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Ricci solitons*

1. The Ricci flow

In 1981 Richard Hamilton [20] initiated the study of the equation

$$\frac{d}{dt}\,g(t) \;=\; -2\,\mathrm{Ric}_{g(t)}\,. \tag{1}$$

Equation (1) – the solutions of which are usually said to form the *Ricci flow* – is a condition imposed on an unknown smooth curve $t \mapsto g(t)$ of Riemannian metrics on a fixed manifold M. The condition consists in requiring the curve to have, at every t in its domain interval, the derivative with respect to t equal to -2 times the Ricci tensor of the metric $g(t)$.

Solutions (trajectories) of the Ricci flow are curves $t \mapsto g(t)$ emanating from a given initial metric $g(0)$, and defined on a maximal interval $[0,T)$ of the variable t, where $0 < T \leqslant \infty$.

In local coordinates x^j, $j = 1,\ldots,\dim M$, (1) constitutes a system of nonlinear partial differential equations of parabolic type, imposed on the componente $g_{jk} = g(e_j, e_k)$ of the metrics $g = g(t)$ belonging to our unknown curve. The symbol e_j denotes here the jth coordinate vector field (so that the directional derivative in the direction of e_j coincides with the partial derivative ∂_j with respect to the jth coordinate). The functions g_{jk} depend on t and on the variables x^j. The coordinate version of condition (1) is rather complicated:

$$\frac{\partial g_{jk}}{\partial t} \;=\; -2R_{jk}, \qquad \text{where} \quad R_{jk} = \partial_p \Gamma_{jk}^p - \partial_j \Gamma_{pk}^p + \Gamma_{qp}^p \Gamma_{jk}^q - \Gamma_{jq}^p \Gamma_{pk}^q,$$

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Γ^p_{jk} being the Christoffel symbols of the metric $g = g(t)$, given by $2\Gamma^p_{jk} = g^{pq}(\partial_j g_{kq} + \partial_k g_{jq} - \partial_q g_{jk})$, with g^{jk} standing for the contravariant components of the metric (meaning that, at every point of the coordinate domain, the matrix $[g^{jk}]$ is the inverse of the matrix $[g_{jk}]$). In the above expressions for R_{jk} and $2\Gamma^p_{jk}$ we have used the Einstein summing convention, so that repeated indices are summed over.

2. The Ricci flow in the proof of Poincaré's conjecture

In [20] Hamilton proved the existence and uniqueness of a maximal Ricci flow trajectory with any given initial metric $g(0)$, on every compact manifold. He also attempted to use this fact for proving the three-dimensional Poincaré conjecture.

His proposed outline of such a proof (known as "the Hamilton programme") consisted of a specific series of steps. The last of those steps were only carried out in 2002 by Grigori Perelman [29–31].

At the same time Perelman also proved the much more general *Thurston geometrization conjecture for three-dimensional manifolds.*

A crucial part of Perelman's argument was provided by surgeries, needed when the Ricci flow runs into a singularity in finite time ($T < \infty$). After the surgery the Ricci flow is used again, in a topologically simpler situation.

3. Ricci solitons – "fixed points" of the Ricci flow

A *Ricci soliton* is a Riemannian metric $g = g(0)$ on a manifold M, evolving under the Ricci flow in an *inessential* manner, in the sense that all the stages $g(t)$ coincide with $g(0)$ up to diffeomorphisms and multiplications by positive constants ("rescalings"). In other words, such a metric represents a fixed point of the Ricci flow in the quotient of the space of metrics on M under the equivalence relation just described.

It is clear what the above definition means in the case of compact manifolds, due to the existence and uniqueness of a maximal Ricci flow trajectory with any given initial metric. Without the compactness assumption, by a Ricci soliton one means a Riemannian metric g on a manifold M, for which equation (1) has a solution satisfying the initial condition $g(0) = g$ and constituting an inessential evolution of the metric.

In §5 we will discuss a different characterization of Ricci solitons, having the form of a differential equation.

4. Blow-up limits

Complete noncompact Ricci solitons often arise as by-products of the Ricci flow on compact manifolds – blow-up limits (or, rescaling limits) of the metrics $g(t)$ restricted to suitable open sets, when the variable $t \in [0, T)$ tends to T (defined in §1), while T is finite.

The following result of Perelman [29] (known as the "no-breathers theorem") states that on compact manifolds one can eqivualently characterize Ricci solitons by a condition seemingly much weaker than their original definition:

Theorem 4.1. *If in a trajectory $t \mapsto g(t)$ of the Ricci flow on a compact manifold there exist two distinct values of t such that the corresponding stages of the flow coincide up to a diffeomorphism and rescaling, then the trajectory is a Ricci soliton.*

In dimension 3 this was first proved by Thomas Ivey [22].

5. The Ricci-soliton equation

One easily verifies that a metric g on a manifold M is a Ricci soliton if and only if some vector field w on M satisfies the *Ricci-soliton equation*

$$\pounds_w g + \mathrm{Ric} = \lambda g, \quad \text{where } \lambda \text{ is a constant,} \tag{2}$$

with Ric denoting the Ricci tensor of g, and $\pounds_w g$ the Lie derivative of g along w.

The term *Ricci soliton* is also used for a Riemannian manifold (M, g) satisfying (2) with some w, as well as for a Ricci flow trajectory $t \mapsto g(t)$ in which the initial stage $g(0)$ (or, equivalently, every stage $g(t)$) has the property (2); the field w may here depend on t.

The objects w and λ appearing in (2) will be called the *soliton vector field* and *soliton constant.*

The coordinate version of condition (2) reads

$$w_{j,k} + w_{k,j} + R_{jk} = \lambda g_{jk}.$$

Instead of $w_{j,k}$ one also writes $\nabla_k w_j$. One may express (2) directly in terms of the components g_{jk} of g, and w^j of w, by replacing R_{jk} with the formula in §1, and the sum $w_{j,k} + w_{k,j}$ with $\partial_k w_j + \partial_j w_k - 2\Gamma_{jk}^p w_p$, for $w_j = g_{jk} w^k$.

6. The topics discussed below

Ricci solitons are of obvious interest, due to their close relation with the Ricci flow (§§3–4). Their characterization as Riemannian metrics satisfying a specific system of partial differential equations (§5) suggests applying to their study the methods of geometric analysis that were used previously in similar situations.

The remaining part of this text deals with a few selected cases in which the approach just mentioned leads to better understanding of Ricci solitons on *compact* manifolds. Examples of such Ricci solitons are provided by Einstein metrics (§7), Kähler-Ricci solitons (§8), and some Riemannian products (§9). In contrast with the lowest dimensions, $n = 2$ and $n = 3$, where compact Ricci solitons have long been classified (§7), the case $n \geqslant 4$ is largely a *terra incognita*, as illustrated by the open problem described in §9.

§§11–15 are devoted to stating and proving a result of Perelman which – despite its rather esoteric nature – is a substantial step toward understanding compact Ricci solitons. In §16 we in turn discuss theorems, proved in the years 1957–2004 by eight mathematicians, and together showing that Kähler-Ricci solitons form a natural class of *canonical metrics* on compact complex surfaces with positive or negative first Chern class.

The final part of this article (§§17–26) presents two classes of examples of Ricci solitons on compact manifolds. They are, namely, Page's and Bérard Bergery's Einstein metrics [4, 28], and the Koiso-Cao Kähler metrics [8, 24]. Their description is rather technical and to go through it one has to "roll up the sleeves" – in contrast, for instance, to the case of homogeneous Einstein manifolds with an irreducible isotropy representation, where the discussion and justification is very brief (§7). The constructions in §§17–26 use conformal changes of Kähler metrics, that is, their multiplication by suitable positive functions. Conditions sufficient for such a change to yield a Ricci soliton, introduced in §20, constitute a system of second-order ordinary differential equations with boundary conditions. Known solutions of this system form three families, two of which correspond (see §§23,25) to the two classes of examples mentioned above, while the third one yields nothing new – the Ricci solitons arising in it are isometric to the Koiso-Cao metrics, as shown by Gideon Maschler [27]. A proof of Maschler's result is given in §26.

7. Einstein metrics

The most obvious class of Ricci solitons on compact manifolds is provided by *Einstein metrics*. They are defined to be Riemannian metrics

g satisfying (2) with $w = 0$, that is, the *Einstein condition*

$$\mathrm{Ric} = \lambda g \quad \text{for some constant } \lambda. \tag{3}$$

The soliton constant λ is then called the *Einstein constant.*

In dimensions $n < 4$, every compact Ricci soliton is an Einstein metric; this was proved by Hamilton [21] for $n = 2$ and by Ivey [22] for $n = 3$. For purely algebraic reasons, when $n < 4$, equation (3) implies that the metric g has constant sectional curvature. Locally, up to isometries and rescalings, such low-dimensional manifolds are thus standard spheres, Euclidean spaces, or hyperbolic spaces.

By an *Einstein manifold* one means a Riemannian manifold (M, g) such that g is an Einstein metric.

The Einstein property of the constant-curvature metrics mentioned above (spherical, Euclidean, hyperbolic) also follows for much more general reasons. Namely, *every homogeneous Riemannian manifold* (M, g) *with an irreducible isotropy representation is an Einstein manifold.* The homogeneity assumption means that the isometry group of g acts on M transitively, while *irreducibility* refers here to the group H_x of those isometries keeping a given point $x \in M$ fixed; H_x acts, infinitesimally, on the tangent space T_xM. The Einstein condition is here a trivial consequence of Schur's lemma and the fact that the Ricci tensor – being a natural invariant of the metric – is preserved by all isometries.

Besides the spherical, Euclidean and hyperbolic metrics, the above theorem also applies, for instance, to the canonical (Fubini-Study) metrics on complex projective spaces $\mathbb{C}\mathrm{P}^m$, showing that, consequently, they are Einstein metrics.

8. Kähler-Ricci solitons

First examples of non-Einstein compact Ricci solitons, representing all even dimensions $n \geqslant 4$, were constructed in the early 1990s by Norihito Koiso [24] and (independently) Huai-Dong Cao [8]. All their examples, as well as generalizations of those examples found by other authors [11, 16, 25], are *Kähler-Ricci solitons*, in the sense of being, simultaneously, Ricci solitons and Kähler metrics.

A special case of Kähler-Ricci solitons is provided by *Kähler-Einstein metrics*, that is, Kähler metrics which are also Einstein metrics.

Recall that one of the possible (mutually equivalent) definitions of a Kähler metric can be phrased as follows. It is a metric g on a manifold M such that some fixed linear automorphism J of the tangent bundle TM (that is, some smooth family $x \mapsto J_x$ of linear automorphisms $J_x : T_xM \to T_xM$) satisfies the conditions $J(Jv) = -v$, $g(Jv, Jw) = g(v, w)$ and $\nabla_v(Jw) = J(\nabla_v w)$ for all smooth vector fields v and w, where ∇ denotes the Levi-Civita connection of the metric g.

By a *Kähler metric on a complex manifold* M we in turn mean a Kähler metric, in the above sense, on M (treated as real manifold), for which the automorphism $J : TM \to TM$ with the required properties is the operator of multiplication by i in the tangent spaces (naturally constituting complex vector spaces).

The details of the Koiso-Cao construction will be discussed in §25. For now it should be mentioned that the Koiso-Cao examples are Kähler metrics on compact complex manifolds M_k^m which, for integers m, k with $m > k > 0$, are defined as follows:

M_k^m is the total space of the holomorphic $\mathbb{C}\mathrm{P}^1$ bundle over the projective space $\mathbb{C}\mathrm{P}^{m-1}$, arising as the projective compactification of the k-th tensor power of the tautological line bundle. (∗)

Thus, m is the complex dimension of M_k^m. Note that M_1^m can also be obtained by blowing up a point in $\mathbb{C}\mathrm{P}^m$.

The same complex manifolds M_k^m carry other, no less interesting Ricci solitons, which are conformally-Kähler (though non-Kähler) Einstein metrics. They were constructed by Don N. Page [28] for $m = 2$ and by Lionel Bérard Bergery [4] in dimensions $m \geqslant 3$.

By a *conformally-Kähler metric* on a manifold M we mean a Riemannian metric g on M admitting a positive function $\mu : M \to \mathbb{R}$ such that the product μg is a Kähler metric.

For details of Page's and Bérard Bergery's examples, see §23.

In complex dimensions $m \geqslant 3$, the constructions presented here can be directly generalized – as pointed out by their authors themselves [4, 8, 24] – to a class of compact complex manifolds slightly larger than the family M_k^m described above. Our discussion focuses on the manifolds M_k^m for a practical reason: the definition of the larger class is rather cumbersome. That definition will, however, be introduced in the proof of Lemma 17.1, which constitutes the initial step of the construction, thus allowing the reader to carry out the generalization just mentioned.

