set $\mathrm{V} \smallsetminus \Lambda^\perp$ in V by $v_\pm = y \pm q - \langle y, y \pm q\rangle\langle y, p\rangle^{-1}p$, where y denotes the radial (identity) vector field on V and $p, q \in$ V are constant vector fields with $p \in \Lambda \smallsetminus \{0\}$ and $q \in \Lambda^+$. Both $v_\pm$ are easily seen to remain unchanged when a different choice of p or q is made, so that they depend only on Λ^+, which makes v_+ and v_- invariant under the action of H. (As H is connected, $C(\Lambda^+) = \Lambda^+$ for all $C \in$ H.) Also, $\langle v_\pm, y\rangle = 0$, $\langle v_\pm, v_\pm\rangle = 1 - \langle y, y\rangle$, $\langle v_+, v_-\rangle = -1 - \langle y, y\rangle$, so that, at every point of the surface $\mathrm{S}'' = \mathrm{S} \smallsetminus \Lambda^\perp$, the vector fields $v_\pm$ are tangent to S'', null, and linearly independent. Therefore, the vector fields $u = 3(v_- - v_+)$ and $w = 2v_-$ trivialize the tangent bundle of S''. As $[u, w] = 2u$, the connection ∇ defined by (10.2) with these u, w and $(a, b) = (0, 1)$ has all the required properties except for being defined just on S'', rather than everywhere in S'.

To show that ∇ has a C^∞ extension to S', let us note that the function $\psi = \langle y, p\rangle$ and the vector field $X = \psi v_+$ are of class C^∞ on S', and hence so is $Z = \psi^{-1}v_-$. (The last conclusion follows as $\langle X, Z\rangle = \langle v_+, v_-\rangle = -2$ on S'', while X, Z are both null, and $X \neq 0$ everywhere in S'.) Furthermore, $d_v\psi = \psi$ both for $v = v_+$ and $v = v_-$. Thus, $\nabla_X X = \psi^3 Z - \psi X$, $\nabla_X Z = -\psi Z$, $\nabla_Z X = \psi Z$, $\nabla_Z Z = 0$, and our assertion follows. $\square$

In the above proof, the (skew-symmetric) Ricci tensor ρ of ∇ is nonzero everywhere in S', as $\rho(X, Z) = \rho(v_+, v_-) = -\rho(u, w)/6 = -1$ by (10.3.a). Also, $w = 2v_- = 2\psi Z$ vanishes on Λ^+, since so does ψ. Thus, Proposition 18.4 illustrates, just like Example 10.5, the necessity of the assumption that $a + b \neq 1$ for conclusion (ii) in Proposition 10.4.

19. Transversal RSTS connections

We discuss here transversal RSTS connections having everywhere-nonzero Ricci tensor. Transversal torsionfree connections which are *flat* were studied, in any codimension, by Wolak [**19**].

Suppose that $\mathcal{F}$ is a codimension m foliation on a manifold M and $\mathcal{V}$ is the codimension m distribution tangent to $\mathcal{F}$. A local section of the quotient bundle $(T\mathrm{M})/\mathcal{V}$ defined on a nonempty open set $U \subset$ M will be called $\mathcal{F}$-*projectable* if it is the image, under the quotient projection $T\mathrm{M} \to (T\mathrm{M})/\mathcal{V}$, of some $\mathcal{V}$-projectable local vector field defined on U (cf. Remark 2.1). Following Molino [**14**], by a *transversal connection* for $\mathcal{F}$ we mean any operation $\overline{\nabla}$ associating with every nonempty open set $U \subset$ M and every pair v, v' of $\mathcal{F}$-projectable

local sections of $(T\mathrm{M})/\mathcal{V}$, defined on U, an $\mathcal{F}$-projectable local section $\overline{\nabla}_v v'$ of $(T\mathrm{M})/\mathcal{V}$, defined on U, in such a way that

(i) the dependence of $\overline{\nabla}_v v'$ on v and v' is local, and, in particular, the operations $\overline{\nabla}$ corresponding to two intersecting open sets agree on their intersection,

(ii) if U satisfies (2.7) then, for some connection ∇ on the base Σ, and any $\mathcal{F}$-projectable local sections v, v' of $(T\mathrm{M})/\mathcal{V}$, defined on U, the π-image of $\overline{\nabla}_v v'$ is the vector field $\nabla_w w'$ on Σ, where w, w' stand for the π-images of v and v'.

