Connections with skew-symmetric Ricci tensor on surfaces

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Abstract. Some known results on torsionfree connections with skew-symmetric Ricci tensor on surfaces are extended to connections with torsion, and Wong's canonical coordinate form of such connections is simplified.

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1. Introduction

This paper generalizes some results concerning the situation where

$$\begin{array}{l}\nabla \text{ is a connection on a surface } \Sigma, \text{ and the Ricci} \\ \text{tensor } \rho \text{ of } \nabla \text{ is skew-symmetric at every point.}\end{array} \tag{1.1}$$

What is known about condition (1.1) can be summarized as follows. Norden [19], [20, §89] showed that, for a torsionfree connection ∇ on a surface, skew-symmetry of the Ricci tensor is equivalent to flatness of the connection obtained by projectivizing ∇, and implies the existence of a fractional-linear first integral for the geodesic equation. Wong [26, Theorem 4.2] found three coordinate expressions which, locally, represent all torsionfree connections ∇ with (1.1) such that $\rho \neq 0$ everywhere in Σ. Kowalski, Opozda and Vlášek [14] used an approach different from Wong's to classify, locally, all torsionfree connections ∇ satisfying (1.1) that are also locally homogeneous, while in [13] they proved that, for real-analytic torsionfree connections ∇ with (1.1), third-order curvature-homogeneity implies local homogeneity (but one cannot replace the word 'third' with 'second'). Blažić and Bokan [2] showed that the torus T^2 is the only closed surface Σ admitting both a torsionfree connection ∇ with (1.1) and a ∇-parallel almost complex structure.

García-Río, Kupeli, Vázquez-Abal and Vázquez-Lorenzo [10] proved that connections ∇ as in (1.1) are equivalently characterized both by being the so-called *affine Osserman connections* on surfaces [10, Theorem 4], and, in the torsionfree case, by the four-dimensional Osserman property of the Riemann extension metric g^∇ [10, Theorem 4]. They also showed that, if such ∇ is torsionfree and $\rho \neq 0$ everywhere, then g^∇ is a curvature-homogeneous self-dual Ricci-flat Walker metric of Petrov type III with the metric signature $(--++)$ on the four-manifold $T^*\Sigma$ [10, Theorem 9], cf. [9, Remark 2.1]. Anderson and Thompson [1, pp. 104–107] proved that, among torsionfree connections ∇ on surfaces, those with (1.1) are characterized by the existence, locally in $T\Sigma$, of a fractional-linear Lagrangian for which the geodesics of ∇ are the solutions of the Euler-Lagrange equations. Bokan, Matzeu and Rakić [3] – [5] studied connections with skew-symmetric Ricci tensor on higher-dimensional manifolds.

The results of this paper begin with Section 5, where we obtain the conclusion of Blažić and Bokan [2] without assuming the existence of a ∇-parallel almost complex structure, while allowing ∇ to have torsion.

Next, in Section 6, we strengthen Wong's theorem [26, Theorem 4.2] by reducing the three cases to just one, and removing the assumption that $\rho \neq 0$.

In Sections 7 – 9 we extend the theorem of Kowalski, Opozda and Vlášek [14] to connections with torsion, proving that a locally homogeneous connection on a surface having skew-symmetric Ricci tensor must be locally equivalent to a left-invariant connection on a Lie group (Theorem 9.1). Note that the last conclusion is also true for all locally homogeneous torsionfree connections on surfaces except the Levi-Civita connection of the standard sphere, as one easily verifies using Opozda's classification of such connections [21, Theorem 1.1]. See Remark 8.2.[1]

Sections 10 and 11 generalize some of Norden's results [20] to connections with torsion. In Section 11 we also give a proof of Anderson and Thompson's theorem [1] based on the Hamiltonian formalism.

Finally, Section 13 describes a class of examples of Ricci-flat Walker four-manifolds which includes those constructed by García-Río, Kupeli, Vázquez-Abal and Vázquez-Lorenzo [10, Theorem 9]. The generalization arises by the use of a more general type of Riemann extensions. However, our examples are not new: they first appeared, in a different form, in Theorem 3.1(ii.3) of Díaz-Ramos, GarcíaRío and Vázquez-Lorenzo's paper [9]. In addition, as we point out in Section 13, Theorem 3.1(ii.3) of [9] states in coordinate language that locally, up to isometries, this larger class of examples consists precisely of all curvature-homogeneous self-dual Ricci-flat Walker $(--++)$ metrics of Petrov type III.

[1] (Added in proof.) Arias-Marco and Kowalski [27] recently extended Opozda's classification to locally homogeneous surface connections with arbitrary torsion. Their Theorem 1 is more general than the results in our Sections 7 – 9. It also implies, for the same reasons as in Remark 8.2, that all locally homogeneous connections on surfaces, with the sole exception of the standard sphere, are locally equivalent to left-invariant connections on Lie groups.

2. Preliminaries

By a ‘manifold’ we always mean a connected manifold. All manifolds, bundles, their sections and subbundles, connections and mappings, including bundle morphisms, are assumed to be C^∞-differentiable, while a bundle morphism, by definition, operates between two bundles with the same base manifold, and acts by identity on the base. For the exterior product of 1-forms ξ,η and a 2-form α on a manifold, the exterior derivative of ξ, and any tangent vector fields u,v,w, we have

$$\begin{array}{ll} \text{a)} & (\xi\wedge\eta)(u,v) \,=\, \xi(u)\eta(v) \,-\, \eta(u)\xi(v)\,,\\ \text{b)} & (\xi\wedge\alpha)(u,v,w) \,=\, \xi(u)\alpha(v,w) \,+\, \xi(v)\alpha(w,u) + \xi(w)\alpha(u,v)\,,\\ \text{c)} & (d\xi)(u,v) \,=\, d_u[\xi(v)] \,-\, d_v[\xi(u)] \,-\, \xi([u,v])\,. \end{array} \tag{2.1}$$

Our sign convention about the curvature tensor $R=R^\nabla$ of a connection ∇ in a real or complex vector bundle $\mathcal{V}$ over a manifold $\varSigma$ is

$$R(u,v)\psi \;\;=\;\; \nabla_{\!v}\nabla_{\!u}\psi \,-\, \nabla_{\!u}\nabla_{\!v}\psi \,+\, \nabla_{[u,v]}\psi \tag{2.2}$$

for sections ψ of $\mathcal{V}$ and vector fields u,v tangent to $\varSigma$. We then denote by

$$\text{i)}\;\; R(u,v):\mathcal{V}\to\mathcal{V}, \qquad\quad \text{ii)}\;\; \varOmega(u,v)=\mathrm{tr}\,[R(u,v)]:\varSigma\to\mathbf{K}, \tag{2.3}$$

the bundle morphism sending any ψ to $R(u,v)\psi$, and, respectively, its pointwise trace, $\mathbf{K}$ being the scalar field ($\mathbf{R}$ or $\mathbf{C}$). Thus, $\varOmega$ is a $\mathbf{K}$-valued 2-form on $\varSigma$. If, in addition, $\mathcal{V}$ is a line bundle (of fibre dimension 1), then $R(u,v)=\varOmega(u,v)$, that is, the morphism $R(u,v)$ acts via multiplication by the $\mathbf{K}$-valued function $\varOmega(u,v)$, and we call $\varOmega$ the *curvature form* of ∇.

The torsion tensor $\varTheta$ of a connection ∇ on a manifold $\varSigma$ is characterized by $\varTheta(v,w)=\nabla_{\!v}w-\nabla_{\!w}v-[v,w]$, for vector fields v,w tangent to $\varSigma$. If $\varSigma$ is a surface, $\varTheta$ is completely determined by the *torsion form* θ, which is the 1-form with $\theta(v)=\mathrm{tr}\,\varTheta(v,\,\cdot\,)$. In fact, $\varTheta=\theta\wedge\mathrm{Id}$, that is, $\varTheta(v,w)=\theta(v)w-\theta(w)v$.

For a connection ∇ on a surface, its torsion form θ, and any 1-form ξ,

$$d\xi \;=\; \nabla\xi \;-\; (\nabla\xi)^* \,+\; \theta\wedge\xi \tag{2.4}$$

in the sense that $(d\xi)(u,v)=(\nabla_{\!u}\xi)(v)-(\nabla_{\!v}\xi)(u)+\theta(u)\xi(v)-\theta(v)\xi(u)$ whenever u,v are tangent vector fields. This is clear from (2.1) and the last paragraph.

Remark 2.1. The *determinant bundle* of a real/complex vector bundle $\mathcal{V}$ of fibre dimension m is its highest real/complex exterior power $\det\mathcal{V}=\mathcal{V}^{\wedge m}$. For any connection ∇ in $\mathcal{V}$, the 2-form $\varOmega$ defined by (2.3.ii) is the curvature form of the connection in the line bundle $\det\mathcal{V}$ induced by ∇.

Remark 2.2. In view of the Bianchi identity and Remark 2.1, the form $\varOmega$ in (2.3.ii) is always closed. Its cohomology class $[\varOmega]\in H^2(\varSigma,\mathbf{K})$ does not depend on the choice of the connection ∇. (This is again immediate from Remark 2.1: two connections in the line bundle $\det\mathcal{V}$ differ by a $\mathbf{K}$-valued 1-form ξ on $\varSigma$, and so, by (2.2) and (2.1.c), their curvature forms differ by $-d\xi$.) Specifically, $[\varOmega]$ equals 2π times $c_1(\mathcal{V})$ when $\mathbf{K}=\mathbf{C}$, cf. [12, Vol. II, p. 311]. On the other hand, $[\varOmega]=0$ in $H^2(\varSigma,\mathbf{K})$ when $\mathbf{K}=\mathbf{R}$, since, choosing a connection ∇ compatible with a Riemannian fibre metric in $\mathcal{V}$, we get $\varOmega=0$.

3. Projectively flat connections

Let ∇ be a connection in a real/complex vector bundle $\mathcal{V}$ over a real manifold Σ. Following Li, Yau and Zheng [15], we call ∇ *projectively flat* if its curvature tensor R equals $\rho\otimes\mathrm{Id}$ for some 2-form ρ on Σ, in the sense that $R(u,v)\psi=\rho(u,v)\psi$ for all sections ψ of $\mathcal{V}$ and vector fields u,v tangent to M. See also Section 10.

This meaning of projective flatness is quite different from what the term traditionally refers to in the case of connections in the tangent bundle [24, p. 915].

Remark 3.1. For a projectively flat connection, a 2-form ρ with $R=\rho\otimes\mathrm{Id}$ is a constant multiple of the form Ω given by (2.3.ii). Thus, ρ is closed, and, in the case of a real vector bundle, $[\rho]=0$ in $H^2(\Sigma,\mathbf{R})$ according to Remark 2.2.