9. An open problem

The Riemannian product of two Ricci solitons having the same soliton constant λ is again a Ricci soliton. Using this fact, the Koiso-Cao examples (§8), and Einstein metrics (§7), one easily constructs non-Einstein, non-Kähler compact Ricci solitons of any dimension $n \geqslant 7$.

Gang Tian [34], in various lectures, and Huai-Dong Cao, in the paper [9], raised the following

Question 9.1. Does there exist a compact Ricci soliton which is neither Einstein nor locally Kähler, and is not locally decomposable into a Riemannian product of lower-dimensional manifolds?

If such an example exists, it must have a positive soliton constant; a finite fundamental group; and a scalar curvature which is both nonconstant and positive. The reason is that compact Ricci solitons which fail to satisfy one of the conditions just listed are necessarily Einstein metrics. These four facts were proved, respectively, by Jean-Pierre Bourguignon [6], back in 1974; Xue-Mei Li [26] in 1993 (and, independently, Manuel Fernández-López and Eduardo García-Río [17] in 2004, as well as Zhenlei Zhang [38] in 2007); Daniel H. Friedan [18] in 1985; and Ivey [22] in 1993. The results of Hamilton [21] and Ivey [22] mentioned in §7 also show that such an example would be of dimension $n \geqslant 4$.

10. Some notations and identities

To simplify our discussion, we introduce some symbols. Given a Riemannian manifold (M, g), we let $\mathcal{F}M, \mathcal{X}M, \varOmega M$ and SM denote the vector spaces of all smooth functions $M \to \mathrm{I\!R}$, smooth vector fields on M, smooth 1-forms on M (that is, sections of the cotangent bundle) and, respectively, smooth symmetric 2-tensor fields on M. Examples of the latter are the metric g, its Ricci tensor Ric, and the Hessian ∇df of any function $f \in \mathcal{F}M$, that is, the covariant derivative of its differential df. Besides the gradient $\nabla : \mathcal{F}M \to \mathcal{X}M$ and the differential $d : \mathcal{F}M \to \varOmega M$, interesting linear operators between pairs of these spaces also include the *g-trace* $\mathrm{tr}_g : SM \to \mathcal{F}M$, as well as the *g-divergence* $\delta : SM \to \varOmega M$, the *interior product* $\imath_v : SM \to \varOmega M$ by any $v \in \mathcal{X}M$, and the *g-Laplacian* $\Delta : \mathcal{F}M \to \mathcal{F}M$, characterized by $\mathrm{tr}_g b = \mathrm{tr}\, B$, if $b \in SM$ and B is the linear endomorphism of the tangent bundle TM such that

$$g(Bv, w) = b(v, w) \quad \text{for} \quad v, w \in \mathcal{X}M, \tag{4}$$

and $(\delta b)_j = g^{pq}\nabla_p b_{pj}$, $\imath_v b = b(v, \cdot)$ and $\Delta f = \mathrm{tr}_g \nabla df$.

The *g-traceless part* of a symmetric 2-tensor field $b \in SM$ is, by definition, the tensor field $\{b\}_0 \in SM$ given by

$$\{b\}_0 = b - (\mathrm{tr}_g b) g/n, \quad \text{where} \quad n = \dim M. \tag{5}$$

The function $\mathrm{s} = \mathrm{tr}_g\mathrm{Ric}$ is called the *scalar curvature* of the metric g. The well-known identities (see, e.g., [12, formulae (2.4) and (2.9)])

$$\text{a)}\ \ 2\delta\mathrm{Ric} = d\mathrm{s}, \qquad \text{b)}\ \ \delta b = dY + \imath_v\mathrm{Ric}, \qquad \text{c)}\ \ 2\imath_v b = dQ \tag{6}$$

hold for any function $f \in \mathcal{F}M$, its gradient $v = \nabla f$, Hessian $b = \nabla df$, its Laplacian $Y = \Delta f$, and $Q = g(v,v)$. We will also use the *nonlinear* differential operator $\mathcal{R} : \mathcal{F}M \to \mathcal{F}M$, given, in any Riemannian manifold (M,g), by

$$\mathcal{R}f \;=\; \Delta f \;-\; |\nabla f|^2/2. \tag{7}$$

One says that a symmetric 2-tensor field $b \in \mathcal{S}M$ on a Kähler manifold (M,g) is *Hermitian* if the endomorphism $B : TM \to TM$ characterized by (4) above is $\mathbb{C}$-linear (that is, commutes with J). This is equivalent to skew-symmetry of the 2-tensor field $b(J\,\cdot\,,\,\cdot\,)$. For every $x \in M$ the operator $B_x : T_xM \to T_xM$ must then be diagonalizable and its eigenvalues have even multiplicities over $\mathbb{R}$. In particular,

$$g \ \text{ and } \ \mathrm{Ric} \ \text{ are always Hermitian.} \tag{8}$$

Below – and in the sequel – the symbols t and T will be used in a way which has nothing to do with their meanings in §§1–5.

For functions $t, \chi : M \to \mathbb{R}$ on a manifold M, we will call χ a *smooth function of* t if t is nonconstant and $\chi = G \circ t$, where G is a smooth function on the interval $\mathit{\Lambda} = t(M)$, that is, on the range of t. Writing $(\ )^{\boldsymbol{\cdot}} = d/dt$, we will form the first and second derivatives $\dot\chi = \dot G \circ t$, $\ddot\chi = \ddot G \circ t$, treating them simultaneously as functions $M \to \mathbb{R}$ and as functions of the variable $t \in \mathit{\Lambda}$. With a fixed Riemannian metric g on M, for χ, σ, which are smooth functions of a nonconstant function $t : M \to \mathbb{R}$, setting $\phi = e^t g(\nabla t, \nabla t)/2$, we have the obvious equalities $\nabla\chi = \dot\chi\nabla t$ and

$$\text{i)}\ \ g(\nabla\sigma, \nabla\chi) = 2e^{-t}\dot\sigma\dot\chi\phi, \qquad \text{ii)}\ \ d\chi = \dot\chi\, dt. \tag{9}$$

In addition, $\nabla d\chi = \dot\chi\nabla dt + \ddot\chi\, dt \otimes dt$. so that $\nabla de^t = e^t(\nabla dt + dt \otimes dt)$ which, for $\tau = e^t$, gives $\nabla dt = e^{-t}\nabla d\tau - dt \otimes dt$, and

$$\nabla d\chi = \dot\chi e^{-t}\nabla d\tau + (\ddot\chi - \dot\chi)\, dt \otimes dt. \tag{10}$$

11. Gradient Ricci solitons

One says that a given Ricci soliton (M,g) is *of the gradient type* when the soliton vector field w with (2) (for some constant λ), may be chosen so as to be the gradient of a function. A Riemannian manifold

(M,g) is a gradient (type) Ricci soliton if and only if there exists a *soliton function* $f: M \to \mathrm{I\!R}$ satisfying the *gradient soliton equation*

$$\nabla df + \mathrm{Ric} = \lambda g, \quad \text{where } \lambda \text{ is a constant.} \tag{11}$$

The symbol ∇ denotes here the Levi-Civita connection of the metric g (but will also be used for the g-gradient operator), while ∇df is, as in §10, the Hessian of f.

That (2) becomes (11) if $2w = \nabla f$ (that is, when $2w$ is the gradient of a function f) follows from the identity $\pounds_v g = 2\nabla df$, valid for any function f and its gradient $v = \nabla f$.

The gradient soliton equation (11) has the following interesting consequences, first noted by Hamilton, cf. also [10, p. 201]:

Lemma 11.1. *Condition* (11) *for a function* f *on a Riemannian manifold* (M,g) *implies constancy of three functions*: $\Delta f + \mathrm{s}$, $\Delta f - g(\nabla f, \nabla f) + 2\lambda f$, *and* $\mathcal{R}f + \lambda f + \mathrm{s}/2$, *where* s *is the scalar curvature, and* $\mathcal{R}f = \Delta f - |\nabla f|^2/2$.

Proof. Applying to both sides of (11) the operators tr_g, $d \circ \mathrm{tr}_g - 2\delta + 2\imath_v$ and $\delta - \imath_v$, we get our three claims as trivial consequences of (6). □

Constancy of the last two functions in the lemma is due to vanishing of their differentials, since all manifolds are here – by definition – connected.

Perelman [29] proved the gradient property of compact Ricci solitons:

Theorem 11.2. *Every compact Ricci soliton is of the gradient type.*

The proof of this theorem in §15 is based on solvability of certain quasi-linear elliptic equations, established by Oscar S. Rothaus [33] in 1981. Rothaus's result may be stated as follows:

Theorem 11.3. *For a compact Riemannian manifold* (M,g) *of dimension* $n \geqslant 3$, *the operator* $\mathcal{R}$ *given by* (7), *and any positive real number* λ, *the assignment* $f \mapsto \mathcal{R}f + \lambda f$ *is a surjective mapping of the space of smooth functions* $M \to \mathrm{I\!R}$ *onto itself.*

The proof of Theorem 11.3, outlined in §14, uses the fact presented in the next section.

12. Logarithmic Sobolev inequalities

The above term refers to a type of estimates, first studied in the late 1960s [15, 19]. Rothaus's version [33] may be phrased as part (c) of the following lemma, in which, for $p \in [1,\infty)$ and a compact Riemannian manifold (M,g) of dimension n, we denote by by $\| \, \|_p$ and $\| \, \|_{p,1}$ the L^p

norm and its associated first Sobolev norm: $\|\varphi\|_{p,1}^p = \int_M [|\nabla\varphi|^p + \varphi^p]\,\mathrm{d}g$ whenever $\varphi \in \mathcal{F}M$, where $\mathcal{F}M$ is the space of all smooth functions $M \to \mathrm{I\!R}$, and $\mathrm{d}g$ the volume element of the metric g. In addition, we use the symbol L_1^pM for the Sobolev space obtained as the completion of $\mathcal{F}M$ in the norm $\|\;\|_{p,1}$, and identified in an obvious manner with a subspace of L^pM. The classical *Sobolev inequality* and the resulting inclusion state that, whenever $p \in (1,n)$ and $\varphi \in \mathcal{F}M$,

$$\|\varphi\|_r \leqslant C\|\varphi\|_{p,1} \text{ and } L_1^pM \subset L^rM, \text{ if } 1 \leqslant r \leqslant np/(n-p), \tag{12}$$

for a constant C depending only on $(M,g), p$ and r.

Lemma 12.1. *For a compact Riemannian manifold (M,g) of dimension $n \geqslant 3$, a constant $\varepsilon \in \mathrm{I\!R}$, and a smooth function $\chi : M \to \mathrm{I\!R}$, let*

$$I_\varepsilon^\chi(\varphi) \,=\, \int_M (\varepsilon|\nabla\varphi|^2 - \varphi^2\log|\varphi| + \chi\varphi^2)\,\mathrm{d}g, \tag{13}$$

where $\varphi\log|\varphi|$ by definition equals 0 on the zero set of φ. Then the functional $I_\varepsilon^\chi : L_1^2M \to \mathrm{I\!R}$, defined by (13),

(a) *is well defined, since $\varphi^2\log|\varphi| \in L^1M$ whenever $\varphi \in L_1^2M$,*
(b) *is continuous with respect to the Sobolev norm $\|\;\|_{2,1}$,*
(c) *assumes a minimum value on the set $\Sigma = \{\varphi \in L_1^2M : \|\varphi\|_2 = 1\}$, provided that $\varepsilon > 0$.*

Every $\varphi \in \Sigma$ realizing the minimum κ of the functional I_ε^χ on Σ, when $\varepsilon > 0$, is in addition a distributional solution of the equation

$$\varepsilon\Delta\varphi \,+\, \varphi\log|\varphi| \,+\, (\kappa-\chi)\varphi \,=\, 0. \tag{14}$$

To prove both Lemma 12.1 (in §13), and Theorem 11.3 (in §14), we will need the obvious equality

$$I_\varepsilon^\chi(\psi) \,=\, \varepsilon\|\psi\|_{2,1}^2 + I_0^0(\psi) \,+\, \langle(\chi-\varepsilon)\psi,\psi\rangle_2 \quad \text{for} \quad \psi \in L_1^2M, \tag{15}$$

where $\langle\,,\rangle_2$ is the inner product of L^2M, along with the (well known)