All local properties of connections on manifolds make sense for transversal connections. One can thus speak of codimension-two foliations on manifolds with transversal torsionfree connections, the Ricci tensor of which is skew-symmetric and nonzero at every point. From now on we refer to them as *transversal RSTS connections with everywhere-nonzero Ricci tensor.*

Example 19.1. Each of these cases leads to a transversal RSTS connection $\overline{\nabla}$ with everywhere-nonzero Ricci tensor for a codimension-two foliation $\mathcal{F}$ on an n-dimensional manifold.

(a) $n = 4$ and $\mathcal{F}$ is the foliation tangent to the vertical distribution $\mathcal{V}$ of a type III SDNE Walker manifold (immediate from Theorem 7.1).

(b) $n = 2$ and $\mathcal{F}$ is the 0-dimensional foliation on a surface Σ with a fixed RSTS connection ∇, the Ricci tensor of which is skew-symmetric and nonzero everywhere (obvious).

(c) $n \geq 3$ is arbitrary and $\mathcal{F}$ is the vertical foliation on the total space of a locally trivial bundle over a surface Σ with ∇ as in (b) (obvious).

(d) $n = 3$ and $\mathcal{F}$ is the horocycle foliation on the unit tangent bundle $T^1\mathrm{Y}$ of the hyperbolic plane Y (immediate from Proposition 18.1, the lines preceding Remark 18.2, and (c)).

In cases (a), (b) and (c) above, *the underlying manifold cannot be compact*, as shown in Theorem 9.3(a) and [**5**, the lines following Theorem 5.1]. In (d), however, although $T^1\mathrm{Y}$ is noncompact, it has compact quotients to which $\mathcal{F}$ descends:

Proposition 19.2. *The horocycle foliation on the unit tangent bundle $T^1\Sigma$ of any closed orientable surface Σ of genus greater than 1, for any hyperbolic metric on Σ, admits a transversal RSTS connection with everywhere-nonzero Ricci tensor.*

This is a direct consequence of Example 19.1(d), Remark 18.2 and Proposition 18.1.

Corollary 19.3. *Compact manifolds with codimension-two foliations that admit transversal RSTS connections having everywhere-nonzero Ricci tensor exist in all dimensions $n \geq 3$, but not in dimension 2.*

In fact, if $n \geq 3$, it suffices to combine Proposition 19.2 with the obvious Cartesian-product construction. For $n = 2$, see the lines preceding Proposition 19.2.

20. Type III SDNE non-Walker manifolds

It is not known whether Theorem 9.3 remains true without the assumption that g is a Walker metric. This section presents a global condition unrelated to compactness, which, although satisfied by some type III SDNE Walker manifolds, can never hold in the non-Walker case.

Specifically, we say that a type III SDNE manifold (M, g) is *vertically complete* if every leaf of its vertical distribution $\mathcal{V}$ is complete as a manifold with the connection induced by the Levi-Civita connection of g. Thus, geodesic completeness of (M, g) implies its vertical

completeness. Note that the leaves of $\mathcal{V}$ are totally geodesic (Section 6), and the connection induced on a leaf is always flat [**6**, Lemma 5.2(i)].

All pairs $(\mathrm{M}, g) = (T^*\Sigma, g^\nabla + 2\pi^*\tau)$ described in the first part of Theorem 7.1 are examples of vertically complete type III SDNE *Walker* manifolds. In fact, by (4.2), the connection induced on each leaf $T_y^*\Sigma$ is the standard flat connection on the vector space $T_y^*\Sigma$. On the other hand, no such examples are possible in the non-Walker case:

Theorem 20.1. *Every vertically complete type* III *SDNE manifold has the Walker property.*

Proof. Suppose that, on the contrary, (M, g) is a vertically complete type III non-Walker SDNE manifold, and so the vertical distribution is not parallel (see Section 6). Let α and β be the 1-forms on M, defined in [**6**, Lemma 5.2]. Thus, according to [**6**, Theorem 6.2(ii)], $\beta \neq 0$ somewhere in M. Formulae (8.1.g) and (8.2.i) in [**6**] now imply that, for some leaf N of $\mathcal{V}$, the 1-form on ξ on N obtained by restricting α to N is not identically zero, while [**6**, formula (8.1.b)] states that $d_u[\alpha(v)] = \alpha(u)\alpha(v)$ for any local sections u, v of $\mathcal{V}$ parallel along $\mathcal{V}$. (The three formulae are established in [**6**, Theorem 8.4], without using the assumption that $\beta \neq 0$ *everywhere*.) Denoting by D the complete flat connection induced on the leaf N, we thus have $\mathrm{D}\xi = \xi \otimes \xi$, and, choosing a geodesic $\mathbb{R} \ni s \mapsto x(s) \in \mathrm{N}$ with the velocity vector field $v(s)$ such that $\mu(s) = \xi_{x(s)}(v(s))$ is nonzero for some $s \in \mathbb{R}$, we obtain $d\mu/ds = \mu^2$. Hence μ cannot be defined everywhere in $\mathbb{R}$. This contradiction completes the proof. $\square$