In the following lemma, the notation $\mathrm{D}=\nabla+\xi\otimes\mathrm{Id}$ means that $\mathrm{D}_v\psi=\nabla_v\psi+\xi(v)\psi$ whenever v is a vector field tangent to Σ and ψ is a section of $\mathcal{V}$. By (2.2) and (2.1.c), the curvature tensors of such connections are related by

$$R^{\mathrm{D}}(u,v)\psi \;=\; R^{\nabla}(u,v)\psi \;-\; [(d\xi)(u,v)]\psi. \tag{3.1}$$

Lemma 3.2. *For a real/complex vector bundle $\mathcal{V}$ over a manifold, the assignment*

$$(\nabla,\xi)\;\mapsto\;(\mathrm{D},\xi),\quad \text{where }\; \mathrm{D}=\nabla+\,\xi\otimes\mathrm{Id} \tag{3.2}$$

defines a bijective correspondence between the set of all pairs (∇,ξ) in which ∇ is a projectively flat connection in $\mathcal{V}$ and ξ is a 1-form on the base manifold such that the curvature tensor of ∇, equals $\rho\otimes\mathrm{Id}$ for $\rho=d\xi$, and the set of all pairs (D,ξ) consisting of a flat connection D in $\mathcal{V}$ and a 1-form ξ on the base.

This is obvious from (3.1). Using Lemma 3.2 and Remark 3.1 we now obtain the following conclusion.

Corollary 3.3. *A real vector bundle over a manifold admits a projectively flat connection if and only if it admits a flat connection.*

4. Skew-symmetry of the Ricci tensor

Lemma 4.1. *A connection ∇ in the tangent bundle $T\Sigma$ of a real surface Σ is projectively flat in the sense of Section 3 if and only if ∇ has skew-symmetric Ricci tensor, and then $R=\rho\otimes\mathrm{Id}$ for the Ricci tensor ρ of ∇.*

Proof. If $R=\rho\otimes\mathrm{Id}$ for *some* 2-form ρ, then the Ricci tensor of ∇ is skew-symmetric, since it equals ρ. Conversely, let the Ricci tensor ρ be skew-symmetric. As the discussion is local and $\dim\Sigma=2$, we may assume that Σ is orientable and choose a 2-form ζ on Σ without zeros. Thus, $R=\zeta\otimes A$ for some bundle morphism $A:T\Sigma\to T\Sigma$. Skew-symmetry of ρ now gives $0=\rho(u,u)=\zeta(u,Au)$ for every vector field u, so that every nonzero vector tangent to Σ at any point y is an eigenvector of A_y, and, consequently, A_y is a multiple of Id, as required. □

We have the following obvious consequence of Lemmas 3.2 and 4.1.

Corollary 4.2. *Given a surface Σ, the assignment $(\nabla,\xi)\mapsto(\mathrm{D},\xi)$, where $\mathrm{D}=\nabla+\xi\otimes\mathrm{Id}$, defines a bijective correspondence between the set of all pairs (∇,ξ) consisting of a connection ∇ on Σ along with a 1-form ξ such that $d\xi$ equals the Ricci tensor of ∇, and the set of all pairs (D,ξ) consisting of any flat connection D on Σ and any 1-form ξ on Σ.*

Remark 4.3. In general, if connections ∇ and D on a surface are related by $\mathrm{D}=\nabla+\xi\otimes\mathrm{Id}$, with a 1-form ξ, then, obviously, $\tau=\theta+\xi$, for the torsion 1-forms θ of ∇ and τ of D, defined as in Section 2.

5. The case of closed surfaces

The next result generalizes a theorem of Blažić and Bokan [2], mentioned in the Introduction.

Theorem 5.1. *A closed surface admitting a connection with skew-symmetric Ricci tensor ρ is diffeomorphic to T^2 or the Klein bottle, and the 2-form ρ is exact.*

Proof. Exactness of ρ is a consequence of Remark 3.1. Thus, in view of Lemma 4.1 and Corollary 3.3, Σ admits a flat connection. Our assertion is now immediate from a result of Milnor [17]. □

Note that, being exact, ρ in Theorem 5.1 must vanish somewhere: if it did not, it would distinguish an orientation of the surface, for which the oriented integral of the positive form ρ would be positive, thus contradicting the exactness of ρ via the Stokes theorem.

Blažić and Bokan [2] exhibited a non-flat torsionfree connection ∇ with skew-symmetric Ricci tensor on the torus T^2, which admits a ∇-parallel almost complex structure, and belongs to a family constructed by Simon [23, p. 322].

Theorem 5.1 and Corollary 4.2 yield the following description of all connections with skew-symmetric Ricci tensor on closed surfaces:

Theorem 5.2. *Let Σ be diffeomorphic to T^2 or the Klein bottle. If D is any flat connection on Σ, and ξ is a 1-form on Σ, then the connection $\nabla=\mathrm{D}-\xi\otimes\mathrm{Id}$ on Σ has skew-symmetric Ricci tensor. Conversely, every connection with skew-symmetric Ricci tensor on Σ equals $\mathrm{D}-\xi\otimes\mathrm{Id}$ for some such D and ξ.*

6. Wong's theorem

Corollary 4.2 leads to an obvious coordinate formula for connections ∇ with skew-symmetric Ricci tensor on surfaces, which produces all local-equivalence classes of such ∇. Specifically, one needs to provide a flat connection D along with a 1-form ξ, and then set $\nabla=\mathrm{D}-\xi\otimes\mathrm{Id}$. In a fixed coordinate system, D can be introduced by prescribing a basis of u,v of D-parallel vector fields, that is, four arbitrary functions subject just to one determinant condition (linear independence of u and v), while ξ amounts to two more arbitrary functions.

Using as many as six arbitrary functions is redundant, and their number is easily reduced. For instance, requiring u to be the first coordinate vector field ∂_1 leaves us with just four arbitrary functions (the first two of the six now being the constants 1 and 0). Another way of replacing six arbitrary functions with four consists in choosing u as well as v to be a product of a positive function and a coordinate vector field. In fact, whenever vector fields u, v on a surface are linearly independent at each point, there exist, locally, coordinates y^1, y^2 with $u = e^\chi \partial_1$, $v = e^\beta \partial_2$ for some functions β, χ. Namely, the distributions spanned by u and v, being one-dimensional, are integrable, and so their leaves are, locally, the level curves of some functions y^1, y^2 without critical points, which means that u, v are functional multiples of ∂_1 and ∂_2, while positivity of the factor functions e^χ and e^β is achieved by adjusting the signs of y^1 and y^2, if necessary.

It is this last approach that allows us to simplify Wong's result [26, Theorem 4.2], by dropping the assumption that $\rho \neq 0$, and reducing the number of separate coordinate expressions from three to one:

Theorem 6.1. *For a torsionfree connection ∇ on a surface Σ, the Ricci tensor ρ of ∇ is skew-symmetric if and only if every point of Σ has a neighborhood U with coordinates y^1, y^2 in which the component functions of ∇ are $\Gamma_{11}^1 = -\partial_1\varphi$, $\Gamma_{22}^2 = \partial_2\varphi$ for some function φ, and $\Gamma_{jk}^l = 0$ unless $j = k = l$.*

Proof. If ρ is skew-symmetric, we may choose, locally, a 1-form ξ with $d\xi = \rho$ and linearly independent vector fields u, v that are D-parallel, for the flat connection $\mathrm{D} = \nabla + \xi \otimes \mathrm{Id}$ (see Corollary 4.2), and then pick local coordinates y^1, y^2 such that $u = e^\chi \partial_1$, $v = e^\beta \partial_2$ for some functions β, χ, as described above. Since ∇ is torsionfree, ξ is the torsion 1-form of D, cf. Remark 4.3. Thus, $\Theta = \xi \wedge \mathrm{Id}$ is the torsion tensor of D, and so $\xi(v)u - \xi(u)v = \Theta(v, u) = [u, v]$, while $[u, v] = [e^\chi \partial_1, e^\beta \partial_2]$, so that the functions $\xi_j = \xi(\partial_j)$ are given by $\xi_1 = -\partial_1\beta$, $\xi_2 = -\partial_2\chi$. As $\mathrm{D}u = \mathrm{D}v = 0$ and $\nabla = \mathrm{D} - \xi \otimes \mathrm{Id}$, we get $\nabla u = -\xi \otimes u$, $\nabla v = -\xi \otimes v$, which, for $\partial_1 = e^{-\chi} u$, $\partial_2 = e^{-\beta} v$, yields $\nabla \partial_1 = -(\xi + d\chi) \otimes \partial_1$, $\nabla \partial_2 = -(\xi + d\beta) \otimes \partial_2$. Since $\xi_1 = -\partial_1\beta$ and $\xi_2 = -\partial_2\chi$, setting $\varphi = \chi - \beta$, we obtain the required expressions for Γ_{jk}^l.

Conversely, for a connection ∇ with Γ_{jk}^l as in the statement of the theorem, using (2.2) with u, v replaced by ∂_1, ∂_2, and $\psi = \partial_1$ or $\psi = \partial_2$, we see that $R = \rho \otimes \mathrm{Id}$ with $\rho_{12} = -\rho_{21} = -\partial_1\partial_2\varphi$, which completes the proof. □

7. Left-invariant connections on Lie groups

We say that a connection ∇ on a manifold Σ is *locally equivalent* to a connection ∇' on a manifold Σ' if every point of Σ has a connected neighborhood U with an affine diffeomorphism $U \to U'$ onto an open subset U' of Σ'.

Here and in the next two sections we describe all local-equivalence types of locally homogeneous connections with skew-symmetric Ricci tensor on surfaces. They all turn out to be represented by left-invariant connections on Lie groups, which is why we discuss the Lie-group case first.

Example 7.1. Given an area form $\alpha \in [\Pi^*]^{\wedge 2} \smallsetminus \{0\}$ in a two-dimensional real vector space Π and a one-dimensional vector subspace Λ of Π, we denote by D the standard (translation-invariant) flat torsionfree connection on Π, by ξ be the 1-form on Π given by $\xi_y(v) = \alpha(y, v)$, for $y \in \Pi$ and $v \in \Pi = T_y\Pi$, and by $\varSigma$ a fixed side of Λ in Π (that is, a connected component of $\Pi \smallsetminus \Lambda$). Let G be the two-dimensional non-Abelian connected subgroup of the group $\mathrm{SL}(\Pi)$, formed by those elements of $\mathrm{SL}(\Pi)$ which leave Λ invariant and operate in Λ via multiplication by positive scalars. Since G acts on $\varSigma$ freely and transitively, choosing a point in $\varSigma$ we identify $\varSigma$ with G and treat the action as consisting of the left translations in G. The restrictions of D and ξ to $\varSigma = G$ then are invariant under all left translations, and hence so is the connection $\nabla = \mathrm{D} - \xi \otimes \mathrm{Id}$ on $\varSigma = G$. As $d\xi = 2\alpha$, where α is now treated as a constant (translation-invariant) 2-form on Π, Corollary 4.2 implies that ∇ has the skew-symmetric Ricci tensor $\rho = 2\alpha \neq 0$ and, by Remark 4.3, the torsion 1-form of ∇ is $\theta = -\xi \neq 0$.