Lemma 12.2. *If a sequence φ_j, $j = 1,2,3,\ldots$, in the Sobolev space L_1^2M of a compact Riemannian manifold (M,g) of dimension $n \geqslant 3$ is bounded in the Sobolev norm $\|\;\|_{2,1}$, then, after φ_j has been replaced by a suitable subsequence, there will exist a function $\psi \in L_1^2M$ such that, as $j \to \infty$, one has simultaneously the convergences $\varphi_j \to \psi$ in the L^2 and L^r norms, with any fixed $r \in (2, 2n/(n-2))$, the weak convergence $\varphi_j \to \psi$ in the norm $\|\;\|_{2,1}$, and convergence of the norms $\|\varphi_j\|_{2,1}$ to some real number $\gamma \geqslant \|\psi\|_{2,1}$.*

Proof. Except for the last inequality $\gamma \geqslant \|\psi\|_{2,1}$, the existence of a subsequence with the required properties is an obvious consequence of the Rellich-Kondrashov and Banach-Alaoglu theorems; note that $L_1^2 M$ is a Hilbert space. On the other hand, the Schwarz inequality $|\langle\psi, \varphi_j\rangle_{2,1}| \leqslant \|\psi\|_{2,1}\|\varphi_j\|_{2,1}$ in $L_1^2 M$ and the weak convergence $\varphi_j \to \psi$ imply, in the limit, that $\|\psi\|_{2,1}^2 \leqslant \gamma\|\psi\|_{2,1}$ and either $\|\psi\|_{2,1} = 0 \leqslant \gamma$, or $\|\psi\|_{2,1} \neq 0$ and both sides may be divided by $\|\psi\|_{2,1}$. □

13. Proof of Lemma 12.1

Setting $H(\varphi) = \varphi^2 \log|\varphi|$ if $\varphi \in \mathrm{I\!R} \smallsetminus \{0\}$ and $H(0) = 0$, we obtain a function H of the variable $\varphi \in \mathrm{I\!R}$ having a continuous derivative H'. With any fixed $r \in (2, \infty)$, both $H(\varphi)/|\varphi|^r$ and $H'(\varphi)/|\varphi|^{r-1}$ tend to zero as $|\varphi| \to \infty$, leading to the estimates $|H(\varphi)| \leqslant c(1 + |\varphi|^r)]$ and $|H'(\varphi)| \leqslant c\max(1, |\varphi|^{r-1})$ for $\varphi \in \mathrm{I\!R}$, where $c > 0$ depends only on r.

Let $\varphi, \psi \in \mathrm{I\!R}$ and $r \in (2, \infty)$. Since $|H'(\varphi)| \leqslant c\max(1, |\varphi|^{r-1})$, Lagrange's classical mean value theorem implies that $|H(\varphi) - H(\psi)| \leqslant c|\varphi - \psi|\max(1, |\zeta|^{r-1})$, for some ζ lying between φ and ψ. Thus, $|H(\varphi) - H(\psi)| \leqslant c|\varphi - \psi|\max(1, |\varphi|^{r-1}, |\psi|^{r-1})$. The Hölder inequality, with q given by $r^{-1} + q^{-1} = 1$, that is, $q = r/(r-1)$, now yields, for *functions* $\varphi, \psi \in L_1^2 M$ and $r \in (2, \infty)$, the integral estimate

$$\|H(\varphi) - H(\psi)\|_1 \leqslant c\|\varphi - \psi\|_r (1 + \|\varphi\|_r^r + \|\psi\|_r^r)^{1/q}, \tag{16}$$

in which c depends on r (and the right-hand side may be infinite).

Let $r = 2n/(n-2)$. The relation $|H(\varphi)| \leqslant c(1 + |\varphi|^r)$, and the inclusion $L_1^2 M \subset L^r M$ in (12) imply integrability of $|\varphi|^r$ for $\varphi \in L_1^2 M$, proving (a). Since $|I_0^0(\varphi) - I_0^0(\psi)| = \|H(\varphi) - H(\psi)\|_1$, while $\|\psi\|_r \leqslant \|\varphi\|_r + \|\varphi - \psi\|_r$, convergence of ψ to φ in the Sobolev norm $\| \ \|_{2,1}$ implies, via (12), convergence in the norm $\| \ \|_r$ and, consequently, also the relation $I_0^0(\varphi) \to I_0^0(\psi)$. The functional $I_0^0 : L_1^2 M \to \mathrm{I\!R}$ is thus continuous, and (b) easily follows in view of (15).

To obtain (c), we begin with convexity of the exponential function, that is, the *Jensen inequality*, which states that $\int_M F\,d\mu \leqslant \log\int_M e^F d\mu$ for any integrable function $F : M \to \mathrm{I\!R}$ on a space M carrying a probability measure μ. Proof: it suffices to show this in the case of simple functions or, equivalently, verify that $q_1^{c_1} \ldots\ q_k^{c_k} \leqslant c_1 q_1 + \ldots + c_k q_k$ whenever $q_j \in (0, \infty)$ and $c_j \in [0, \infty)$, $j = 1, \ldots, k$, with $\sum_{j=1}^k c_j = 1$, which is easily achieved by applying $\partial/\partial q_1$ to maximize the difference $q_1^{c_1} \ldots q_k^{c_k} - c_1 q_1 - \ldots - c_k q_k$ for fixed $q_2, \ldots, q_k$ and $c_1, \ldots, c_k$.

Obviously, $a\int_M \varphi^2 \log\varphi\,\mathrm{d}g = \int_M \varphi^2 \log\varphi^a \mathrm{d}g$ if $a, \varepsilon \in (0, \infty)$ and $\varphi \in \Sigma$ (notation as in (c)). Thus, $\int_M \varphi^2 \log\varphi^a \mathrm{d}g \leqslant (2 + a)\log\|\varphi\|_{2+a}$ from Jensen's inequality $\int_M F\,d\mu \leqslant \log\int_M e^F d\mu$ for $d\mu = \varphi^2 \mathrm{d}g$ and $F = \log\varphi^a$. Setting $a = 4/(n-2)$ and choosing $p = 2$ in the Sobolev inequality (12), we get $2\int_M \varphi^2 \log\varphi\,\mathrm{d}g \leqslant n\log C\|\varphi\|_{2,1}$. Since $\|\varphi\|_{2,1}^2 = \int_M(|\nabla\varphi|^2+\varphi^2)\,\mathrm{d}g$, this last inequality yields $I_\varepsilon^\chi(\varphi) \geqslant \Phi(\xi) + \min\chi$, where $\Phi(\xi) = \varepsilon(\xi^2-1) - (1+2/a)\log C\xi$ for $\xi = \|\varphi\|_{2,1} \geqslant \|\varphi\|_2 = 1$. In addition, $\inf\{\Phi(\xi) : \xi \in [1,\infty)\} > -\infty$, so that, *if* $\varepsilon > 0$, *the functional* I_ε^χ *is bounded from below on the set* Σ.

Let $\varepsilon > 0$. Using the above italicized conclusion, we denote by κ the infimum of the functional I_ε^χ on Σ and fix a sequence $\varphi_j \in \Sigma$, $j = 1,2,3,\ldots$, for which $I_\varepsilon^\chi(\varphi_j) \to \kappa$ as $j \to \infty$. The sequence φ_j is thus bounded in the Sobolev norm $\|\;\|_{2,1}$, since in the obvious equality $\varepsilon\|\nabla\varphi\|_2 = 2I_\varepsilon^\chi(\varphi) - 2I_{\varepsilon/2}^\chi(\varphi)$, for $\varphi = \varphi_j$, the terms $2I_\varepsilon^\chi(\varphi)$ converge, and $2I_{\varepsilon/2}^\chi(\varphi)$ are bounded from below (as we just saw). Let us now replace the sequence φ_j with a subsequence chosen as in Lemma 12.2 for some real γ and a limit function in L_1^2M, denoted by φ (rather than ψ). The estimate (16) and the inclusion $L_1^2M \subset L^rM$ in (12) give $I_0^0(\varphi_j) \to I_0^0(\varphi)$. As $j \to \infty$, (15) for $\psi = \varphi_j$ therefore yields $\kappa = \varepsilon\gamma + I_0^0(\varphi) + \langle(\chi-\varepsilon)\varphi,\varphi\rangle_2 = I_\varepsilon^\chi(\varphi) + \varepsilon(\gamma^2 - \|\varphi\|_{2,1}^2)$. Consequently,

$$\kappa - I_\varepsilon^\chi(\varphi) = \varepsilon(\gamma^2 - \|\varphi\|_{2,1}^2). \tag{17}$$

Since $\kappa \leqslant I_\varepsilon^\chi(\varphi)$, we have $\gamma \leqslant \|\varphi\|_{2,1}$ while, by Lemma 12.2, $\gamma \geqslant \|\varphi\|_{2,1}$, and so $\gamma = \|\varphi\|_{2,1}$. Thus, (17) gives $\kappa = I_\varepsilon^\chi(\varphi)$, proving (c).

Let us now fix $\varphi \in \Sigma$ such that $I_\varepsilon^\chi(\varphi)$ has the minimum value κ. If $\psi \in \Sigma$ and $\langle\varphi,\psi\rangle_2 = 0$, putting $\psi_\theta = (\cos\theta)\varphi + (\sin\theta)\psi$, for $\theta \in \mathbb{R}$, we obtain a curve $\theta \mapsto \psi_\theta \in \Sigma$. The function $\theta \mapsto I_\varepsilon^\chi(\psi_\theta)$ is then differentiable and its derivative can be evaluated by differentiation under the integral symbol, yielding $d[I_\varepsilon^\chi(\psi_\theta)]/d\theta = \int_M(\partial\Pi_\theta/\partial\theta)\,\mathrm{d}g$, where Π_θ denotes the integrand in (13) with φ replaced by ψ_θ.

This is immediate from Lebesgue's dominated convergence theorem, since the absolute value $|\partial\Pi_\theta/\partial\theta|$ is bounded from above, uniformly in θ, by an integrable function. In fact, the first and third terms in Π_θ, differentiated with respect to θ, yield a linear combination of $\cos 2\theta$ and $\sin 2\theta$ with coefficients that are integrable functions; note that due to the definition of L_1^pM, preceding formula (12), the distributional gradient $\nabla\psi$ of any function $\psi \in L_1^pM$ is a measurable vector field with a square-integrable g-norm $|\nabla\psi| : M \to \mathbb{R}$. The derivative with respect to θ of the second term in Π_θ has in turn the absolute value $|\partial\psi_\theta/\partial\theta|\,|H'(\psi_\theta)|$, for H as at the beginning of the proof; the inequality $|H'(\varphi)| \leqslant c\max(1,|\varphi|^{r-1})$ established there, along with the obvious fact that $|\psi_\theta|$ and $|\partial\psi_\theta/\partial\theta|$ do not exceed $2\max(|\varphi|,|\psi|)$, gives

$$|\partial\psi_\theta/\partial\theta|\,|H'(\psi_\theta)| \leqslant c\max(1,|\varphi|^r,|\psi|^r) \leqslant c(1+|\varphi|^r+|\psi|^r)$$

with a new constant $c>0$. For $r=2n/(n-2)$ we now note that the inclusion $L^2_1M\subset L^rM$ in (12) yields integrability of $|\varphi|^r$ and $|\psi|^r$.

As φ minimizes I^χ_ε on Σ, the derivative $d[I^\chi_\varepsilon(\psi_\theta)]/d\theta$ at $\theta=0$ equals 0. Differentiating the integrand, one consequently gets

$$-\varepsilon\langle\nabla\varphi,\nabla\psi\rangle_2 + \langle\varphi\log|\varphi|+(\kappa-\chi)\varphi,\psi\rangle_2 = 0 \tag{18}$$

for $\psi\in L^2_1M$ such that $\langle\varphi,\psi\rangle_2=0$. On the other hand, (18) also holds for $\psi=\varphi$, since $I^\chi_\varepsilon(\varphi)=\kappa$. We thus have (18) whenever $\psi\in L^2_1M$ and – in particular – for test functions $\psi\in\mathcal{F}M$, which proves (14).