Acknowledgements

References

[1] Z. Afifi, Riemann extensions of affine connected spaces, *Quart. J. Math. Oxford (2)* **5** (1954), 312–320.

[2] T. Arias-Marco and O. Kowalski, Classification of locally homogeneous affine connections with arbitrary torsion on 2-dimensional manifolds, *Monatsh. Math.* **153** (2008), 1–18.

[3] N. Blažić, N. Bokan and Z. Rakić, A note on the Osserman conjecture and isotropic covariant derivative of curvature, *Proc. Amer. Math. Soc.* **128** (2000), 245–253.

[4] E. Calviño-Louzao, E. García-Río, P. Gilkey and R. Vázquez-Lorenzo, The geometry of modified Riemannian extensions, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, online first (2009), doi:10.1098/rspa.2009.0046.

[5] A. Derdzinski, Connections with skew-symmetric Ricci tensor on surfaces, *Results Math.* **52** (2008), 223-245.

[6] A. Derdzinski, Non-Walker self-dual neutral Einstein four-manifolds of Petrov type III, *J. Geom. Anal.* **19** (2009), 301–357.

[7] A. Derdzinski and W. Roter, Projectively flat surfaces, null parallel distributions, and conformally symmetric manifolds, *Tohoku Math. J.* **59** (2007), 565–602.

[8] J. C. Díaz-Ramos, E. García-Río and R. Vázquez-Lorenzo, Four-dimensional Osserman metrics with nondiagonalizable Jacobi operators, *J. Geom. Anal.* **16** (2006), 39–52.

[9] F. J. E. Dillen and L. C. A. Verstraelen (Eds.), *Handbook of Differential Geometry, Vol. I*, North-Holland, Amsterdam, 2000.

[10] E. García-Río, D. N. Kupeli, M. E. Vázquez-Abal and R. Vázquez-Lorenzo, Affine Osserman connections and their Riemann extensions, *Differential Geom. Appl.* **11** (1999), 145–153.

[11] E. García-Río, D. N. Kupeli and R. Vázquez-Lorenzo, *Osserman Manifolds in Semi-Riemannian Geometry*, Lecture Notes in Math. 1777, Springer, Berlin, 2002.

[12] P. B. Gilkey, *The Geometry of Curvature Homogeneous Pseudo-Riemannian Manifolds*, Imperial College Press, London, 2007.

[13] O. Kowalski, B. Opozda and Z. Vlášek, A classification of locally homogeneous affine connections with skew-symmetric Ricci tensor on 2-dimensional manifolds, *Monatsh. Math.* **130** (2000), 109–125.

[14] P. Molino, Connexions et *G*-structures sur les variétés feuilletées, *Bull. Sci. Math. (2)* **92** (1968), 59–63.

[15] B. Opozda, A classification of locally homogeneous connections on 2-dimensional manifolds, *Differential Geom. Appl.* **21** (2004), 173–198.

[16] E. M. Patterson and A. G. Walker, Riemann extensions, *Quart. J. Math. Oxford (2)* **3** (1952), 19–28.

[17] M. Toomanian, Killing vectorfields and infinitesimal affine transformations on a generalised Riemann extension, *Tensor (N.S.)* **32** (1978), 335–338.

[18] A. G. Walker, Canonical form for a Riemannian space with a parallel field of null planes, *Quart. J. Math. Oxford (2)* **1** (1950), 69–79.

[19] R. A. Wolak, Transversely affine foliations compared with affine manifolds, *Quart. J. Math. Oxford (2)* **41** (1990), 369-384.

[20] Y.-C. Wong, Two dimensional linear connexions with zero torsion and recurrent curvature, *Monatsh. Math.* **68** (1964), 175–184.