We always identify the elements of the Lie algebra $\mathfrak{g}$ of any Lie group G with left-invariant vector fields on G. The symbol $\mathfrak{sl}(\Pi)$ denotes the Lie algebra of traceless endomorphisms of a real vector space Π.

Theorem 7.2. *For any connected Lie group G, the left-invariant connections ∇ on G which are projectively flat in the sense of Section 3 are in a bijective correspondence with pairs (Ψ, f) formed by any Lie-algebra homomorphism $\Psi : \mathfrak{g} \to \mathfrak{sl}(\mathfrak{g})$ and any linear functional $f \in \mathfrak{g}^*$. For $u, v \in \mathfrak{g}$, this correspondence is given by $\nabla_u v = (\Psi u)v + f(u)v$, and ∇ has the Ricci tensor ρ with $\rho(u, v) = f([u, v])$.*

Proof. A left-invariant connection ∇ on G clearly amounts to a linear operator $\mathfrak{g} \ni u \mapsto \nabla_u$ valued in linear endomorphisms of $\mathfrak{g}$. Decomposing ∇_u into a traceless part and a multiple of Id, we obtain the formula $\nabla_u v = (\Psi u)v + f(u)v$ describing a bijective correspondence between left-invariant connections ∇ on G and pairs (Ψ, f), in which $\Psi : \mathfrak{g} \to \mathfrak{sl}(\mathfrak{g})$ is a linear operator, and $f \in \mathfrak{g}^*$. By (2.2), the curvature tensor of ∇ is given by $R(u, v) = \nabla_{[u,v]} + \nabla_v \nabla_u - \nabla_u \nabla_v$, cf. (2.3.i), for $u, v, w \in \mathfrak{g}$, that is, $R(u, v) = f([u, v])\mathrm{Id} + \Psi[u, v] - [\Psi u, \Psi v]$, where the first two occurrences of $[\,,\,]$ represent the Lie-algebra operation in $\mathfrak{g}$, and the last one stands for the commutator in $\mathfrak{sl}(\mathfrak{g})$. On the other hand, projective flatness of ∇ means that $R(u, v) = \rho(u, v)\mathrm{Id}$ for some 2-form ρ (which must then coincide with the Ricci tensor of ∇). Equating the traceless parts of the last two expressions for $R(u, v)$, we see that ∇ is projectively flat if and only if $\Psi[u, v] = [\Psi u, \Psi v]$ for all $u, v \in \mathfrak{g}$, and then $\rho(u, v) = f([u, v])$. This completes the proof. □

Theorem 7.2 leads to an explicit description of all local-equivalence types of left-invariant connections ∇ with skew-symmetric Ricci tensor on two-dimensional Lie groups G. It is convenient to distinguish three cases, based on the rank (dimension of the image) of the Lie-algebra homomorphism Ψ associated with ∇, which assumes the values $0, 1$ and 2.Note that the Lie algebras $\mathfrak{g}$ of the groups G in question represent just two isomorphism types (Abelian and non-Abelian).

First, connections ∇ as above with $\operatorname{rank}\Psi = 0$ (that is, $\Psi = 0$) are, by Theorem 7.2, in a one-to-one correspondence with linear functionals $f \in \mathfrak{g}^*$.

Secondly, those of our connections having $\operatorname{rank}\Psi = 1$ are precisely the connections ∇ of the form $\nabla_u v = q(u)Bv + f(u)v$, for all $u, v \in \mathfrak{g}$, with any fixed $B \in \mathfrak{sl}(\mathfrak{g}) \smallsetminus \{0\}$ and $q, f \in \mathfrak{g}^*$ such that $q \neq 0$ and $\operatorname{Ker} q$ contains the commutant ideal $[\mathfrak{g}, \mathfrak{g}]$. (Note that $[\mathfrak{g}, \mathfrak{g}] = \{0\}$ if $\mathfrak{g}$ is Abelian, and $\dim [\mathfrak{g}, \mathfrak{g}] = 1$ if it is not, while, for q and B which are both nonzero, $u \mapsto q(u)B$ is a Lie-algebra homomorphism $\mathfrak{g} \to \mathfrak{sl}(\mathfrak{g})$ if and only if $[\mathfrak{g}, \mathfrak{g}] \subset \operatorname{Ker} q$.)

Finally, the case $\operatorname{rank}\Psi = 2$ occurs only for non-Abelian $\mathfrak{g}$ (by Theorem A.1(i) in Appendix A). Our connections then have the form $\nabla_u v = (\Psi u)v + f(u)v$, for $u, v \in \mathfrak{g}$, with Ψ explicitly described as follows: $\Psi u = A$ and $\Psi v = B$, where u, v is a fixed basis of $\mathfrak{g}$ with $[u, v] = u$ and $A, B \in \mathfrak{sl}(\mathfrak{g})$ are given by

$$Aw = w', \quad Aw' = 0, \quad Bw = w/2, \quad Bw' = -w'/2, \tag{7.1}$$

in an arbitrary basis w, w' of $\mathfrak{g}$. See Theorem A.1(ii) in Appendix A.

8. Flat locally homogeneous connections

Let the Ricci tensor ρ of a connection ∇ on a surface Σ be skew-symmetric. In the open set U where $\rho \neq 0$, the determinant bundle $[T\Sigma]^{\wedge 2}$ is trivialized by ρ, and so ρ restricted to U is *recurrent* in the sense that $\nabla\rho = \phi \otimes \rho$ for some 1-form ϕ defined just on U. If U is nonempty, we call ϕ the *recurrence form* of ρ. On U one then has

$$d\phi = 2\rho. \tag{8.1}$$

In fact, the local-coordinate form $\rho_{jk,l} = \phi_l \rho_{jk}$ of the recurrence relation, combined with the Ricci identity, gives $(\phi_{l,m} - \phi_{m,l})\rho_{jk} = \rho_{jk,lm} - \rho_{jk,ml} = R_{mlj}{}^s \rho_{sk} + R_{mlk}{}^s \rho_{js} + \Theta^s_{lm} \rho_{jk,s} = (2\rho_{ml} + \theta_l \phi_m - \theta_m \phi_l)\rho_{jk}$ (where θ is the torsion 1-form of ∇), the last equality being immediate as $R_{mlj}{}^s = \rho_{ml}\delta^s_j$ in view of Lemma 4.1, and $\Theta^s_{lm} = \theta_l \delta^s_m - \theta_m \delta^s_l$ (cf. Section 2). Cancelling the factor ρ_{jk} and noting that $\phi_{l,m} - \phi_{m,l} = (d\phi)_{ml} + \theta_l \phi_m - \theta_m \phi_l$ (by (2.4) and (2.1.a)), we obtain (8.1).

García-Río, Kupeli and Vázquez-Lorenzo [11, p. 144] showed that a torsionfree connection with skew-symmetric Ricci tensor on a surface can be locally symmetric only if it is flat. By (8.1), this remains true for connections with torsion.

Lemma 8.1. *For a connection ∇ on an n-dimensional manifold Σ, if*

(i) $\nabla_{e_j} e_k = \Gamma^l_{jk} e_l$ *and* $\Theta(e_j, e_k) = \Theta^l_{jk} e_l$ *for some vector fields $e_1, \dots, e_n$ trivializing $T\Sigma$ and some constants $\Gamma^l_{jk}, \Theta^l_{jk}$, where $j, k, l \in \{1, \dots, n\}$, repeated indices are summed over, and Θ is the torsion tensor of ∇, or*

(ii) ∇ *is flat and has parallel torsion, or*

(iii) $n = 2$ *and ∇ is a part of a locally homogeneous triple (∇, ξ, g) also including a nonzero 1-form ξ and a pseudo-Riemannian metric g on Σ,*

then ∇ is locally equivalent to a left-invariant connection on some Lie group.

Proof. As $[u,v]=\nabla_u v-\nabla_v u-\Theta(u,v)$ for vector fields u,v, our e_j in (i) span a Lie algebra $\mathfrak{h}$ of vector fields, trivializing $T\Sigma$, and so (i) follows from Theorem B.1 in Appendix B. Next, if ∇ is flat and $\nabla\Theta=0$, choosing, locally, ∇-parallel vector fields $e_1,\dots,e_n$ trivializing $T\Sigma$, we see that the assumptions in (i) hold with $\Gamma_{jk}^l=0$, and so (i) implies (ii). Finally, let (∇,ξ,g) be as in (iii), with $n=2$. We denote by u the unique vector field with $g(u,\cdot\,)=\xi$. A second vector field v is defined by $g(v,v)=0$ and $g(u,v)=1$ (if $g(u,u)=0$), or $|g(v,v)|=1$ and $g(u,v)=0$ (if $g(u,u)\neq 0$); in the latter case, v is determined only up to a sign. In both cases, for reasons of naturality, the triple $(\nabla,u,\pm v)$ is locally homogeneous, and so, since, locally, u and $\pm v$ trivialize $T\Sigma$, the covariant derivatives $\nabla_u u,\pm\nabla_u v,\pm\nabla_v u,\nabla_v v$, as well as $\Theta(u,v)$, are linear combinations of u and $\pm v$ with constant coefficients. Fixing, locally, the sign $\pm$, we obtain (iii) from (i) for $e_1=u$ and $e_2=\pm v$. □

Remark 8.2. Every locally homogeneous torsionfree connection on a surface is locally equivalent either to the Levi-Civita connection of the standard sphere, or to a left-invariant connection on a Lie group.[2] This is obvious from Opozda's local classification of such connections [21, Theorem 1.1]: for u,v,U,V as in [21, formula (1.4)], we may apply Lemma 8.1(i) to $e_1=uU$ and $e_2=vV$, while left-invariant pseudo-Riemannian metrics on a two-dimensional non-Abelian Lie group realize all non-flat constant-curvature metric types other than the standard sphere.

Example 8.3. Let the tangent bundle $T\Sigma$ of a simply connected surface Σ be trivialized by vector fields u,v such that $[u,v]=2(u-v)$. The 1-form ξ on Σ with $\xi(u)=\xi(v)=4$ is closed, as $(d\xi)(u,v)=0$ by (2.1.c). For any fixed function $\varphi:\Sigma\to\mathbf{R}$ with $\xi=d\varphi$, the connection ∇ on Σ with

$$\nabla_u u=u,\quad \nabla_u v=-v,\quad \nabla_v u=u+v,\quad \nabla_v v=e^{-\varphi}u-v, \tag{8.2}$$

is locally equivalent to a left-invariant connection on a Lie group, although, as φ is nonconstant, this is not immediate from Lemma 8.1(i) for $e_1=u$ and $e_2=v$.