14. Outline of the proof of Rothaus's theorem (Thm. 11.3)

We need another, well-known

Lemma 14.1. *Let φ be any function in the Sobolev space L^2_1M of a compact Riemannian manifold (M,g) of dimension $n\geqslant 3$. The function $\psi=|\varphi|$ then satisfies the conditions $\psi\in L^2_1M$ and $\|\psi\|_{2,1}\leqslant\|\varphi\|_{2,1}$.*

Proof. Suppose first that φ is smooth, and denote by θ any term of a fixed sequence of positive numbers tending to 0. For the sequence $\varphi_\theta=\sqrt{\varphi^2+\theta^2}$ of positive smooth functions, the obvious equality $\varphi_\theta-\psi=\theta^2/(\varphi_\theta+\psi)$ gives $|\varphi_\theta-\psi|\leqslant\theta$. One thus has uniform convergence $\varphi_\theta\to\psi$, which also yields $\|\varphi_\theta-\psi\|_2\to 0$ and $\|\varphi_\theta\|_2\to\|\psi\|_2$. On the other hand, $\nabla\varphi_\theta=(\varphi/\varphi_\theta)\nabla\varphi$, while $|\varphi/\varphi_\theta|\leqslant 1$. so that $|\nabla\varphi_\theta|\leqslant|\nabla\varphi|$ and $\|\nabla\varphi_\theta\|_2\leqslant\|\nabla\varphi\|_2$. Replacing our sequence with a suitable subsequence, we obtain $\|\nabla\varphi_\theta\|_2\to c$ for some $c\leqslant\|\nabla\varphi\|_2$. The sequence $\|\varphi_\theta\|^2_{2,1}=\|\nabla\varphi_\theta\|^2_2+\|\varphi_\theta\|^2_2$ thus converges to γ^2, for $\gamma=(\|\psi\|^2_2+c^2)^{1/2}$. In view of Lemma 12.2, replacing our subsequence by a further subsequence allows us to assume convergence of φ_θ in the norm L^2 to some limit function lying in L^2_1M. Since we already know that $\|\varphi_\theta-\psi\|_2\to 0$, this limit function must be $\psi=|\varphi|$, which implies that $\psi\in L^2_1M$. The inequalities $\gamma\geqslant\|\psi\|_{2,1}$ (in Lemma 12.2) and $c\leqslant\|\nabla\varphi\|_2$ in turn show that $\|\psi\|^2_{2,1}\leqslant\gamma^2=\|\psi\|^2_2+c^2\leqslant\|\psi\|^2_2+\|\nabla\varphi\|_2=\|\varphi\|^2_{2,1}$, proving our claim in the case where φ is smooth.

For any function $\varphi\in L^2_1M$ we have $\varphi_j\to\varphi$ as $j\to\infty$, in the norm $\|\;\|_{2,1}$, with some sequence φ_j, $j=1,2,3,\ldots$, of smooth functions. Let $\psi_j=|\varphi_j|$ and $\psi=|\varphi|$. Thus, $\psi_j\in L^2_1M$, and the sequence $\|\psi_j\|_{2,1}\leqslant\|\varphi_j\|_{2,1}$ of norms is bounded; replacing it by a convergent subsequence, we obtain $\|\psi_j\|_{2,1}\to\gamma\leqslant\|\varphi\|_{2,1}$ (since $\|\varphi_j\|_{2,1}\to\|\varphi\|_{2,1}$).

Lemma 12.2 for the sequence ψ_j allows us to choose a limit function in $L^2_1 M$, which must coincide with ψ due to the convergence $\psi_j \to \psi$ in the L^2 norm (obvious in view of the convergence $\varphi_j \to \varphi$ in $L^2 M$, as $|\psi_j - \psi| \leqslant |\varphi_j - \varphi|$), while the inequality $\|\psi\|_{2,1} \leqslant \gamma$ in Lemma 12.2 gives $\|\psi\|_{2,1} \leqslant \|\varphi\|_{2,1}$. □

With the same assumptions and notations as in Lemma 12.1, given $\varepsilon > 0$, let φ minimize I^χ_ε on the set Σ. Then $\psi = |\varphi|$ also minimizes I^χ_ε on Σ. Namely, Lemma 14.1 shows that $\psi \in \Sigma$ and ψ satisfies the inequality $\|\psi\|_{2,1} \leqslant \|\varphi\|_{2,1}$, while $I^\chi_\varepsilon(\psi) \leqslant I^\chi_\varepsilon(\varphi)$ as a consequence of this inequality and (15), since the two final terms in (15) remain unchanged if ψ is replaced by $|\psi|$.

In other words, we may also assume that the function φ minimizing I^χ_ε on Σ is nonnegative. It then follows that φ is positive everywhere and smooth. The reasons for the last conclusion can only be outlined here. First, using the De Giorgi and Nash method, one shows that φ is of class C^2 with locally-Hölder second partial derivatives. Next, a suitable version of a local maximum principle, using geodesic coordinates, proves that the zero set of φ is open. Its openness – along with the condition $\|\varphi\|_2 = 1$ (which is a part of the definition of Σ, and prevents φ from vanishing identically) – implies positivity of φ. A bootstrapping-type argument in spaces of multiply differentiable functions with Hölder derivatives yields in turn smoothness of φ.

Proof of Theorem 11.3. Let us fix (M, g) and λ satisfying the hypotheses of the theorem, and a smooth function $\psi : M \to \mathbb{R}$. For $\varepsilon = 1/\lambda$ and $\chi = -\psi/(2\lambda)$ there exists – as we saw earlier – a positive smooth solution φ of (14) with some constant κ. Setting $f = -2(\kappa + \log \varphi)$, one easily verifies that (14) takes the form $\mathcal{R}f + \lambda f = \psi$. □

15. Proof of Perelman's gradient-type theorem (Thm. 11.2)

Assumption. (M, g) is a compact Riemannian manifold and (2) holds for a fixed real number λ and a fixed vector field w.

Objective. To find a function f satisfying equation (11): $\nabla df + \mathrm{Ric} = \lambda g$. We may assume here that $n = \dim M \geqslant 3$ and the soliton constant λ is positive, since – as shown by Hamilton [21] and, back in 1974, Bourguignon [6] – in the compact case condition (2) with $n = 2$, or with $\lambda \leqslant 0$, always gives (3), that is, (11) for $f = 0$.

How to proceed. The function f is supposed to satisfy (11); where to get it from? Lemma 11.1 suggests an answer – three functions, naturally associated with f should, *ex post facto*, turn out to be constant, which (with the right choice of their constant values) leads to three second-order elliptic equations, imposed on the required function f. One can thus try to show first that one of the three equations has a solution f, and then – that such f also satisfies (11). In the first equation, the constant in question must obviously be the mean value $\mathrm{s}_{\mathrm{avg}}$ of the scalar curvature s (since $\int_M \Delta f \, \mathrm{d}g = 0$), and f such that $\Delta f + \mathrm{s} = \mathrm{s}_{\mathrm{avg}}$ exists due to the general solvability criterion for linear elliptic equations. However, nobody knows how to get from here to (11). Things look better for the third equation, in which, since $\lambda > 0$, by adding to f a suitable constant we may require that the constant on the right-hand side be zero. As we will see, solvability follows here from Rothaus's theorem.

Argument. For any compact Riemannian manifold (M, g), any function f, constant λ and vector field w, let us set

$$h = \nabla df + \mathrm{Ric} - \lambda g, \quad b = \pounds_w \, g + \mathrm{Ric} - \lambda g, \quad \psi = \Delta e^{-f} + 2\delta[e^{-f} w],$$

where δ is the divergence operator acting on vector fields. Then (with no further assumptions!), for the operator $\mathcal{R}$ given by (7),

$$\int_M |h|^2 e^{-f} \mathrm{d}g \; + \int_M (\mathcal{R}f + \lambda f + \mathrm{s}/2)\psi \, \mathrm{d}g \; = \int_M \langle h, b\rangle e^{-f} \mathrm{d}g.$$

(Proof – a trivial, though tedious, integration by parts.) In our situation, with λ and w satisfying (2), the integral on the right-hand side vanishes, since $b = 0$, and Rothaus's theorem (Thm. 11.3) allows us to choose f such that $\mathcal{R}f + \lambda f + \mathrm{s}/2 = 0$. Thus, $h = 0$, which completes the proof.

16. Canonical Kähler metrics

Let g be a Kähler metric on a fixed compact complex manifold M. The formulae $\omega = g(J\,\cdot\,,\,\cdot\,)$ and $\rho = \mathrm{Ric}(J\,\cdot\,,\,\cdot\,)$ then define the *Kähler form* and *Ricci form* of the metric g. Both of them are closed 2-forms, that is, skew-symmetric 2-tensor fields (due to (8)) with $d\omega = d\rho = 0$. If a function f on M satisfies the gradient-soliton equation (11), then $i\partial\overline{\partial}f + \rho = \lambda\omega$, since the form $i\partial\overline{\partial}f$ is exact (being the exterior derivative of $i\overline{\partial}f$). This implies equality of de Rham cohomology classes: $[\rho] = \lambda[\omega] \in H^2(M, \mathrm{I\!R})$. On the other hand, $[\rho]$ is always equal to the first Chern class c_1 of M multiplied by 2π, so that it only depends on the complex structure – and not, for instance, on g.

For a compact complex manifold to admit a Kähler-Ricci soliton with a positive (or negative) soliton constant λ it is thus necessary that its first Chern class c_1 be positive (or negative), in the sense of realizability of c_1 (or $-c_1$) as the cohomology class of the Kähler form of some Kähler metric.

In complex dimension 2 this necessary condition is also sufficient. In addition, on compact complex surfaces with positive or negative first Chern class c_1, Kähler-Ricci solitons constitute a natural choice of a *distinguished*, or *canonical*, Kähler metric.

More precisely, *such a surface always admits a Kähler-Ricci soliton, which is in addition unique up to the action of the identity component of its complex automorphism group, and rescalings.*

The above statement summarizes a series of results, both classical and more recent. Here belong the theorems establishing:

(a) uniqueness of Kähler-Einstein metrics (1957: Eugenio Calabi [7], for $c_1 < 0$, 1987: Shigetoshi Bando, Toshiki Mabuchi [2], for $c_1 > 0$),
(b) truth of Calabi's conjecture about the existence of a Kähler-Einstein metric when $c_1 < 0$ (1978: Thierry Aubin [1], Shing-Tung Yau [37]),
(c) existence Kähler-Ricci solitons on compact toric complex manifolds with $c_1 > 0$ (2004: Xu-Jia Wang, Xiaohua Zhu [36]),
(d) uniqueness of a Kähler-Ricci soliton when $c_1 > 0$, modulo automorphisms from the identity component (2002: Tian and Zhu [35]).

17. Ricci-Hesian equations

From now on all functions are – by definition – *smooth.*

Using Maschler's terminology [27], we say that a function τ on a Riemannian manifold (M, g) *satisfies a Ricci-Hessian equation* if, for some function α on M, nonzero at all points of a dense subset, one has, with the notation introduced in (5),

$$\{\alpha\nabla d\tau + \mathrm{Ric}\}_0 = 0 \tag{19}$$

or, equivalently: $\alpha\nabla d\tau + \mathrm{Ric} = \eta g$ for some function η.

Solutions of equations of type (19) exist, for instance, on certain Kähler manifolds, forming the family of *special Kähler-Ricci potentials*, which is completely classified, both locally [12, §18] and in the compact case [13, §16]. Their definition and a discussion of how they are related to condition (19) can be found in [12, §7] and [27].

Another special case of a Ricci-Hessian equation (19) is the gradient-soliton equation (11), in which $\tau = f$ and $\alpha = 1$. A further connection between equations (19) and (11) is due to the fact that many solutions of Ricci-Hessian equations (19), which themselves do not satisfy (11), may be used to construct solutions of (11) by suitably modifying the metric g and the functions appearing in (19). The modifications of metrics consist here in their *conformal changes*, a detailed discussion of which will be given later (§19).

The approach just described was developed by Maschler in [27]. His initial solutions τ of equations of type (19) belonged to the class – mentioned earlier in this section – of special Kähler-Ricci potentials.

The following lemma, using the convention borrowed from the end of §1, is a slight variation on Maschler's argument [27, Remark 4.2]. The solution τ of a Ricci-Hessian equation (19), arising here, is a special Kähler-Ricci potential on the compact Kähler manifold (M_k^m, g), that is, belongs to a family of examples constructed in [13, §5].

Lemma 17.1. *Let $m, k \in \mathbb{Z}$ and smooth functions $x, \phi : [0, T] \to \mathbb{R}$ of the variable t, with $T \in (0, \infty)$, satisfy the conditions $m > k > 0$ and*
(a) $\ddot{\phi} = (m-1)\dot{x}\dot{\phi} + m\phi - m$,
(b) $\phi(0) = \phi(T) = 0$, *while* $\phi > 0$ *on the interval* $(0, T)$,
(c) $\dot{\phi}(0) = k$ *and* $\dot{\phi}(T) = -k$,
where $(\ )^{\cdot} = d/dt$. *Then the complex manifold $M = M_k^m$ described by* $(*)$ *in* §8 *admits a Kähler metric g with scalar curvature* s *and a smooth surjective function $t : M \to [0, T]$ with the following properties*:
(i) *the function $\tau = e^t$ is a solution of the Ricci-Hessian equation* (19), *that is,* $\{\alpha\nabla d\tau + \mathrm{Ric}\}_0 = 0$, *where* $\alpha = (m-1)(\dot{x}+1)e^{-t}$,
(ii) $\Delta\tau = 2(\dot{\phi} + m\phi)$ *and* $g(\nabla\tau, \nabla\tau) = 2e^t\phi$, *for the function* $\tau = e^t$,
(iii) $e^t\mathrm{s}/2 \,=\, m(m-1) \,-\, m(m-1)\phi \,-\, (2m-1)\dot{\phi} \,-\, \ddot{\phi}$.

Outline of proof. The metric g and function t will be shown to exist via an explicit construction, carried out in a slightly larger class of complex manifolds than just the family M_k^m (see also the comment at the end of §8). To be specific, suppose that N is a compact complex manifold of dimension $m-1$ and its canonical bundle (the top exterior power of cotangent bundle) can be raised to the fractional tensor power with the exponent k/m. When m and k are relatively prime, this amounts to realizability of the canonical bundle as the mth tensor power of some line bundle; it is known [23] that N must then be biholomorphic to the projective space $\mathbb{C}\mathrm{P}^{m-1}$, for which the tautological bundle is an mth tensor root of the canonical bundle. If, however, the fraction k/m can be simplified, there are more such examples, as illustrated by the case of odd Cartesian powers of $\mathbb{C}\mathrm{P}^1$.

Let us also assume that N carries a Kähler-Einstein metric h with the Ricci tensor $2mh$. In other words, the Einstein constant is required to be positive, and its value becomes $2m$ after h is suitably rescaled.

We denote by M the projective compactification of the line bundle $\mathcal{E}$ over N arising as the (k/m)th tensor power of the canonical bundle. Our assumptions guarantee the existence in $\mathcal{E}$ of a Hermitian Chern connection with the curvature form equal to the Ricci form of h (see §16) multiplied by $-k/m$. The tangent bundle of the total space $\mathcal{E}$ is thus decomposed into the direct sum of the vertical subbundle $\mathcal{V}$, tangent to the fibres, and the horizontal distribution $\mathcal{H}$ of the connection.

The fibre norm in $\mathcal{E}$ is a nonnegative function r on the total space $\mathcal{E}$, which we simultaneously treat as an independent variable. Our variable t, restricted to the interval $(0,T)$, may now be turned into a function of the variable r, characterized by a choice of one of the diffeomorphisms $r \mapsto t(r)$ between the intervals $(0,\infty)$ and $(0,T)$ satisfying the equation

$$\frac{d}{dr}\,t(r) \;=\; \frac{2\phi(t(r))}{kr}\,. \tag{20}$$

This allows us to identify t and ϕ with functions $\mathcal{E}\smallsetminus N \to \mathrm{I\!R}$, depending only on the fibre norm r, and defined on the complement of the zero section $N \subset \mathcal{E}$. Our metric g on $\mathcal{E} \smallsetminus N$ is now defined by requiring that $\mathcal{V}$ be g-orthogonal to $\mathcal{H}$, that g restricted to each fibre of $\mathcal{E}$ be the Euclidean metric multiplied by the function $2(kr)^{-2}e^t\phi$, and that g restricted to $\mathcal{H}$ be the product of the function $2e^t$ and the pullback of the base metric h under the bundle projection $\mathcal{E} \to N$.

A solution $t(r)$ of (20) is obviously nonunique; other solutions arise from it by rescaling the independent variable r. Any resulting new metric, is, however, isometric to g, an isometry being provided by a suitable rescaling in every fibre of the bundle $\mathcal{E}$.

The rest of the proof – verifying that g and t have smooth extensions to M and satisfy the required conditions – is trivial (though tedious). $\square$

18. Another description of gradient Ricci solitons

Let us restrict our consideration to gradient Ricci solitons with *positive* soliton constants λ. A soliton function f satisfying (11) can thus be *normalized* by adding a suitable constant so as to make the function $\Delta f - g(\nabla f, \nabla f) + 2\lambda f$ (constant in view of Lemma 11.1) equal to zero. We also normalize the metric g, replacing it with λg (and in effect assuming that $\lambda = 1$), which leaves the Ricci tensor Ric and the Hessian ∇df unchanged. These normalizations result in the equalities

$$\text{i)}\;\; \nabla df \,+\, \mathrm{Ric} \,=\, g, \qquad \text{ii)}\;\; \Delta f \,-\, g(\nabla f, \nabla f) \,+\, 2f \,=\, 0. \tag{21}$$

For a function f on a Riemannian manifold (M,g) of dimension $n \geqslant 3$, the normalized version (21) of the gradient-soliton equation (11) is equivalent to the following three-equation system:

$$\begin{array}{ll} \text{(a)} & \{\nabla df + \mathrm{Ric}\}_0 = 0\,, \\ \text{(b)} & \Delta f - g(\nabla f, \nabla f) + 2f = 0\,, \\ \text{(c)} & (n-2)(\Delta f + \mathrm{s} - n) + n[\Delta f - g(\nabla f, \nabla f) + 2f] = 0\,. \end{array}$$

The symbol $\{\ \}_0$ denotes here the g-traceless part, given by (5), while $\mathrm{s} = \mathrm{tr}_g\mathrm{Ric}$ is the scalar curvature of g.

The equivalence between (21) and the system (a) – (c) is obvious from the fact that equality of the two sides in (21.i) amounts to simultaneous equalities of their g-traceless parts, and their g-traces.

19. Conformal changes of Riemannian metrics

By a *conformal change* of a Riemannian metric g on a manifold M one means its replacement by the product μg, where $\mu : M \to (0,\infty)$.

For a fixed n-dimensional Riemannian manifold (M,g), let Ric and s denote the Ricci tensor and scalar curvature of g. Their counterparts $\overline{\mathrm{Ric}}, \bar{\mathrm{s}}$ for the conformally related metric $\bar{g} = g/\sigma^2$, with any function $\sigma : M \to (0,\infty)$, are easily verified to be given by

$$\begin{aligned} \overline{\mathrm{Ric}} \,&=\, \mathrm{Ric} \,+\, (n-2)\sigma^{-1}\nabla d\sigma \,+\, \big[\sigma^{-1}\Delta\sigma \,-\, (n-1)\sigma^{-2}g(\nabla\sigma,\nabla\sigma)\big]\,g\,, \\ \bar{\mathrm{s}} \,&=\, \sigma^2\mathrm{s} \,+\, 2(n-1)\,\sigma\Delta\sigma \,-\, n(n-1)g(\nabla\sigma,\nabla\sigma)\,. \end{aligned}$$

See, for instance, [14, pp. 528–529].

Suppose now that both σ and f are smooth functions of a given nonconstant function $t : M \to \mathrm{I\!R}$ (cf. §1), and consider the functions x, y of the variable t defined by $x = 2\log\sigma - t + (m-1)^{-1}f$ and $y = (m-1)^{-1}f$, where $m = n/2$ (and n need not be even). Let us further assume that $g(\nabla t, \nabla t)$ is a smooth function of the function t, and set $\phi = e^t g(\nabla t, \nabla t)/2$. Thus,

$$\text{i)}\;\; \sigma = e^{(x-y+t)/2}, \quad \text{ii)}\;\; f = (m-1)y, \quad \text{iii)}\;\; g(\nabla t, \nabla t) = 2e^{-t}\phi. \quad (22)$$

Our goal is to find sufficient conditions for a conformal change of a metric g (which – so far – is completely arbitrary) to yield a gradient Ricci soliton $\bar{g}$ with a soliton constant $\lambda > 0$ (that is, $\lambda = 1$, after the normalization described in §18). These conditions will be imposed on the unknown functions x, y of the variable t, corresponding – via (22) – to the unknown functions σ and f, which in turn will constitute the functional factor in the conformal change $\bar{g} = g/\sigma^2$, and the soliton function for $\bar{g}$.

We first evaluate the ingredients of (a) – (c) in §18 for f and the metric $\bar{g}$ (instead of g), expressing them in terms of: the functions x, y, ϕ (and their derivatives with respect to t, with the notation $(\ )^{\boldsymbol{\cdot}} = d/dt$); the g-Hessian $\nabla d\tau$ of the function $\tau = e^t$; its g-Laplacian $\Delta\tau$; the Ricci tensor Ric of g; and its scalar curvature s, so that

$$\{\overline{\nabla} df + \overline{\mathrm{Ric}}\}_0 = \{\alpha \nabla d\tau + \mathrm{Ric} + \beta\, dt \otimes dt\}_0 \tag{23}$$

where $\alpha = (m-1)(\dot{x}+1)e^{-t}$ and $\beta = (m-1)(2\ddot{x} - \dot{y}^2 + \dot{x}^2 - 1)/2$,

$$\begin{aligned} &(m-1)^{-1} e^{y-x}[\overline{\Delta} f - \bar{g}(\overline{\nabla} f, \overline{\nabla} f) + 2f] \\ &\qquad = \dot{y}[\Delta\tau - 2(\dot{\phi} + m\phi)] + 2[\phi\ddot{y} - (m-1)\phi\dot{x}\dot{y} + \dot{\phi}\dot{y} + y e^{y-x}], \end{aligned} \tag{24}$$

and, still with $n = 2m$,

$$\begin{aligned} &e^{y-x}[(\overline{\Delta} f + \bar{\mathrm{s}} - 2m) + m(m-1)^{-1}\{\overline{\Delta} f - \bar{g}(\overline{\nabla} f, \overline{\nabla} f) + 2f\}] \\ &\quad = m[2\dot{x}\dot{\phi} - (2m-1)\phi\dot{x}^2 + \phi\dot{y}^2 + 2(y-1)e^{y-x} - \phi + 2m] \\ &\quad - 2[m(m-1) - m(m-1)\phi - (2m-1)\dot{\phi} - \ddot{\phi} - e^t\mathrm{s}/2] \\ &\quad + (2m-1)(\dot{x}+1)[\Delta\tau - 2(\dot{\phi} + m\phi)] \\ &\quad - 2[\ddot{\phi} - (m-1)\dot{x}\dot{\phi} - m\phi + m] \\ &\quad + (2m-1)[2\ddot{x} - \dot{y}^2 + \dot{x}^2 - 1]\phi. \end{aligned} \tag{25}$$

Verifying (23) – (25) is, as before, easy, though tedious, and may be simplified by using the following intermediate steps. First, due to (22.i),

$$\text{a) } 2\dot{\sigma}\sigma^{-1} = \dot{x} - \dot{y} + 1, \quad \text{b) } 2(\ddot{\sigma} - \dot{\sigma})\sigma^{-1} = \ddot{x} - \ddot{y} + [(\dot{x} - \dot{y})^2 - 1]/2. \tag{26}$$

From the above expression for $\overline{\mathrm{Ric}}$ and $\overline{\nabla} df$, (9.ii) – (10) with $\chi = \sigma$ or $\chi = f$, (22.ii), and (26) with $m = n/2$ and $\tau = e^t$, we obtain[1]

$$\begin{aligned} &(m-1)^{-1}\{\overline{\mathrm{Ric}} - \mathrm{Ric}\}_0 \\ &\qquad = \{(\dot{x} - \dot{y} + 1)e^{-t}\nabla d\tau + 2(\ddot{\sigma} - \dot{\sigma})\sigma^{-1} dt \otimes dt\}_0, \\ &(m-1)^{-1}\overline{\nabla} df = \dot{y}e^{-t}\nabla d\tau \\ &\qquad + (\ddot{y} + \dot{x}\dot{y} - \dot{y}^2)\, dt \otimes dt - (\dot{x} - \dot{y} + 1)\dot{y}e^{-t}\phi g. \end{aligned} \tag{27}$$

Now (26) and (27) trivially imply (23). Applying to the second equality in (27) the operator $\mathrm{tr}_{\bar{g}} = \sigma^2 \mathrm{tr}_g$ and, separately, setting $\chi = f$ in (9.i), then using the fact that $\mathrm{tr}_g(dt \otimes dt) = g(\nabla t, \nabla t)$, and finally expressing σ and $g(\nabla t, \nabla t)$ through (22), we get

$$\begin{aligned} &(m-1)^{-1} e^{y-x}\overline{\Delta} f = \dot{y}\Delta\tau + 2[\ddot{y} - m\dot{y} + (m-1)(\dot{y} - \dot{x})\dot{y}]\phi, \\ &(m-1)^{-2} e^{y-x}\bar{g}(\overline{\nabla} f, \overline{\nabla} f) = 2\dot{y}^2\phi, \end{aligned} \tag{28}$$

which gives (24). From the formula for $\bar{\mathrm{s}}$, (9) – (10) with $\chi = \sigma$, and (26), we have

$$\begin{aligned} &(2m-1)^{-1} e^{y-x}\bar{\mathrm{s}} = (2m-1)^{-1} e^t \mathrm{s} + (\dot{x} - \dot{y} + 1)\Delta\tau \\ &\qquad + [2\ddot{x} - 2\ddot{y} - (m-1)(\dot{x} - \dot{y})^2 - 2m(\dot{x} - \dot{y}) - m - 1]\phi. \end{aligned} \tag{29}$$

The last equality, (28) and (24) easily yield (25).