In fact, the 1-form η on Σ with $\eta(u)=0$ and $\eta(v)=e^{-\varphi/2}$ is closed, as $d_u\varphi=d_v\varphi=4$, and so (2.1.c) gives $(d\eta)(u,v)=0$. Choosing a function γ with $d\gamma=\eta$ and setting $\chi=e^{-\varphi/2}\tanh\gamma$, we have $d_u\chi=-2\chi$ and $d_v\chi=\chi^2-2\chi-e^{-\varphi}$. Now, for $w=v+\chi u$, the definition of ∇ yields $\nabla_u u=u$, $\nabla_w u=u+w$ and $\nabla_u w=\nabla_w w=-w$. Our claim now follows from Lemma 8.1(i) with $e_1=u$ and $e_2=w$, as $\Theta(u,w)=\Theta(u,v)=\nabla_u v-\nabla_v u-[u,v]=-3u$.

We will not use the easily-verified fact that ∇ is flat.

Lemma 8.4. *Let* (D,ξ) *be a locally homogeneous pair consisting of a flat connection* D *on a surface* Σ *and a* 1*-form* ξ *on* Σ*. Suppose that* D *is torsionfree, or* ξ *is the torsion* 1*-form of* D*, cf. Section* 2*. Decomposing the* 2*-tensor* $\mathrm{D}\xi$ *uniquely as* $\mathrm{D}\xi=\alpha+g$*, where* α *is skew-symmetric and* g *is symmetric, we then have* $d\xi=2\alpha$*. Furthermore, unless*

[2] (Added in proof.) The conclusion of Remark 8.2 holds even without requiring the connection in question to be torsionfree, as one easily verifies using Arias-Marco and Kowalski's recent generalization [27] of Opozda's result. See also the footnote in the Introduction.

(a) D *is locally equivalent to a left-invariant connection on a Lie group* G *in such a way that* ξ *corresponds to a left-invariant* 1*-form on* G,

the following three conditions must be satisfied:

(b) ξ *and* $\mathrm{D}\xi$ *are nonzero everywhere,*
(c) $\mathrm{D}\xi = \alpha - c\,\xi\otimes\xi$ *for some constant* c,
(d) *the* 2*-form* $\alpha = d\xi/2$ *is parallel and nonzero.*

Proof. By (2.4), $d\xi = 2\alpha$, as $\theta\wedge\xi = 0$. Suppose that (a) does not hold. This gives (b): the case where D is torsionfree and $\mathrm{D}\xi = 0$ is excluded since it would imply (a) with an Abelian group G, while the case of parallel torsion would lead to (a) in view of Lemma 8.1(ii).

The rank of g, the symmetric part of $\mathrm{D}\xi$, is constant on Σ and equal to $0, 1$ or 2. If (a) fails, we must have rank $g \le 1$, and so, locally, $g = \pm\eta\otimes\eta$ for some 1-form η. In fact, if g were of rank 2, Lemma 8.1(iii) applied to the triple (D,ξ,g) would yield (a) (as $\xi\ne 0$ by (b)). Furthermore, η must be a constant multiple of ξ, for if it were not, Lemma 8.1(iii) for the triple $(\mathrm{D},\xi,g+\xi\otimes\xi)$ would imply (a) again. This gives (c).

Still assuming that (a) is not satisfied, we will now prove (d). Namely, if $\alpha = d\xi/2$ were not parallel, α would be nonzero everywhere due to local homogeneity of (D,ξ). Hence α would be recurrent, in the sense that $\mathrm{D}\alpha = \zeta\otimes\alpha$ for some nonzero 1-form ζ (cf. the lines preceding (8.1)). As D is flat, so is the connection induced by D in the bundle $[T^*\Sigma]^{\wedge 2}$. Thus, locally, $e^{-\chi}\alpha$ is parallel for some function χ, and so $\mathrm{D}\alpha = d\chi\otimes\alpha$. It would now follow that $\zeta = d\chi$ and $d\zeta = 0$. However, since we assumed that $\alpha = d\xi/2$ is *not* parallel, ξ cannot be a constant multiple of ζ, so that Lemma 8.1(iii) applied to $(\mathrm{D},\xi,\xi\otimes\xi+\zeta\otimes\zeta)$ would give (a) as before. Thus, $\alpha = d\xi/2$ is parallel.

To show that $\alpha\ne 0$, suppose, on the contrary, that $\alpha = d\xi/2$ vanishes identically (and (a) does not hold). Choosing, locally, a function φ with $d\varphi = \xi$, we can now rewrite the equality $\mathrm{D}\xi = -c\,\xi\otimes\xi$ (cf. (c)) as $\mathrm{D}\eta = 0$ for $\eta = e^{c\varphi}\xi$. Since D is flat, we may now select, locally, D-parallel vector fields u, v with $\eta(u) = 1$ and $\eta(v) = 0$. Setting $w = e^{c\varphi}u$ we have $\mathrm{D}_v v = \mathrm{D}_w v = \mathrm{D}_v w = 0$ and $\mathrm{D}_w w = cw$, since $\mathrm{D}u = \mathrm{D}v = 0$, while $d\varphi = \xi = e^{-c\varphi}\eta$, so that $d_u\varphi = e^{-c\varphi}$ and $d_v\varphi = 0$. Similarly, as the torsion tensor Θ of D equals 0 (when D is torsionfree), or $\Theta = \xi\wedge\mathrm{Id}$ (when ξ is the torsion 1-form of D), we get $\Theta(v,w) = 0$ or, respectively, $\Theta(v,w) = -\xi(w)v = -\eta(u)v = -v$. Lemma 8.1(i) with $n = 2$, $e_1 = v$ and $e_2 = w$ now yields (a), and the resulting contradiction proves (d). □

Lemma 8.5. *Every flat locally homogeneous connection* ∇ *on a surface* Σ *is locally equivalent to a left-invariant connection on some Lie group.*

Proof. Let θ be the torsion form of ∇ (see Section 2). We assume that conditions (b) – (d) in Lemma 8.4 are satisfied by $\mathrm{D} = \nabla$ and $\xi = \theta$, since otherwise our assertion follows from Lemma 8.4(a). Thus, $\nabla\theta = \alpha - c\,\theta\otimes\theta$ for the nonzero parallel 2-form $\alpha = d\theta/2$ and a constant c.

As $\nabla\alpha = 0$, while $\alpha \neq 0$ and $\theta \neq 0$, on some neighborhood U of any given point of Σ, we may choose ∇-parallel vector fields w, w' such that $\alpha(w, w') = 1$ and $\theta(w)\theta(w') \neq 0$ everywhere in U. Let us now define functions P, Q, φ, vector fields u, v, and 1-forms ζ, η, ξ on U by $P = \theta(w)$, $Q = \theta(w')$, $\varphi = 2\log|PQ|$, $u = Qw - Pw'$, $v = 3(P^{-1}w + Q^{-1}w')/2$, $\zeta = dP$, $\eta = dQ$, and $\xi = d\varphi$. We have

$$3\zeta(w) = P^2, \quad 3\zeta(w') = PQ - 3, \quad 3\eta(w) = PQ + 3, \quad 3\eta(w') = Q^2. \tag{8.3}$$

In fact, the relation $\nabla\theta = \alpha - c\,\theta\otimes\theta$ evaluated on the pairs (w, w), (w, w'), (w', w) and (w', w') yields $\zeta(w) = -cP^2$, $\zeta(w') = -cPQ - 1$, $\eta(w) = -cPQ + 1$, $\eta(w') = -cQ^2$. As $d\zeta = ddP = 0$, (2.1.c) gives $0 = (d\zeta)(w, w') = -(3c + 1)P$, and hence $c = -1/3$, so that the preceding equalities become (8.3). (As $\Theta = \theta\wedge\mathrm{Id}$, cf. Section 2, $[w, w'] = \Theta(w', w) = Qw - Pw'$.) Combining (8.3) with the relations $\xi = d\varphi = 2(Q^{-1}\eta + P^{-1}\zeta)$ and $\nabla w = \nabla w' = 0$, we get $\xi(u) = \xi(v) = 4$ and (8.2) (for our ∇, u, v, φ). Our assertion now follows from Example 8.3. □

9. The general locally homogeneous case

The following theorem, combined with the discussion, at the end of Section 7, of certain left-invariant connections on Lie groups, completes our description of all locally homogeneous connections with skew-symmetric Ricci tensor on surfaces.

Theorem 9.1. *Every locally homogeneous connection ∇ with skew-symmetric Ricci tensor on a surface Σ is locally equivalent to a left-invariant connection on a Lie group.*

Proof. If ∇ is flat, we can use Lemma 8.5. Suppose now that ∇ is not flat. As the Ricci tensor ρ is nonzero at every point (cf. Lemma 4.1), the recurrence form ϕ with (8.1) is defined and nonzero everywhere in Σ. (If ϕ were zero somewhere, it would vanish identically in view of local homogeneity of ∇.) Let $\xi = \phi/2$. By (8.1) and Corollary 4.2, the connection $\mathrm{D} = \nabla + \xi\otimes\mathrm{Id}$ is flat, as well as locally homogeneous (due to naturality of ξ), and has the torsion 1-form $\theta + \xi$, where θ is the torsion 1-form of ∇, cf. Remark 4.3.

If ξ and θ are linearly independent at each point, our claim follows from Lemma 8.1(iii) for the triple (D, ξ, g) with $g = \xi\otimes\xi + \theta\otimes\theta$. Let us now consider the remaining case where $\theta = s\xi$ for some $s \in \mathbf{R}$, and D has the torsion 1-form $\theta + \xi = (s + 1)\xi$.

By Lemma 8.5, D is locally equivalent to a left-invariant connection on some Lie group. If $s \neq -1$, our ξ equals $(s + 1)^{-1}$ times the torsion 1-form of D, and the original connection $\nabla = \mathrm{D} - \xi\otimes\mathrm{Id}$, obtained from D via a natural formula, is also locally equivalent to a left-invariant connection on a Lie group, as required.

Suppose now that $s = -1$. Thus, the flat connection D is torsionfree, and the pair (D, ξ) is locally homogeneous, since so is ∇. We may identify Σ, locally, with an open convex set U in a real affine plane Π, in such a way that D equals, on U, the standard translation-invariant flat torsionfree connection of Π.

As $\nabla = \mathrm{D} - \xi \otimes \mathrm{Id}$, our assertion will follow if we show that the pair (D, ξ) satisfies condition (a) in Lemma 8.4. To this end, let us assume that, on the contrary, (a) in Lemma 8.4 fails to hold. By Lemma 8.4, $\mathrm{D}\xi = \alpha - c\xi \otimes \xi$ for some constant c and a parallel nonzero 2-form α. Since α is parallel, $\alpha = \mathrm{D}\eta$ for the 1-form η given by $\eta_y(v) = \alpha(y - o, v)$, for $y \in U \subset \Pi$ and $v \in T_y\Pi$, where o is any fixed origin in the affine plane Π. Hence $\mathrm{D}(\xi - \eta) = -c\xi \otimes \xi$ is symmetric, that is, $\xi - \eta = d\varphi$ for some function φ, and the 3-tensor $-c\mathrm{D}(\xi \otimes \xi) = \mathrm{DD}d\varphi$ is totally symmetric. It follows now that $c = 0$. Namely, if we had $c \neq 0$, the resulting symmetry of $[\mathrm{D}(\xi \otimes \xi)](u, v, \cdot\,) = \mathrm{D}_u[\xi(v)\xi]$ in the D-parallel vector fields u, v, combined with the equality $\mathrm{D}\xi = \alpha - c\xi \otimes \xi$ and (2.1.b), would give $3\xi \otimes \alpha = \xi \wedge \alpha$, while $\xi \wedge \alpha = 0$ as $\dim \Pi = 2$, and so $\xi \otimes \alpha = 0$, which is impossible, since $\xi \neq 0$ and $\alpha \neq 0$ by Lemma 8.4(b),(d). This contradiction shows that $c = 0$ and $\xi - \eta$ is D-parallel. The way η depends on the origin o shows that a suitable choice of o will yield $\xi = \eta$. We may thus treat Π as a two-dimensional real vector space in which the new origin o is the zero vector, and ξ is defined as in Example 7.1. Let us now choose a one-dimensional vector subspace Λ of Π such that $U \subset \Pi \smallsetminus \Lambda$. (Note that $o \notin U$, as $\xi \neq 0$ everywhere in U, while $\xi_o = 0$.) According to Example 7.1, condition (a) in Lemma 8.4 is actually satisfied, contrary to what we assumed earlier in this paragraph. This new contradiction completes the proof. □

10. Norden's theorems

Given a connection ∇ in a vector bundle $\mathcal{V}$, a vector subbundle $\mathcal{L} \subset \mathcal{V}$ is called ∇-*parallel* when it has the property that, for any vector field v tangent to the base manifold, if ψ is a section of $\mathcal{L}$, so is $\nabla_v\psi$.

Let ∇ be a connection in a vector bundle $\mathcal{V}$ over a manifold Σ. We say that *the projectivization of* ∇ *is flat* if every one-dimensional vector subspace of the fibre $\mathcal{V}_y$ at any point $y \in \Sigma$ is the fibre at y of some ∇-parallel line subbundle of the restriction of $\mathcal{V}$ to a neighborhood of y. This is clearly the same as requiring flatness (integrability of the horizontal distribution) of the connection induced by ∇ in the bundle of real projective spaces obtained by projectivizing $\mathcal{V}$.

Lemma 10.1. *A connection* ∇ *in a real/complex vector bundle* $\mathcal{V}$ *over a manifold* Σ *is projectively flat in the sense of Section* 3 *if and only if the projectivization of* ∇ *is flat.*

Proof. If ∇ is projectively flat, its curvature tensor equals $\rho \otimes \mathrm{Id}$ for some exact 2-form ρ (see Remark 3.1) and we may choose a 1-form ξ with $\rho = d\xi$. By Lemma 3.2, the connection D defined as in (3.2) is flat. Let U be a contractible neighborhood of any given point of Σ. Since any D-parallel nonzero section of $\mathcal{V}$ over U spans a ∇-parallel line subbundle, the projectivization of ∇ is flat.

Conversely, let the projectivization of ∇ be flat. If a section ψ of $\mathcal{V}$ defined on an open set $U \subset \Sigma$ spans a ∇-parallel line subbundle, that is, $\nabla_v\psi = \xi(v)\psi$ for some 1-form ξ on U and all vector fields v on U, (2.2) and (2.1.c) give

$R(u,v)\psi = [(d\xi)(v,u)]\psi$. Thus, for fixed vectors u, v tangent to Σ at a point y, *every* $\psi \in \mathcal{V}_y \smallsetminus \{0\}$ is an eigenvector of the operator $R(u,v) : \mathcal{V}_y \to \mathcal{V}_y$, which, consequently, is a multiple of the identity. Hence ∇ is projectively flat. □

The following immediate consequence of Lemmas 4.1 and 10.1 was first proved by Norden [19], [20, §89, formula (5)] for torsionfree connections.

Theorem 10.2. *For a connection* ∇ *in the tangent bundle of a surface, the projectivization of* ∇ *is flat if and only if the Ricci tensor of* ∇ *is skew-symmetric.*

The next result is also due to Norden [20, §49]:

Theorem 10.3. *If* ∇ *is a torsionfree connection in the tangent bundle of a manifold of dimension* $n > 2$ *and the projectivization of* ∇ *is flat, then* ∇ *itself is flat.*

Proof. Let R be the curvature tensor of ∇, and let u, v, w be tangent vectors at any point. If u, v are linearly dependent, $R(u,v)w = 0$. If they are not, choosing w such that u, v, w are linearly independent, we have $\rho(u,v)w + \rho(v,w)u + \rho(w,u)v = 0$ from the first Bianchi identity for $R = \rho \otimes \mathrm{Id}$ (cf. Lemma 10.1), so that $\rho(u,v) = 0$ and, again, $R(u,v)w = \rho(u,v)w = 0$ for *every* vector w. □

Some more results of Norden's are discussed at the end of the next section.

The conclusion of Theorem 10.3 fails, in general, for connections with torsion: to obtain a counterexample, we set $\nabla = \mathrm{D} - \xi \otimes \mathrm{Id}$, where D is a flat connection in the tangent bundle of a manifold Σ of any dimension, and the 1-form ξ on Σ is not closed. In fact, by Lemmas 10.1 and 3.2, the projectivization of ∇ is flat, while (3.1) shows that ∇ itself is not flat.

Counterexamples as above cannot contradict Theorem 10.3 by producing a *torsionfree* connection ∇ in any dimension $n > 2$. In fact, if the torsion tensor Θ of a flat connection D in $T\Sigma$ equals $\xi \wedge \mathrm{Id}$, for a 1-form ξ (and so $\nabla = \mathrm{D} - \xi \otimes \mathrm{Id}$ is torsionfree), then, expressing $[\,,\,]$ in terms of ∇ and Θ we obtain $[u,v] = \xi(v)u - \xi(u)v$ and $\xi([u,v]) = 0$ for any D-parallel vector fields u, v. The Jacobi identity and (2.1.c) now give $(d\xi)(u,v)w + (d\xi)(v,w)u + (d\xi)(w,u)v = 0$ whenever u, v, w are D-parallel, which, as in the proof of Theorem 10.3, implies that $d\xi = 0$ when $n > 2$.

11. Fractional-linear Lagrangians and first integrals

Among torsionfree connections ∇ on surfaces, those with skew-symmetric Ricci tensor have an interesting characterization in terms of the geodesic flow, discovered by Anderson and Thompson [1, pp. 104–107]. It consists, locally, in the existence of a fractional-linear Lagrangian for which the ∇-geodesics are the Euler-Lagrange trajectories. A related result of Norden [20, §89] establishes the existence of a fractional-linear first integral for the geodesic equation as a consequence of skew-symmetry of the Ricci tensor (cf. also [18]). The definition of a fractional-linear function can be found in Appendix C.

Both results are presented below. First, what Anderson and Thompson proved in [1, pp. 104–107] can be phrased as follows.

Theorem 11.1. *For a torsionfree connection* ∇ *on a surface* Σ*, the Ricci tensor* ρ *of* ∇ *is skew-symmetric if and only if every point in* $T\Sigma \smallsetminus \Sigma$ *has a neighborhood* U *with a fractional-linear Lagrangian* $L : U \to \mathbf{R}$ *such that the solutions of the Euler-Lagrange equations for* L *coincide with those geodesics of* ∇ *which, lifted to* $T\Sigma$*, lie in* U*.*

Proof. Let ρ be skew-symmetric. Locally, in Σ, we may choose a 1-form ξ with $d\xi = \rho$ and D-parallel vector fields v, w trivializing $T\Sigma$, where D is the flat connection given by $\mathrm{D} = \nabla + \xi \otimes \mathrm{Id}$ (see Corollary 4.2). Since the ∇-geodesic equation $\nabla_{\dot y}\dot y = 0$ for a curve $t \mapsto y(t) \in \Sigma$ now amounts to $\mathrm{D}_{\dot y}\dot y = \xi(\dot y)\,\dot y$, while, as ∇ is torsionfree, Remark 4.3 states that the torsion 1-form τ of D coincides with ξ, the required Lagrangian L exists in view of Theorem C.3 in Appendix C.

Conversely, if such a Lagrangian exists, it has, locally, the form $L = \eta/\zeta$ appearing in Theorem C.3, and so, by Theorem C.3, the Euler-Lagrange trajectories for L (that is, the ∇-geodesics) are characterized by $\mathrm{D}_{\dot y}\dot y = \tau(\dot y)\,\dot y$, where D is a flat connection and τ is the torsion 1-form of D. The connections ∇ and $\mathrm{D} - \tau \otimes \mathrm{Id}$ thus have the same geodesics, and are both torsionfree according to Remark 4.3, so that they coincide due to the coordinate form of the geodesic equation. Thus, ∇ has skew-symmetric Ricci tensor by Corollary 4.2. □

The next theorem, proved by Norden [20, §89] in the torsionfree case, remains valid for connections with torsion. The fractional-linear first integral for the geodesic equation, appearing in Norden's original statement, arises as the composite, in which ω, viewed as a function $T\Sigma \to \mathbf{C}$ real-linear on the fibres, is followed by a fractional-linear function defined on an open subset of $\mathbf{C}$.