[1] noting that $\overline{\nabla} df = \nabla df - \sigma^{-1}[d\sigma \otimes df + df \otimes d\sigma - g(\nabla\sigma, \nabla f) g]$

20. Ricci solitons conformal to Kähler metrics

Having fixed $T \in (0,\infty)$ and integers m, k such that $m > k > 0$, let us consider the following system of second-order differential equations:

$$\begin{array}{rrcl} \text{i)} & 2\ddot{x} & = & \dot{y}^2 - \dot{x}^2 + 1, \\ \text{ii)} & \phi\ddot{y} & = & (m-1)\phi\dot{x}\dot{y} - \dot{\phi}\dot{y} - ye^{y-x}, \\ \text{iii)} & \ddot{\phi} & = & (m-1)\dot{x}\dot{\phi} + m\phi - m, \end{array} \tag{30}$$

imposed on unknown C^∞ functions $x, y, \phi : [0,T] \to \mathrm{I\!R}$ of the variable $t \in [0,T]$, where $(\ )^\cdot = d/dt$, and the additional first-order equation

$$2\dot{x}\dot{\phi} - (2m-1)\phi\dot{x}^2 + \phi\dot{y}^2 + 2(y-1)e^{y-x} - \phi + 2m = 0 \tag{31}$$

along with the boundary conditions

$$\phi(0) = \phi(T) = 0, \quad \phi > 0 \text{ on } (0,T), \quad \dot{\phi}(0) = k = -\dot{\phi}(T). \tag{32}$$

Equation (31) does not reduce too drastically the solution set of (30), since the left-hand side of (31) multiplied by e^x is an integral of (30).

Theorem 20.1. *With any solution $(x,y,\phi) : [0,T] \to \mathrm{I\!R}^3$ of the system* (30) – (32)*, for T, m, k fixed as above, one can associate a Ricci soliton $\bar{g}$ conformal to a Kähler metric on the compact complex manifold $M = M_k^m$ defined by* $(*)$ *in* §8*. To obtain $\bar{g}$, we use the Kähler metric g on M and the function $t : M \to \mathrm{I\!R}$, arising from our x, ϕ, T, m, k via Lemma* 17.1*. We then set $\bar{g} = g/\sigma^2$ for the function σ in* (22.i)*.*

The resulting metric $\bar{g}$ and the function f defined by (22.ii) *satisfy conditions* (a) – (c) *of* §18*, equivalent to the gradient-soliton equation* (11) *for $\lambda = 1$ and for $\bar{g}$ instead of g.*

Proof. The right-hand sides in equations (23) – (25) are all equal to zero – in (23) we have $\{\alpha\nabla d\tau + \mathrm{Ric}\}_0 = 0$ and $\beta = 0$, by Lemma 17.1(i) and (30.i); in (24), both terms on the right-hand side vanish as a consequence of Lemma 17.1(ii) and (30.ii); while in (25) each of the final five lines is zero due to (31), Lemma 17.1(iii), Lemma 17.1(ii), (30.iii) and (30.i). □

For all known examples of even-dimensional compact Ricci solitons (listed in Question 9.1), the multiplicities of eigenvalues of the Ricci tensor are even at every point. This follows from (8) and the fact that the Ricci tensor of a Riemannian product is, in a natural sense, the direct sum of the Ricci tensors of the factor manifolds.

Suppose that, for some solution $(x,y,\phi) : [0,T] \to \mathrm{I\!R}^3$ of the system (30) – (32), and the function σ defined by (22.i), one has $\ddot{\sigma} \neq \dot{\sigma}$ somewhere in the interval $[0,T]$ (whether such solutions exists, is not known; the issue is further complicated by the singularities of equation (30.ii) at $t = 0$ and $t = T$, due to (32)). For reasons named below, the compact Ricci soliton $(M,\bar{g})$ arising then from Theorem 20.1 would be non-Einstein, non-Kähler (even locally), and not locally isometric to a Riemannian product with Einstein or locally-Kähler factor manifolds. The result would be an affirmative answer to Question 9.1.

The reasons just mentioned are as follows. Since the real dimension of M is even, at every point satisfying the condition $(\ddot{\sigma}-\dot{\sigma})\,dt \neq 0$ the Ricci tensor $\overline{\mathrm{Ric}}$ has an eigenvalue of odd multiplicity, for which the gradient $\overline{\nabla}\tau$ is an eigenvector. This is immediate from (27) since, in view of Lemma 17.1(i) and (8), $\nabla d\tau$ and Ric are Hermitian and simultaneously diagonalizable at each point, with even-dimensional eigenspaces, while from (6.c) and the second equality in Lemma 17.1(ii) it follows that $\nabla\tau$ is a common eigenvector of both $\nabla d\tau$ and Ric.

Solutions of the system (30) – (31) for which $\ddot{\sigma}=\dot{\sigma}$ are in turn completely understood. We will describe them in §§21–25.

Theorem 20.1 also has a *local version*, in which instead of the boundary conditions (32) one only assumes positivity of ϕ on the open interval forming the domain of the solution (x,y,ϕ) to (30) – (31) with $m\geqslant 2$. The construction of Lemma 17.1 then gives a metric g and function t which, although defined just on some noncompact manifold, still satisfy the conclusions (i) – (iii) in Lemma 17.1.

Yet another generalization of Theorem 20.1 arises when one removes equation (30.ii) from the system (30) – (32), keeping all the remaining requirements. The right-hand side of (23) will then still be equal to zero. This leads to a weaker version of the gradient-soliton equation (11): $\overline{\nabla}df+\overline{\mathrm{Ric}}=\lambda\bar{g}$ for some function λ that need not be constant. Metrics $\bar{g}$ for which such functions f and λ exist were studied by several authors [27, p. 369], [3], [32]; they are sometimes called (gradient) *Ricci almost-solitons.*

21. Symmetries of (30) – (31) and the condition $\ddot{\sigma}=\dot{\sigma}$

In the equations forming the system (30) – (31), no term depends explicitly on the variable t, while the terms containing the first derivatives $\dot{x},\dot{y},\dot{\phi}$ are homogeneous quadratic in them. The system will thus remain satisfied if in a solution, defined on any interval, one replaces the variable t by $c\pm t$, with a constant c.

The set of solutions defined on the interval $[0,T]$ and satisfying the boundary conditions (32) is therefore invariant under the substitution of $T-t$ for the variable t.

We now discuss the solutions of (30) – (31) such that $\ddot{\sigma}=\dot{\sigma}$.

Lemma 21.1. *Let a solution* (x, y, ϕ) *of* (30) – (31)*, with* $m \geqslant 2$*, defined on an open interval of the variable* t*, satisfy in addition the condition* $\ddot{\sigma} = \dot{\sigma}$*, where the function* σ *is given by* (22.i)*. Then* $\sigma = q_0 + q_1 e^t$ *for some constants* q_0, q_1*, at least one of which is positive. Furthermore,*

$$\text{i)}\ \ \dot{y}\dot{\phi} \,=\, (m\dot{x} - \dot{y})\dot{y}\phi \,-\, ye^{y-x}\,, \qquad \text{ii)}\ \ \ddot{y} \,=\, (\dot{y} - \dot{x})\dot{y}, \qquad (33)$$

and one of the following three cases must occur:
(i) $y = 0$ *on the entire interval of the variable* t*,*
(ii) $q_1 = 0$*, so that* $\sigma = q_0$ *is a positive constant,*
(iii) $q_0 = 0 < q_1$ *and* $\sigma = q_1 e^t$*.*
In the solution set of the full system (30) – (32)*, with* T, m, k *fixed as before, the variable substitution of* $T - t$ *for* t *preserves the condition* $\ddot{\sigma} = \dot{\sigma}$*, leaves case* (i) *unchanged, and switches case* (ii) *with* (iii)*.*

Proof. Let us fix a solution (x, y, ϕ) of (30). For the function η defined to be the difference between the left-hand and right-hand sides of (33.i), subtracting (30.i) multiplied by $\phi/2$ from (30.ii) we see, using (26.b) and (30), that $\eta = 2(\ddot{\sigma} - \dot{\sigma})\phi\sigma^{-1}$. Also, by (30), the expression

$$2(\phi\eta)^{\cdot} - 2(m-1)\phi\eta\dot{x} + 4(m-1)(\dot{\sigma} - \sigma)\phi^2\sigma^{-2}\dot{y}\dot{\sigma}$$

is the product of $-\phi\dot{y}$ and the left-hand side of (31), as (26.a) gives $4(\dot{\sigma} - \sigma)\sigma^{-2}\dot{\sigma} = (2\dot{\sigma}\sigma^{-1})^2 - 2(2\dot{\sigma}\sigma^{-1}) = (\dot{x} - \dot{y} + 1)(\dot{x} - \dot{y} - 1)$. From (30) – (31) with $\ddot{\sigma} = \dot{\sigma}$ we thus get (33.i) (that is, vanishing of η) and

$$(\dot{\sigma} - \sigma)\phi\dot{y}\dot{\sigma} \,=\, 0. \qquad (34)$$

By (33.i) and (30.ii), $(m\dot{x}\dot{y} - \dot{y}^2)\phi = \dot{\phi}\dot{y} + ye^{y-x} = [(m-1)\dot{x}\dot{y} - \ddot{y}]\phi$ while, from (30.iii), $\phi \neq 0$ on some dense subset of the domain interval. This proves (33.ii).

On the other hand, if $\ddot{\sigma} = \dot{\sigma}$, then $\dot{\sigma} = q_1 e^t$ and $\sigma = q_0 + q_1 e^t$ with constants q_0, q_1, which cannot be both nonpositive, since $\sigma > 0$ due to (22.i). Under the hypotheses of the lemma, equation (34) gives rise (in view of analyticity of the solution (x, y, ϕ) of (30) on every connected component of the dense set on which $\phi \neq 0$) to four possible cases: $\phi = 0$, $\dot{y} = 0$, $\dot{\sigma} = 0$ and $\dot{\sigma} = \sigma$. The first one is excluded by (30.iii); the second, as a consequence of (30.ii), amounts to requiring that $y = 0$; the third case gives (ii); the fourth – (iii).

If (30) – (32) are assumed, the substitution of $T - t$ for t obviously preserves condition (i) while, by (22.i), it causes σ to be replaced with the function $t \mapsto e^{t - T/2}\sigma(T - t)$, which completes the proof. □

22. Case (i) in Lemma 21.1

Let us consider the case $y = 0$ in Lemma 21.1. In view of (22.ii), this is nothing else than vanishing of the normalized soliton function f in Theorem 20.1 (or – more precisely – in its local version; see §20). Our construction thus leads now to Einstein metrics $\bar{g}$, cf. §7.

We describe below the solutions (x, y, ϕ) of the system formed by (30) – (31) and equation $y = 0$ on any open interval; translating the variable t (as in §21), we may assume that the interval contains 0.

Theorem 22.1. *For all solutions (x, y, ϕ) of (30) – (31) such that $y = 0$, on an open interval containing 0, the initial data*

$$(x_0,\, y_0,\, \phi_0,\, \dot{x}_0,\, \dot{y}_0,\, \dot{\phi}_0) \;=\; (x(0),\, y(0),\, \phi(0),\, \dot{x}(0),\, \dot{y}(0),\, \dot{\phi}(0)) \qquad (35)$$

satisfy the conditions

$$y_0 = \dot{y}_0 = 0, \quad 2\dot{x}_0\dot{\phi}_0 = [(2m-1)\dot{x}_0^2 + 1]\,\phi_0 + 2e^{-x_0} - 2m. \qquad (36)$$

Conversely, every choice of the data (35) satisfying (36) is realized by a unique solution (x, y, ϕ) of (30) – (31) with $y = 0$, defined on a maximal interval containing 0. One then has

$$x = x_0 + 2\log|\Theta|, \quad \text{where} \quad \Theta = \cosh(t/2) + \dot{x}_0 \sinh(t/2), \qquad (37)$$

$y = 0$, and ϕ is the unique solution of the first-order linear equation

$$2\dot{x}\dot{\phi} = [(2m-1)\dot{x}^2 + 1]\,\phi + 2e^{-x} - 2m \qquad (38)$$

with the initial conditions $\phi(0) = \phi_0$, $\dot{\phi}(0) = \dot{\phi}_0$.