Theorem 11.2. *Let* ∇ *be a connection on a surface* Σ*. If the Ricci tensor* ρ *of* ∇ *is skew-symmetric, then every point of* Σ *has a neighborhood* U *with a complex-valued* 1*-form* ω *on* U *such that* $\omega_y : T_y\Sigma \to \mathbf{C}$ *is a real-linear isomorphism for each* $y \in U$ *and, for any geodesic* $t \mapsto y(t)$ *of* ∇ *contained in* U*, either* $\omega(\dot y) = 0$ *for all* t*, or* $\omega(\dot y) \neq 0$ *for all* t *and* $\omega(\dot y)/|\omega(\dot y)|$ *is constant as a function of* t*.*

Proof. We may choose, locally, a 1-form ξ with $d\xi = \rho$ and a complex-valued 1-form ω such that $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega$ trivialize $T^*\Sigma$ and are D-parallel, for the flat connection $\mathrm{D} = \nabla + \xi \otimes \mathrm{Id}$ (see Corollary 4.2). The ∇-geodesic equation for a curve $t \mapsto y(t)$ reads $\nabla_{\dot y}\dot y = 0$, that is, $\mathrm{D}_{\dot y}\dot y = \xi(\dot y)\,\dot y$. Applying the D-parallel form ω to both sides, we may rewrite the last equation as $[\omega(\dot y)]^{\cdot} = \xi(\dot y)\,\omega(\dot y)$. Since ξ is real-valued, ω has all the required properties. □

Finally, Norden also showed in [20, §89] that, if ω is a complex-valued 1-form on a surface Σ and, for every $y \in \Sigma$, the real-linear operator $\omega_y : T_y\Sigma \to \mathbf{C}$ is an isomorphism, then the conclusion of Theorem 11.2 holds for *some* torsionfree connection ∇ with skew-symmetric Ricci tensor on Σ. This is immediate since

ω is D-parallel for a unique flat connection D. Setting $\nabla = \mathrm{D} - \tau \otimes \mathrm{Id}$, where τ is the torsion 1-form of D, we easily obtain our assertion using Corollary 4.2, Remark 4.3 and the proof of Theorem 11.2.

12. Walker metrics and projectability

Suppose that $\mathcal{V}$ is a null parallel distribution on a pseudo-Riemannian manifold (M, g). As the parallel distribution $\mathcal{V}^\perp$ is integrable, replacing M with a suitable neighborhood of any given point, we may assume that

the leaves of $\mathcal{V}^\perp$ are all contractible and constitute the fibres of a bundle projection $\pi : M \to \Sigma$ over some manifold Σ. (12.1)

Let a null parallel distribution $\mathcal{V}$ on a pseudo-Riemannian manifold (M, g) satisfy (12.1) and the additional curvature condition

$$R(v, \cdot\,)u = 0 \quad \text{for all sections } v \text{ of } \mathcal{V} \text{ and } u \text{ of } \mathcal{V}^\perp. \tag{12.2}$$

Then, by [8, p. 587, assertions (ii) and (iv) in Section 14],

(a) the requirement that $\pi^*\xi = g(v, \cdot\,)$ defines a natural bijective correspondence between sections v of $\mathcal{V}$ parallel along $\mathcal{V}^\perp$ and sections ξ of $T^*\Sigma$,
(b) there exists a unique torsionfree connection ∇ on Σ such that, for any π-projectable vector field w on M, if v and ξ realize the correspondence in (a), then so do $v' = \overline{\nabla}_w v$ and $\xi' = \nabla_{\pi w}\xi$, where the vector field πw on Σ is the π-image of w, and $\overline{\nabla}$ denotes the Levi-Civita connection of g.

We refer to ∇ as the *projected connection* on Σ, corresponding to g and $\mathcal{V}$.

13. A theorem of Díaz-Ramos, García-Río and Vázquez-Lorenzo

One says that an endomorphism of a pseudo-Euclidean 3-space is of *Petrov type* III if it is self-adjoint and sends some ordered basis (X, Y, Z) to $(0, X, Y)$. We are interested in the case where this endomorphism is the self-dual Weyl tensor of an oriented pseudo-Riemannian four-manifold (M, g) of the neutral signature $(--++)$, acting in the 3-space of self-dual bivectors at a point of M.

In [9, Theorem 3.1(ii.3)] Díaz-Ramos, García-Río and Vázquez-Lorenzo described the local-isometry types of all those curvature-homogeneous self-dual oriented Einstein four-manifolds of the neutral metric signature $(--++)$, which are of Petrov type III, in the sense that so is the self-dual Weyl tensor at each point of the manifold, and admit a two-dimensional null parallel distribution compatible with the orientation. (Compatibility is defined at the end of this section.)

The metrics mentioned above can also be characterized as *the type* III *Jordan-Osserman Walker metrics in dimension four*. See [9, Remark 2.1].

The description in [9, Theorem 3.1(ii.3)] had the form of a local-coordinate expression. We rephrase it below (see Theorem 13.1) using coordinate-free language. The first part of Theorem 13.1 slightly generalizes a result of García-Río, Kupeli, Vázquez-Abal and Vázquez-Lorenzo [10, Theorem 9], which also uses a coordinate-free formula and assumes that, in our notation, $\lambda = 0$.

First we need some definitions. Let $M = T^*\Sigma$ be the total space of the cotangent bundle of a manifold Σ, and let $\pi : T^*\Sigma \to \Sigma$ be the bundle projection. Any connection ∇ on Σ gives rise to the Patterson-Walker *Riemann extension metric* [22], which is the pseudo-Riemannian metric g^∇ on $T^*\Sigma$ defined by requiring that all vertical and all ∇-horizontal vectors be g^∇-null, while $g_x^\nabla(\xi, w) = \xi(d\pi_x w)$ for any $x \in T^*\Sigma = M$, any vertical vector $\xi \in \operatorname{Ker} d\pi_x = T_y^*\Sigma$, with $y = \pi(x)$, and any $w \in T_x M$. Patterson and Walker also studied in [22] metrics of the form $g = g^\nabla + \pi^*\lambda$, where λ is any fixed twice-covariant symmetric tensor field on Σ.

Theorem 13.1. *Let there be given a surface Σ, a torsionfree connection ∇ on Σ such that the Ricci tensor ρ of ∇ is skew-symmetric and nonzero everywhere, and a twice-covariant symmetric tensor field λ on Σ. Then, for a suitable orientation of the four-manifold $M = T^*\Sigma$, the metric $g = g^\nabla + \pi^*\lambda$ on M, with the neutral signature $(--++)$, is Ricci-flat and self-dual of Petrov type* III*, the vertical distribution $\mathcal{V} = \operatorname{Ker} d\pi$ is g-null, g-parallel and compatible with the orientation, while g and $\mathcal{V}$ satisfy the curvature condition* (12.2)*, and the corresponding projected connection on Σ, characterized by* (b) *in Section* 12*, coincides with our original ∇.*

Conversely, if (M, g) is a neutral-signature oriented self-dual Einstein four-manifold of Petrov type III *admitting a two-dimensional null parallel distribution $\mathcal{V}$ compatible with the orientation, then, for every $x \in M$, there exist Σ, ∇, λ as above and a diffeomorphism of a neighborhood of x onto an open subset of $T^*\Sigma$, under which g corresponds to the metric $g^\nabla + \pi^*\lambda$, and $\mathcal{V}$ to the vertical distribution* $\operatorname{Ker} d\pi$.

Proof. In the coordinates y^j, ξ_j for $T^*\Sigma$ arising from a local coordinate system y^j in Σ, if we let the products of differentials stand for symmetric products and Γ_{kl}^j for the components of ∇, then $g = g^\nabla + \pi^*\lambda$ can be expressed as

$$g \,=\, 2d\xi_j\, dy^j + (\lambda_{kl} - 2\xi_j \Gamma_{kl}^j w)\, dy^k\, dy^l. \tag{13.1}$$

Setting $x_1 = \xi_1$, $x_2 = \xi_2$, $x_3 = y^1$, $x_4 = y^2$, $\xi = \lambda_{11}$, $\eta = \lambda_{22}$, $\gamma = \lambda_{12}$, $P = -2\Gamma_{11}^1$, $Q = -2\Gamma_{11}^2$, $S = -2\Gamma_{22}^1$, $T = -2\Gamma_{22}^2$, $U = -2\Gamma_{12}^1$ and $V = -2\Gamma_{12}^2$, one easily sees that (13.1) amounts to formula (2.1) in Díaz-Ramos, García-Río and Vázquez-Lorenzo's paper [9], with (a, b, c) given by formula (3.2) in [9]. Skew-symmetry of ρ is in turn equivalent to condition (3.3) in [9]: the three equations forming (3.3) state that $\rho_{11} = 0$, $\rho_{22} = 0$, and, respectively, $\rho_{12} + \rho_{21} = 0$. Relation (3.4) in [9] is, however, equivalent to *symmetry* of ρ, so that, under our assumptions about ρ, (3.4) is not satisfied at any point. Our claim now follows from [9, Theorem 3.1(ii.3)] and Walker's theorem [25], cf. [7, p. 062504-7]. □

The notion of *compatibility* between a two-dimensional null distribution and the orientation of the underlying four-manifold refers to the following well-known fact (see, e.g., [6, Proposition 37.1(i) on p. 638]): any null plane Π in a pseudo-Euclidean 4-space E of the neutral signature $(--++)$ naturally distinguishes an orientation of E, namely, the one which, for some/any basis u, v of Π, makes the bivector $u \wedge v$ self-dual.

A bivector in E equals $u \wedge v$ for some basis u, v of some null plane if and only if it is nonzero, null, and self-dual or anti-self-dual [6, Lemma 37.8 on p. 645]. For a self-dual oriented Ricci-flat four-manifold (M, g) of the neutral metric signature, the Levi-Civita connection in the bundle of anti-self-dual bivectors is flat, and so, using null parallel anti-self-dual bivectors, we see that, locally, (M, g) admits a whole family, diffeomorphic to the circle, of two-dimensional null parallel distributions which are *not* compatible with the orientation.

Appendix A. Lie subalgebras of $\mathfrak{sl}(2, \mathbf{R})$

In a real vector space Π with $\dim \Pi = 2$, two-dimensional Lie subalgebras $\mathfrak{g}$ of $\mathfrak{sl}(\Pi)$ are in a bijective correspondence with one-dimensional vector subspaces Λ of Π. The correspondence assigns to Λ the set $\mathfrak{g}$ of all $B \in \mathfrak{sl}(\Pi)$ which leave Λ invariant. The commutant ideal $[\mathfrak{g}, \mathfrak{g}]$ then consists of all $B \in \mathfrak{sl}(\Pi)$ with $\Lambda \subset \operatorname{Ker} B$. This is immediate from the following well-known fact.

Theorem A.1. *Let Π be a two-dimensional real vector space.*

(i) *No two-dimensional Lie subalgebra of $\mathfrak{sl}(\Pi)$ is Abelian.*
(ii) *Every two-dimensional Lie subalgebra of $\mathfrak{sl}(\Pi)$ has a basis A, B with $[A, B] = A$, and any such A, B have the form* (7.1) *in some basis w, w' of Π.*

Conversely, if $A, B \in \mathfrak{sl}(\Pi)$ are given by (7.1) *in some basis w, w' of Π, then $[A, B] = A$, and* $\operatorname{span}\{A, B\}$ *is a two-dimensional Lie subalgebra of $\mathfrak{sl}(\Pi)$.*

Proof. A bilinear form $\langle\,,\rangle$ in $\mathfrak{sl}(\Pi)$ defined by $2\langle A, B\rangle = \operatorname{tr} AB$, or, equivalently, $\langle A, A\rangle = -\det A$, has the Lorentzian signature $(-++)$, as one sees using the matrix representation. Also, $[A, B]$ is $\langle\,,\rangle$-orthogonal to A and B, for A, B in $\mathfrak{sl}(\Pi)$, as $\langle [A, B], A\rangle = \langle A, BA\rangle - \langle BA, A\rangle = 0$ (which is nothing else than bi-invariance of the Killing form $\langle\,,\rangle$). Thus, the 3-form μ on $\mathfrak{sl}(\Pi)$ with $2\mu(A, B, C) = \langle [A, B], C\rangle$ is skew-symmetric. Furthermore, $\mathfrak{sl}(\Pi)$ carries a unique orientation such that $\mu(A, B, C) = 1$ for any positive-oriented $(-++)$-orthonormal basis A, B, C of $\mathfrak{sl}(\Pi)$, since this is the case for the basis

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

when $\Pi = \mathbf{R}^2$. As $\langle [A, B], \,\cdot\,\rangle = 2\mu(A, B, \,\cdot\,)$, the commutator operation in $\mathfrak{sl}(\Pi)$ equals twice the vector product in our oriented pseudo-Euclidean 3-space, with the volume form μ. Hence $[A, B] \neq 0$ when A, B are linearly independent: completing them to a basis A, B, C, we get $\langle [A, B], C\rangle = 2\mu(A, B, C) \neq 0$. This proves (i).

By assigning to every two-dimensional (non-Abelian) Lie subalgebra $\mathfrak{g}$ of $\mathfrak{sl}(\Pi)$ its $\langle\,,\rangle$-orthogonal complement $\mathfrak{g}^\perp$, which coincides with $[\mathfrak{g}, \mathfrak{g}]$, we obtain a bijective correspondence between the set of such $\mathfrak{g}$ and the set of all $\langle\,,\rangle$-null lines (one-dimensional vector subspaces) in $\mathfrak{sl}(\Pi)$. In fact, if A, B is a basis of $\mathfrak{g}$ and $[A,B] = A$, skew-symmetry of μ shows that A is orthogonal to both A and B and hence spans the null line $\mathfrak{g}^\perp = [\mathfrak{g},\mathfrak{g}]$. Conversely, a vector subspace $\mathfrak{g}$ of $\mathfrak{sl}(\Pi)$ such that $\mathfrak{g}^\perp$ is a null line must be a Lie subalgebra, since $\mathfrak{g}^\perp \subset \mathfrak{g}$, and a basis A, B of $\mathfrak{g}$ with $A \in \mathfrak{g}^\perp$ has $\langle [A,B], A\rangle = 2\mu(A,B,A) = 0$, and so $[A,B] \in \mathfrak{g}^{\perp\perp} = \mathfrak{g}$.

That any linearly independent pair A, B in $\mathfrak{sl}(\Pi)$ with $[A,B] = A$ has the form (7.1) in some basis w, w' of Π can be seen as follows. We have $A \in [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}^\perp$, where $\mathfrak{g} = \mathrm{span}\{A, B\}$, and so A is $\langle\,,\rangle$-null, that is, $\mathrm{tr}\,A = \det A = 0$. In a basis of Π containing an element of $\mathrm{Ker}\,A$, the matrix representing A is therefore triangular, with zeros on the diagonal, so that $A^2 = 0$, while $A \neq 0$. Thus, $A(\Pi) \subset \mathrm{Ker}\,A$ and, as both spaces are one-dimensional, $A(\Pi) = \mathrm{Ker}\,A$. The relation $[A,B] = A$ implies in turn that $\mathrm{Ker}\,A$ invariant under B, and so B has real characteristic roots. Since $\mathrm{tr}\,B = 0$, the two roots must be nonzero, or else we would have $\mathrm{Ker}\,B = \mathrm{Ker}\,A$ and, in a basis containing an element of $\mathrm{Ker}\,A$, the matrices of both A and B would be triangular, with zeros on the diagonal, contradicting the linear independence of A and B. Thus, B is diagonalizable, with some nonzero eigenvalues $\pm c$ such that $\mathrm{Ker}\,A = \mathrm{Ker}\,(B + c)$. Choosing a basis w, w' of Π diagonalizing B with $w' \in \mathrm{Ker}\,A$, we may rescale w so that $Aw = w'$ (since $A(\Pi) = A(\mathrm{Ker}\,(B - c)) = \mathrm{Ker}\,A$). Applying $[A,B] = A$ to w we now get $c = 1/2$, which yields (7.1), proving (b). □

Appendix B. Local Lie-group structures

In this appendix we state and prove Theorem B.1, a well-known result, included here to provide a convenient reference for the proof of Lemma 8.1(i).

Given a real/complex vector space $\mathfrak{h}$ of sections of a real/complex vector bundle $\mathcal{V}$ over a manifold Σ, we will say that $\mathfrak{h}$ *trivializes* $\mathcal{V}$ if, for every $y \in \Sigma$, the evaluation operator $\psi \mapsto \psi_y$ is an isomorphism $\mathfrak{h} \to \mathcal{V}_y$. This amounts to requiring that $\dim \mathfrak{h}$ coincide with the fibre dimension of $\mathcal{V}$ and each $v \in \mathfrak{h}$ be either identically zero, or nonzero at every point of Σ. In other words, some (or any) basis of $\mathfrak{h}$ should form a trivialization of $\mathcal{V}$.

Theorem B.1. *Let a Lie algebra $\mathfrak{h}$ of vector fields on a simply connected manifold Σ trivialize its tangent bundle $T\Sigma$, and let $\Psi : \mathfrak{h} \to \mathfrak{g}$ be any Lie-algebra isomorphism between $\mathfrak{h}$ and the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on a Lie group G. Then there exists a mapping $F : \Sigma \to G$ such that every $v \in \mathfrak{h}$ is F-projectable onto Ψv. Any such mapping F is, locally, a diffeomorphism, and Ψ determines F uniquely up to compositions with left translations in G.*

Proof. Given $(y,z) \in \Sigma \times G$, let $K_{y,z} : T_y\Sigma \to T_{(y,z)}(\Sigma \times G) = T_y\Sigma \times T_zG$ be the linear operator with $K_{y,z}u = (u, (\Psi u')_z)$ for $u' \in \mathfrak{h}$ characterized by $u'_y = u \in T_y\Sigma$. Since $\Psi u'$ is left-invariant, the formula $\mathcal{H}_{(y,z)} = K_{y,z}(T_y\Sigma)$ defines a vector subbundle of $T(\Sigma\times G)$, invariant under the left action of G on $\Sigma\times G$. Thus,

$\mathcal{H}$ is (the horizontal distribution of) a G-connection in the trivial G-principal bundle over Σ with the total space $\Sigma \times G$.

The distribution $\mathcal{H}$ on $\Sigma \times G$ is integrable, that is, our G-connection is flat. In other words, the $\mathcal{H}$-horizontal lift operation $v \mapsto \tilde{v}$, applied to vector fields v, w on Σ is a Lie-algebra homomorphism. In fact, $\tilde{v}_{(y,z)} = (v_y, (\Psi v')_z)$, with $v' \in \mathfrak{h}$ such that $v'_y = v_y$. Choosing in $\mathfrak{h}$ a basis e_j, $j = 1, \ldots, n$, we have $v = v^j e_j$, $w = w^j e_j$, $[e_j, e_k] = c^l_{jk} e_l$ and $[\Psi e_j, \Psi e_k] = c^l_{jk} \Psi e_l$ for some real numbers c^l_{jk} and functions v^j, w^j. (The indices $j, k, l = 1, \ldots, n$, if repeated, are summed over.) Thus, $\tilde{v} = (v, v^j \Psi e_j)$, that is, $\tilde{v}_{(y,z)} = (v_y, v^j(y)(\Psi e_j)_z)$, and similarly for w. Hence $[\tilde{v}, \tilde{w}] = ([v,w], (d_v w^l - d_w v^l + v^j w^k c^l_{jk}) \Psi e_l)$, as required: namely, $[v,w] = [v^j e_j, w^k e_k] = (d_v w^l - d_w v^l + v^j w^k c^l_{jk}) e_l$, and so $[v,w]^l = d_v w^l - d_w v^l + v^j w^k c^l_{jk}$.

Therefore, as Σ is simply connected, $\Sigma \times G$ is the disjoint union of the leaves of $\mathcal{H}$, and the projection $\pi : \Sigma \times G \to \Sigma$ maps each leaf N diffeomorphically onto Σ (cf. [12, Vol. I, Corollary 9.2, p. 92]). On the other hand, one easily sees that a mapping F has the properties claimed in our assertion if and only if $d\Xi_y = K_{y,F(y)}$ for all $y \in \Sigma$, where $\Xi : \Sigma \to \Sigma \times G$ is given by $\Xi(y) = (y, F(y))$. Equivalently, Ξ is required to be an $\mathcal{H}$-horizontal section of the G-bundle $\Sigma \times G$, that is, the inverse diffeomorphism $\Sigma \to N$ of $\pi : N \to \Sigma$ for some leaf N of $\mathcal{H}$. The existence of F and its uniqueness up to left translations are now immediate, while such F is, locally, a diffeomorphism in view of the inverse maping theorem. This completes the proof. $\square$

Appendix C. Lagrangians and Hamiltonians

A more detailed exposition of the topics oulined here can be found in [16].

We use the same symbol $\mathcal{V}$, for the total space of a vector bundle $\mathcal{V}$ over a manifold Σ as for the bundle itself, identifying each fibre $\mathcal{V}_y$, $y \in \Sigma$, with the submanifold $\pi^{-1}(y)$ of $\mathcal{V}$, where $\pi : \mathcal{V} \to \Sigma$ is the bundle projection. (Thus, $T\Sigma$ and $T^*\Sigma$ are manifolds.) As a set, $\mathcal{V} = \{(y, \psi) : y \in \Sigma, \;\; \psi \in \mathcal{V}_y\}$.

The identity mapping $\Pi \to \Pi$ in a real vector space Π with $\dim \Pi < \infty$, treated as a vector field on Π, is called the *radial vector field* on Π. On the total space $\mathcal{V}$ of any vector bundle over a manifold Σ we have the *radial vector field*, denoted here by $\mathbf{x}$, which is vertical (tangent to the fibres) and, restricted to each fibre of $\mathcal{V}$, coincides with the radial field on the fibre.