The linear equation (38) has an obvious integrating factor, which allows us to rewrite it, for t with $\Theta(t)\dot{\Theta}(t) \neq 0$, as $(G\phi)^{\cdot} = F$, where

$$G \;=\; 2(\Theta^{2m-1}\dot{\Theta})^{-1}, \qquad F \;=\; (\Theta^m\dot{\Theta})^{-2}(e^{-x_0} - m\Theta^2). \qquad (39)$$

Proof. Our system consists of (30), (31) and the condition $y = 0$ (which, due to (30.i) and (26.b), imply the equality $\ddot{\sigma} = \dot{\sigma}$). In other words, we have two unknown functions, x and ϕ, subject to just two equations: $2\ddot{x} = 1 - \dot{x}^2$ and (38); note that (30.iii) follows from them. Furthermore, the existence and uniqueness of a solution ϕ to (38) with the stated initial conditions are obvious; this is so even in the singular case, that is, when $\dot{x}_0 = 0$, as one easily verifies using the integrating factor $[\cosh(t/2)]^{-2m}\coth(t/2)$. □

23. Page's and Bérard Bergery's examples

Having fixed $m, k \in \mathbb{Z}$ such that $m > k > 0$ and $a \in (-1, 0)$, we define the data (35) with (36), functions Θ, G, F, and a constant $T > 0$, by $x_0 = -\log(ka+m)$, $y_0 = \phi_0 = \dot{y}_0 = 0$, $\dot{x}_0 = a$, $\dot{\phi}_0 = k$, formulae (37) and (39), and $T = 2\log[(1-a)/(1+a)]$. Thus

$$\begin{array}{lrcl} \text{i)} & 2\Theta(t) & = & (1+a)e^{t/2} + (1-a)e^{-t/2} > 0, \\ \text{ii)} & 4\dot{\Theta}(t) & = & (1+a)e^{t/2} - (1-a)e^{-t/2}, \\ \text{iii)} & \Theta^2 - 4\dot{\Theta}^2 & = & 1 - a^2 > 0, \qquad 4\ddot{\Theta} = \Theta > 0, \\ \text{iv)} & F & = & [ka + m - m\Theta^2](\Theta^m\dot{\Theta})^{-2}. \end{array} \tag{40}$$

For the polynomial S in the variables a, ξ given by

$$S(a,\xi) = \sum_{j=0}^{m} \frac{(-1)^j \xi^{2j}}{2j-1}\left[\tbinom{m-1}{j}(ma+k)a + \tbinom{m-1}{j-1}(ka+m)\right],$$

where $\binom{m-1}{j} = 0$ if $j < 0$ or $j \geqslant m$, the expression $P(a) = a^{-1}S(a,a)$ is a polynomial in the variable a, for which

$$\begin{aligned} &P(0) = -k < 0, \\ &P(-k/m) = (1-k/m)(1+m/k)\int_0^{k/m}(1-a^2)^{m-1}\,da > 0. \end{aligned} \tag{41}$$

The first equality in (41) is obvious due to the definition of P, while the second one is easily obtained from our formula for $S(a,\xi)$ by using a binomial expansion of the integrand and the fact that the factor $ma+k$ in $S(a,\xi)$ vanishes when $a = -k/m$.

As a consequence of (41), we may now fix $a \in (-k/m, 0)$ such that $P(a) = 0$, that is, $S(a,a) = 0$. Since $S(a,\xi)$ is an even function of ξ, we also get $S(a,-a) = 0$.

Replace $t \in [0,\infty)$ by the new variable $\xi = 2\dot{\Theta}(t)/\Theta(t) \in [a,1)$. From (40.i), (40.iii) and (40.i-ii) it follows, respectively, that this makes sense, and that $|\xi| < 1$ and $2\dot{\xi} = 1 - \xi^2 > 0$, while $\xi \to 1$ as $t \to \infty$. Thus, $t \mapsto \xi$ is a diffeomorphism of the interval $[0,\infty)$ onto $[a,1)$, easily verified – if one uses (40.i-ii) again – to send $t = 0$ and $t = T$, for our positive constant $T = 2\log[(1-a)/(1+a)]$, to $\xi = a$ and, respectively, $\xi = -a$. The first part of (40.iii) gives $\xi^2 = 1 + (a^2-1)\Theta^{-2}$, that is, $\Theta^2 = (1-a^2)/(1-\xi^2)$, and $4\dot{\Theta}^2 = (1-a^2)\xi^2/(1-\xi^2)$ (since $4\dot{\Theta}^2/\Theta^2 = \xi^2$). Expressing G and F through ξ, we obtain, from (39),

$$G = \frac{4(1-\xi^2)^m}{(1-a^2)^m\xi}, \tag{42}$$

as $\Theta^{2m-1}\dot\Theta = \Theta^{2m}\dot\Theta/\Theta = (\Theta^2)^m\xi/2$, and, in view of (40.iv),

$$F \,=\, 4(1-a^2)^{-m-1}[(ma+k)a\xi^{-2} - (ka+m)](1-\xi^2)^m. \qquad (43)$$

The above formula for $S(a,\xi)$ and (43) easily show that

$$4(1-\xi^2)\, d[S(a,\xi)/\xi]/d\xi \;=\; (1-a^2)^{m+1}F, \qquad (44)$$

The relation $F = (G\phi)^{\cdot}$ in Theorem 22.1 amounts to the equality $2F = (1-\xi^2)d(G\phi)/d\xi$. It is therefore satisfied by ϕ such that

$$(1-a^2)(1-\xi^2)\phi \;=\; 2S(a,\xi). \qquad (45)$$

Consequently, our choice of a causes this function ϕ of the variable ξ to vanish for $\xi = \pm a$. In other words, treating ϕ as a function of t, we have $\phi(0) = \phi(T) = 0$, where $T = 2\log[(1-a)/(1+a)]$.

Next, ϕ given by (45) also satisfies the remaining boundary conditions (32). Namely, the definition of $S(a,\xi)$ and (42) – (43) yield

$$(1-a^2)^{m+1}(1-\xi^2)^{-m}\xi^2(F \pm kG) \,=\, -4(\xi \pm a)[(ka+m)\xi \mp (ma+k)],$$

and so $F/G = \pm k$ for $\xi = \pm a$. Switching to the variable t, we obtain $F(0)/G(0) = k$ and $F(T)/G(T) = -k$. The equalities $\phi(0) = \phi(T) = 0$ and $F = (G\phi)^{\cdot}$ now show that $\dot\phi(0) = k$ and $\dot\phi(T) = -k$.

Positivity of ϕ on $(0,T)$ is thus reduced to nonvanishing of ϕ when $\xi \in (a,-a)$. If, however, ϕ vanished for some $\xi \in (a,-a)$, (45) would – first – yield $\xi \neq 0$ (as the definition of $S(a,\xi)$ and the inequalities $-k/m < a < 0$ give $S(a,0) = -(ma+k)a > 0$) and – secondly – allow us to assume that $\xi \in (a,0)$ (since $S(a,\xi)$ is an even function of ξ). Vanishing of ϕ, and consequently of $S(a,\xi)/\xi$, for both this value of ξ and for $\xi = a$, would imply vanishing of the derivative $d[S(a,\xi)/\xi]/d\xi$ somewhere in $[a,0)$ which, combined with (44), leads to a contradiction: by (43), $F < 0$ when $a \in (-k/m,0)$ and $0 < |\xi| < 1$.

Theorem 20.1 now implies that the solution (x,y,ϕ) of (30) – (31), corresponding to the above data (35), with our $a \in (-k/m,0)$, allows us to construct a Kähler-Ricci soliton on each of the compact complex manifolds M_k^m. We thus obtain the *examples found by Page* (for $m = 2$) *and by Bérard Bergery* (if $m > 2$).

The above constructions can also be found in Besse's book [5, pp. 273–275] and, obviously, the original papers [4, 28].

24. Case (ii) in Lemma 21.1

Constancy of σ in Lemma 21.1 implies that $\bar g = g/\sigma^2$ is a Kähler metric, since – in the local version of Theorem 20.1 – it arises from a *trivial* conformal change of the Kähler metric g.

In the following description of solutions (x, y, ϕ) to (30) – (31) with a constant function σ we are assuming, without loss of generality (cf. §25) that the domain interval contains 0.

Theorem 24.1. *If a solution (x, y, ϕ) of (30) – (31) defined on an open interval containing 0 has the property that the function σ is constant, then the initial data (35) must satisfy the conditions*

$$\begin{aligned} &\dot{y}_0 - \dot{x}_0 = 1, \qquad (1 - me^{x_0 - y_0})\dot{y}_0 = y_0, \\ &\dot{\phi}_0 = [(m-1)\dot{y}_0 - m]\phi_0 + m - e^{y_0 - x_0}. \end{aligned} \tag{46}$$

Conversely, any data (35) with (46) are realized by a unique solution (x, y, ϕ) of (30) – (31) with a constant function σ, defined on the whole real line. Explicitly,

$$x = x_0 - t + \dot{y}_0(e^t - 1), \qquad y = y_0 + \dot{y}_0(e^t - 1), \tag{47}$$

and ϕ is the unique solution of the first-order linear equation

$$\dot{\phi} = [(m-1)\dot{y}_0 e^t - m]\phi + m - e^{y_0 - x_0 + t} \tag{48}$$

with the initial conditions $\phi(0) = \phi_0$, $\dot{\phi}(0) = \dot{\phi}_0$.

Equation (48) may also be expressed as $(G\phi)^{\cdot} = F$, where

$$G(t) = \exp[mt - (m-1)\dot{y}_0 e^t], \quad F(t) = (m - e^{y_0 - x_0 + t})G(t). \tag{49}$$

Proof. Constancy of σ and (26.a) give $\dot{y} - \dot{x} = 1$, so that, by (33.ii), $\ddot{y} = \dot{y}$, which yields (47) and the first equality in (46).

To prove (48), consider two possible cases. In the first one, $\dot{y}_0 = 0$. From (47) we thus obtain constancy of y and the equality $x = x_0 - t$. Using (30.ii) we see that $y = 0$. Conclusion (38) in Theorem 22.1 for $x = x_0 - t$ now implies (48) for $y_0 = \dot{y}_0 = 0$.

In the remaining case, $\dot{y}_0 \neq 0$. Dividing (30.ii) by $\dot{y} = \ddot{y} = \dot{y}_0 e^t \neq 0$, and then replacing $\dot{x}$ (or, ye^{y-x}) with $\dot{y}_0 e^t - 1$ (or, respectively, $[y_0 + \dot{y}_0(e^t - 1)]e^{y_0 - x_0 + t}$), we obtain

$$\dot{\phi} = [(m-1)\dot{y}_0 e^t - m]\phi - e^{y_0 - x_0}(e^t - 1 + y_0/\dot{y}_0). \tag{50}$$

Let us use (50) and the equation obtained by differentiating (50) to express $\dot{\phi}$ and $\ddot{\phi}$ in terms of $x_0, y_0, \dot{y}_0$ and ϕ. The resulting expressions and the equality $x = \dot{y}_0 e^t - 1$ allow us to rewrite the difference between the two sides of (30.iii), so as to obtain both the second equality of (46), and (48). We also see that the second equality in (46), combined with (48), implies (50), and hence (30.iii). The third equality of (46) is in turn obvious from (48) for $t = 0$.

We have thus proved (46) and (48), as well as the fact that they imply (30.iii). They similarly yield the remaining equations in (30) – (31).

More precisely, (30.ii) is nothing else than (50) (that is, (48)) multiplied by $\dot{y} = \ddot{y} = \dot{y}_0 e^t \neq 0$, (30.i) is an obvious consequence of (47) – (48), while (31) can easily be verified directly. □

25. The Koiso-Cao examples

We fix $m, k \in \mathbb{Z}$ with $m > k > 0$ and define $S : (0, \infty) \to \mathrm{I\!R}$ by

$$S(a) \;=\; \int_0^Q [\tau^{m-1} + (k-m)\tau^m] e^{-a\tau}\, d\tau, \quad \text{where } Q \;=\; \frac{m+k}{m-k}\,. \tag{51}$$

One then has – as shown below – the inequalities

$$S(0) \;<\; 0 \;<\; S(a) \quad \text{whenever} \quad a \geqslant m(m-k). \tag{52}$$

We may thus choose $a \in (0, m(m-k))$ such that $S(a) = 0$. Next, we introduce data (35) satisfying (46) by setting $y_0 = (1 - m/k)a/(m-1)$, $x_0 = y_0 - \log(m-k)$, $\phi_0 = 0$, $\dot{y}_0 = a/(m-1)$, $\dot{x}_0 = \dot{y}_0 - 1$, and $\dot{\phi}_0 = k$. The solution $(x, y, \phi) : \mathrm{I\!R} \to \mathrm{I\!R}^3$ of (30) – (31) with a constant function σ, corresponding to these data as in Theorem 24.1, then also satisfies the boundary conditions (32) with $T = \log[(m+k)/(m-k)]$. In fact – first, our choice of ϕ_0 and $\dot{\phi}_0$ gives $\phi(0) = 0$ and $\dot{\phi}(0) = k$. Second, $\phi(T) = 0$ since, due to Theorem 24.1, $G(T)\phi(T) = \int_0^T F(t)\, dt$ while, evaluating the last integral in terms of the new variable $\tau = e^t$ and using the equality $\dot{y}_0 = a/(m-1)$ along with (51) and (49), as well as our choice of a, we obtain $\int_0^T F(t)\, dt = S(a) = 0$. Third, the equalities $(G\phi)^{\cdot} = F$ (in Theorem 24.1), $\phi(T) = 0$ and (49) imply that $\dot{\phi}(T) = F(T)/G(T) = m - e^{y_0 - x_0 + T}$, and so $\dot{\phi}(T) = -k$, since $e^{y_0 - x_0} = m - k$ and $e^T = (m+k)/(m-k)$. Fourth, the derivative of $G\phi$, that is, the function F given by (49), has only one zero in $\mathrm{I\!R}$ and, consequently, ϕ may have at most two zeros; the relations $\phi(0) = \phi(T) = 0 < \dot{\phi}(0)$ thus imply the inequality $\phi > 0$ on $(0, T)$.