By a *Lagrangian* $L : U \to \mathbf{R}$, or, respectively, a *Hamiltonian* $H : U_* \to \mathbf{R}$ in a manifold Σ one means a function on a nonempty open set $U \subset T\Sigma$ or $U_* \subset T\Sigma$. The *Legendre mapping* $U \to T^*\Sigma$, or $U_* \to T\Sigma$, associated with L or H, is defined by requiring that, for each $y \in \Sigma$, it send any $v \in U \cap T_y\Sigma$ or $\xi \in U_* \cap T^*_y\Sigma$ to the differential of $L : U \cap T_y\Sigma \to \mathbf{R}$ (or, of $H : U_* \cap T_y\Sigma \to \mathbf{R}$) at v (or at ξ), which is an element of $T^*_v(U \cap T_y\Sigma) = T^*_y\Sigma \subset T^*\Sigma$ or, respectively, of $T^*_\xi(U_* \cap T^*_y\Sigma) = T_y\Sigma \subset T\Sigma$. We call such a Lagrangian $L : U \to \mathbf{R}$ or Hamiltonian $H : U_* \to \mathbf{R}$ in Σ *nonsingular* if the associated Legendre mapping is a diffeomorphism $U \to U_*$, or $U \to U_*$ (then referred to as the Legendre *transformation*), for some open set $U_* \subset T^*\Sigma$ or, respectively, $U \subset T\Sigma$. Nonsingular Lagrangians L in Σ are in a natural bijective correspondence with nonsingular Hamiltonians H in Σ. Namely, if $L : U \to \mathbf{R}$ is nonsingular, we define $H : U \to \mathbf{R}$ by $H = d_{\mathbf{x}}L - L$, for the radial vector field $\mathbf{x}$ mentioned above, and then use the Legendre transformation to identify U with U_*, so that H becomes a function $U_* \to \mathbf{R}$. A nonsingular Hamiltonian $H : U_* \to \mathbf{R}$ similarly gives rise to $L : U_* \to \mathbf{R}$ with $L = d_{\mathbf{x}}H - H$ that may be viewed as a function $L : U \to \mathbf{R}$. We will write $L \leftrightarrow H$ if L and H correspond to each other under the assignments $L \mapsto H$ and $H \mapsto L$ (easily seen to be each other's inverses).

A Lagrangian $L : U \to \mathbf{R}$ and a Hamiltonian $H : U_* \to \mathbf{R}$ in Σ both give rise to *equations of motion.* For L these are the *Euler-Lagrange equations*, imposed on curves $t \mapsto y(t) \in \Sigma$ the velocity of which, viewed as a curve $t \mapsto v(t) \in T\Sigma$, lies entirely in U, while H leads to *Hamilton's equations*, imposed on curves $t \mapsto (y(t), \xi(t)) \in U_*$. In the coordinates y^j, v^j for $T\Sigma$ (or, y^j, ξ_j for $T^*\Sigma$), induced by a local coordinate system y^j in Σ, the former read $[\partial L/\partial v^j]^{\cdot} = \partial L/\partial y^j$, and the latter $\dot y^j = \partial H/\partial \xi_j$, $\dot\xi_j = -\partial H/\partial y^j$, with $(\,)^{\cdot} = d/dt$. Both systems of equations can be rephrased in coordinate-free terms: the former characterizes curves parametrized by closed intervals $[a, b]$ which are fixed-ends critical points of the *action functional* given by $\int_a^b L(v(t))\,dt$, while the latter describes the integral curves of the unique vector field X_H on U_* with $\sigma(X_H, \cdot) = dH$, where σ is the symplectic form on $T^*\Sigma$ (see Remark C.2 below). If such L and H are both nonsingular and $L \leftrightarrow H$, the Legendre transformation maps the set of solutions of the Euler-Lagrange equations for L bijectively onto the set of solutions of Hamilton's equations for H.

By a *fractional-linear function* in a two-dimensional real vector space Π we mean any rational function of the form η/ζ, defined on a nonempty open subset of $\Pi \smallsetminus \operatorname{Ker}\zeta$, where $\zeta, \eta \in \Pi^*$ are linearly independent functionals. Similarly, given a real vector bundle $\mathcal{P}$ of fibre dimension 2 over a manifold, a function $U \to \mathbf{R}$ on an open set U in the total space $\mathcal{P}$ will be called *fractional-linear* if its restriction to every nonempty intersection $U \cap \mathcal{P}_y$, for $y \in \Sigma$, is fractional-linear.

Remark C.1. For Π, ζ, η as above, $d(\eta/\zeta) = \zeta^{-2}(\zeta d\eta - \eta d\zeta)$ is easily verified to be a diffeomorphism $\Pi \smallsetminus \operatorname{Ker}\zeta \to \Pi^* \smallsetminus \operatorname{Ker} w$ with the inverse diffeomorphism $d(v/w)$, where v, w is the basis of Π dual to ζ, η. Note that v/w then is a fractional-linear function $\Pi^* \smallsetminus \operatorname{Ker} w \to \mathbf{R}$.

Remark C.2. The total space $T^*\Sigma$ of the cotangent bundle of any manifold Σ carries the symplectic form $\sigma = d\kappa$, where κ is the *canonical* 1*-form*, defined by $\kappa_\xi(u) = \xi(d\pi_\xi u)$ for any $\xi \in T^*\Sigma$ and $u \in T_\xi(T^*\Sigma)$ (so that, at the same time, $\xi \in T^*_y\Sigma$ for $y = \pi(\xi)$). In coordinates y^j, ξ_j as above, $\kappa = \xi_j\, dy^j$ and $\sigma = d\xi_j \wedge dy^j$.

Theorem C.3. *Given vector fields v, w trivializing the tangent bundle $T\Sigma$ of a surface Σ, let us define a Lagrangian $L : U \to \mathbf{R}$ and Hamiltonian $H : U_* \to \mathbf{R}$*

in Σ, both fractional-linear, by $L = \eta/\zeta$ and $H = v/w$, where ζ, η are the 1*-forms dual to v, w at each point, treated as functions $T\Sigma \to \mathbf{R}$ linear on each fibre, and v, w are similarly viewed as functions $T^*\Sigma \to \mathbf{R}$ linear on the fibres, while $U = T\Sigma \smallsetminus \operatorname{Ker}\zeta$ and $U_* = T^*\Sigma \smallsetminus \operatorname{Ker} w$ are the complements in $T\Sigma$ and $T^*\Sigma$ of the total spaces of the line subbundles $\operatorname{Ker}\zeta$ and $\operatorname{Ker} w$. Then L, H are both nonsingular, $L \leftrightarrow H$ under the Legendre transformation, and the solutions of the Euler-Lagrange equations for L are precisely the curves $t \mapsto y(t) \in \Sigma$ with*

$$\mathrm{D}_{\dot y}\dot y \;=\; \tau(\dot y)\,\dot y\,, \tag{C.1}$$

where D *is the flat connection on Σ such that u, v are* D*-parallel, and τ is the torsion* 1*-form of* D*, cf. Section* 2*.*

Proof. Setting $P = \tau(v)$, $Q = \tau(w)$ for the torsion 1-form τ of D, we get $\tau = P\zeta + Q\eta$, and so (2.4) with $\nabla = \mathrm{D}$, $\xi = \zeta$ or $\xi = \eta$, and $\theta = \tau$ gives $d\zeta = Q\eta\wedge\zeta$, $d\eta = P\zeta\wedge\eta$. Let us identify $T^*\Sigma$ (and $T\Sigma$) with $\Sigma\times\mathbf{R}^2$ with the aid of the diffeomorphism $\Sigma\times\mathbf{R}^2 \to T^*\Sigma$ (or, $\Sigma\times\mathbf{R}^2 \to T\Sigma$) that sends (y, r, s) to $(y, r\zeta_y + s\eta_y)$ (or, (y, a, b) to $(y, av_y + bw_y)$). We use the same symbols for differential forms (including functions) on the factor manifolds Σ and $\mathbf{R}^2$ as for their pullbacks to $T^*\Sigma = \Sigma\times\mathbf{R}^2$ or $T\Sigma = \Sigma\times\mathbf{R}^2$. For instance, P, Q, ζ, η and $\zeta\wedge\eta$ also stand for the pullbacks of these functions/forms from Σ to $\Sigma\times\mathbf{R}^2$. Similarly, the vector fields v, w on Σ are also treated as vector fields on $\Sigma\times\mathbf{R}^2$, tangent to the Σ factor. To avoid confusion, we will refrain from viewing ζ, η (or v, w) as functions on $T\Sigma$ (or $T^*\Sigma$), linear on the fibres, and instead denote those functions by r, s (or, respectively, a, b), which is consistent with our convention, since it means nothing else than treating the coordinate functions r, s or a, b on the $\mathbf{R}^2$ factor as functions on $\Sigma\times\mathbf{R}^2$.

The vector field $Z = av + bw + (aP + bQ)\mathbf{x}$ on $T\Sigma$, with $\mathbf{x}$ denoting the radial vector field, generates the flow of equation (C.1) imposed on curves $t \mapsto y(t) \in \Sigma$. In fact, writing $\dot y = av + bw$, where a, b are functions of t, we have $\tau(\dot y) = aP + bQ$, and so (C.1) amounts to $\dot a = (aP + bQ)a$ and $\dot b = (aP + bQ)b$ (with $\dot y = av + bw$).

Next, in view of Remark C.1, $L = b/a$, $H = r/s$, the Legendre transformation $U \to U_*$ sends (y, a, b) to (y, r, s) with $r = -b/a^2$ and $s = 1/a$, while L, H are both nonsingular and $L \leftrightarrow H$. (Note that $d_{\mathbf{x}}L = 0$ and $d_{\mathbf{x}}H = 0$ due to homogeneity of L and H.) Consequently, the Legendre transformation pushes the radial vector field $\mathbf{x}$ in $T\Sigma$ and the functions $a, b, aP + bQ$ forward onto $-\mathbf{x}$ in $T^*\Sigma$ and the functions $1/s, -r/s^2, \varphi/s^2$, with $\varphi = sP - rQ$. Hence it pushes the vector field $-Z$ (for Z generating the flow of (C.1)) forward onto $s^{-2}(\varphi\mathbf{x} - sv + rw)$ in $T^*\Sigma$. However, $s^{-2}(\varphi\mathbf{x} - sv + rw)$ equals X_H, the unique vector field with $\sigma(X_H, \,\cdot\,) = dH$. Namely, $dH = d(r/s) = s^{-2}(s\,dr - r\,ds)$, while the canonical 1-form κ on $T^*\Sigma = \Sigma\times\mathbf{R}^2$ can be expressed as $\kappa = r\zeta + s\eta$, and so the symplectic form $\sigma = d\kappa$ is given by $\sigma = \varphi\,\zeta\wedge\eta - \zeta\wedge dr - \eta\wedge ds$, with $\varphi = sP - rQ$. Due to the invariance of the solutions of (C.1) under the parameter reversal, the distinction between $-Z$ and Z is of no significance, and our assertion follows. □

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