Theorem 20.1 states in turn that the above solution (x, y, ϕ) leads to a construction of a Kähler-Ricci soliton on each of the compact complex manifolds M_k^m. These are the *Koiso-Cao examples.*

Proof of the inequality (52). For $m, k, a, Q \in \mathrm{I\!R}$ such that $m \geqslant 1$ and $Q \in (0, \infty)$, defining $S(a)$ by (51) (even without assuming that $Q = (m+k)/(m-k)$), we have $S(a) = H_{m-1} + (k-m)H_m$, where $H_m = \int_0^Q \tau^m e^{-a\tau}\, d\tau$ for $m \geqslant 0$. If $m \geqslant 1$, integration by parts yields $mH_{m-1} = aH_m + Q^m e^{-aQ}$, and so $mS(a) = [a - m(m-k)]H_m + Q^m e^{-aQ}$. Positivity of H_m thus gives $S(a) > 0$ for $a \geqslant m(m-k)$. However, $H_m = Q^{m+1}/(m+1)$ if $a = 0$ and $m \geqslant 1$. Our formula for $mS(a)$

now states that $m(m+1)Q^{-m}S(0) = -m(m-k)Q + m + 1$. With $Q = (m+k)/(m-k)$ and $m > k \geqslant 1$, it follows that $S(0) < 0$. $\square$

26. Case (iii) in Lemma 21.1

The assumptions of Lemma 17.1 will still hold if the quintuple m, k, T, x, ϕ is replaced by $m, k, T, \hat{x}, \hat{\phi}$, where $\hat{x}(t) = x(T-t)$ and $\hat{\phi}(t) = \phi(T-t)$. Performing the construction described in the proof of Lemma 17.1 for $m, k, T, \hat{x}, \hat{\phi}$, with the same objects $N, h, \mathcal{E}, M$, the same fibre norm, and the same Hermitian connection in $\mathcal{E}$ as before, we obtain a new metric $\hat{g}$ and function $\hat{t} : M \to \mathrm{I\!R}$, for which we must use a symbol other than t, since it differs in general from the function $t : M \to \mathrm{I\!R}$ associated with the original data m, k, T, x, ϕ. To describe how the pair (g, t) is related to $(\hat{g}, \hat{t})$, we use the diffeomorphism $Z : M \to M$ which, restricted to $\mathcal{E} \smallsetminus N$, acts in every fibre as the standard inversion, that is, the division of any nonzero vector by the square of its norm. Letting $Z^*\hat{g}$ and $Z^*\chi$ denote the pullback of the metric $\hat{g}$ under Z and the composition $\chi \circ Z$, for any function χ on $\mathcal{E} \smallsetminus N$, we then have

$$\text{a)}\quad Z^*r \,=\, 1/r\,, \qquad \text{b)}\quad Z^*\hat{t} = T - t\,, \qquad \text{c)}\quad Z^*\hat{g} = e^{T-2t}g\,, \tag{53}$$

where, as before, $r : \mathcal{E} \smallsetminus N \to \mathrm{I\!R}$ is the fibre norm.

In fact, (53.a) is a trivial consequence of the definition of Z. To justify (53.b) – (53.c), note that (20) remains valid after one has replaced $\phi(t)$ by $\hat{\phi}(t) = \phi(T-t)$ and $t(r)$ by $\hat{t}(r) = T - t(1/r)$. This last choice of $\hat{t} : M \to \mathrm{I\!R}$ easily yields (53.b). The metric $\hat{g}$ arises, in turn, from a modified version of the formulae for g in the proof of Lemma 17.1; the modification amounts to using $e^{\hat{t}(r)}$ and $\hat{\phi}(\hat{t}(r)) = \phi(t(1/r))$ instead of e^t and ϕ (that is, instead of $e^{t(r)}$ and $\phi(t(r))$), which gives (53.c).

Relation (53.c) states that Z is an isometry between the Riemannian manifolds $(M, e^{T-2t}g)$ and $(M, \hat{g})$. Thus, $\hat{g}$ is isometric to a metric resulting from a specific conformal change of g.

If, in addition, the original data m, k, T, x, ϕ arise from a solution (x, y, ϕ) of (30) – (32), the function $\sigma : M \to \mathrm{I\!R}$ is defined by (22.i), and $\hat{\sigma}$ denotes its analog for the new solution $(\hat{x}, \hat{y}, \hat{\phi})$ obtained by substituting $T - t$ for the variable t, then

$$\text{i)}\quad Z^*\hat{\sigma} \,=\, e^{-t+T/2}\sigma\,, \qquad \text{ii)}\quad Z^*(\hat{g}/\hat{\sigma}^2) \,=\, g/\sigma^2. \tag{54}$$

The first equality is obvious here due to (22.i) and (53.b), the second – in view of the first one combined with (53.c) and multiplicativity of the operation Z^* with respect to products of functions and tensor fields.

By (54.ii), applying Theorem 20.1 to these solutions (x, y, ϕ) and $(\hat{x}, \hat{y}, \hat{\phi})$ results in *compact Ricci solitons* $(M, g/\sigma^2)$ *and* $(M, \hat{g}/\hat{\sigma}^2)$, *which are isometric to each other.*

If, furthermore, (x, y, ϕ) represents case (iii) in Lemma 21.1 then, according to the final clause of Lemma 21.1, the solution $(\hat{x}, \hat{y}, \hat{\phi})$ is an example of case (ii), and so the manifold $(M, \hat{g}/\hat{\sigma}^2)$ must be isometric to one of the Koiso-Cao examples. The same therefore holds for $(M, g/\sigma^2)$. We have in this way obtained a proof of Maschler's result [27].

Since the standard inversion of the plane is not holomorphic (while being antiholomorphic), Z transforms the original complex structure onto a different one, biholomorphic to it.

References

[1] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. (2) 102 (1978), no. 1, 63–95.

[2] S. Bando, T. Mabuchi, *Uniqueness of Einstein Kähler metrics modulo connected group actions*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, 11–40.

[3] A. Barros, E. Ribeiro, *Some characterizations for compact almost Ricci solitons*, Proc. Amer. Math. Soc. (electronic), posted on July 22, 2011, DOI 10.1090/S0002-9939-2011-11029-3.

[4] L. Bérard-Bergery, *Sur de nouvelles variétés riemanniennes d'Einstein*, Institut Élie Cartan, 6, Inst. Élie Cartan, vol. 6, Univ. Nancy, Nancy, 1982, 1–60.

[5] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987.

[6] J. P. Bourguignon, *L'espace des métriques riemanniennes d'une variété compacte*, Thèse d'Etat, Université Paris VII (1974).

[7] E. Calabi, *On Kähler manifolds with vanishing canonical class*, Algebraic geometry and topology. A symposium in honor of S. Lefschetz, Princeton University Press, Princeton, NJ, 1957, 78–89.

[8] H.-D. Cao, *Existence of gradient Kähler-Ricci solitons*, Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), A. K. Peters, Wellesley, MA, 1996, 1–16.

[9] H.-D. Cao, *Recent progress on Ricci solitons*, Recent advances in geometric analysis, Adv. Lect. Math. (ALM), vol. 11, Int. Press, Somerville, MA, 2010, 1–38.

[10] B. Chow, *Ricci flow and Einstein metrics in low dimensions*, Surveys in differential geometry: essays on Einstein manifolds, Surv. Differ. Geom., VI, Int. Press, Boston, MA, 1999, 187–220.

[11] A. Dancer, M. Wang, *On Ricci solitons of cohomogeneity one*, Ann. Global Anal. Geom. 39 (2011), no. 3, 259–292, DOI 10.1007/s10455-010-9233-1.

[12] A. Derdzinski, G. Maschler, *Local classification of conformally-Einstein Kähler metrics in higher dimensions*, Proc. London Math. Soc. (3) 87 (2003), no. 3, 779–819, DOI 10.1112/S0024611503014175.

[13] A. Derdzinski, G. Maschler, *Special Kähler-Ricci potentials on compact Kähler manifolds*, J. Reine Angew. Math. 593 (2006), 73–116, DOI 10.1515/CRELLE.2006.030.

[14] F. J. E. Dillen, L. C. A. Verstraelen (eds.), *Handbook of differential geometry. Vol. I*, North-Holland, Amsterdam, 2000.

[15] P. Federbush, *Partially alternate derivation of a result of Nelson*, J. Mathematical Phys. 10 (1969), 50–52, DOI 10.1063/1.1664760.

[16] M. Feldman, T. Ilmanen, D. Knopf, *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Differential Geom. 65 (2003), no. 2, 169–209.

[17] M. Fernández-López, E. García-Río, *A remark on compact Ricci solitons*, Math. Ann. 340 (2008), no. 4, 893–896, DOI 10.1007/s00208-007-0173-4.

[18] D. H. Friedan, *Nonlinear models in* $2+\varepsilon$ *dimensions*, Ann. Physics 163 (1985), no. 2, 318–419.

[19] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. 97 (1975), no. 4, 1061–1083.

[20] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), no. 2, 255–306.

[21] R. S. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), Contemp. Math., vol. 71, Amer. Math. Soc., Providence, RI, 1988, 237–262.

[22] T. Ivey, *Ricci solitons on compact three-manifolds*, Differential Geom. Appl. 3 (1993), no. 4, 301–307, DOI 10.1016/0926-2245(93)90008-O.

[23] S. Kobayashi, S. Ochiai, *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ. 13 (1973), 31–47.

[24] N. Koiso, *On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics*, Recent topics in differential and analytic geometry, Adv. Stud. Pure Math., vol. 18, Academic Press, Boston, MA, 1990, 327–337.

[25] C. Li, *On rotationally symmetric Kähler-Ricci solitons*, preprint arXiv:1004.4049 (2010).

[26] X.-M. Li, *On extensions of Myers' theorem*, Bull. London Math. Soc. **27** (1995), 392–396.

[27] G. Maschler, *Special Kähler-Ricci potentials and Ricci solitons*, Ann. Global Anal. Geom. 34 (2008), no. 4, 367–380, DOI 10.1007/s10455-008-9114-z.

[28] D. N. Page, *A compact rotating gravitational instanton*, Phys. Lett. B 79 (1978), no. 3, 235–238, DOI 10.1016/0370-2693(78)90231-9.

[29] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, preprint, arXiv:math.DG/0211159.

[30] G. Perelman, *Ricci flow with surgery on three-manifolds*, preprint, arXiv:math.DG/0303109.

[31] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, preprint, arXiv:math.DG/0307245.

[32] S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti, *Ricci almost solitons*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 10 (2011), no. 4, 757–799, DOI 10.2422/2036-2145.2011.4.01.

[33] O. S. Rothaus, *Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators*, J. Funct. Anal. 42 (1981), no. 1, 110–120, DOI 10.1016/0022-1236(81)90050-1.

[34] G. Tian, private communication (2005).

[35] G. Tian, X. Zhu, *A new holomorphic invariant and uniqueness of Kähler-Ricci solitons*, Comment. Math. Helv. 77 (2002), no. 2, 297–325, DOI 10.1007/s00014-002-8341-3.

[36] X.-J. Wang, X. Zhu, *Kähler-Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math. 188 (2004), no. 1, 87–103, DOI 10.1016/j.aim.2003.09.009.

[37] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339–411, DOI 10.1002/cpa.3160310304.

[38] Z. Zhang, *On the finiteness of the fundamental group of a compact shrinking Ricci soliton*, Colloq. Math. 107 (2007), no. 2, 297–299, DOI 10.4064/cm107-2-9.

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