Special Ricci-Hessian equations on Kähler manifolds

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Abstract. Special Ricci-Hessian equations on Kähler manifolds (M, g), as defined by Maschler [Ann. Global Anal. Geom. 34 (2008), 367–380] involve functions τ on M and state that, for some function α of the real variable τ, the sum of $\alpha\nabla d\tau$ and the Ricci tensor equals a functional multiple of the metric g, while $\alpha\nabla d\tau$ itself is nonzero almost everywhere. Three well-known obvious cases are provided by (non-Einstein) gradient Kähler-Ricci solitons, conformally-Einstein Kähler metrics, and special Kähler-Ricci potentials. We show that, outside of these three cases, such an equation can only occur in complex dimension two and, at generic points, it must then represent one of three types, for which, up to normalizations, $\alpha = 2\cot\tau$, or $\alpha = 2\coth\tau$, or $\alpha = 2\tanh\tau$. We also use the Cartan-Kähler theorem to prove that these three types are actually realized.

Introduction

Following Maschler [**17**, p. 367], one says that functions τ, α, σ on a Riemannian manifold (M, g) with the Ricci tensor r satisfy a *Ricci-Hessian equation* if

$$\alpha\nabla d\tau + \mathrm{r} = \sigma g \quad \text{for some function } \sigma : M \to \mathrm{I\!R}, \tag{0.1}$$

∇ being the Levi-Civita connection of g. We call equation (0.1) *special* when

$$\alpha\nabla d\tau \neq 0 \text{ on a dense set, } \dim M = n > 2, \text{ and } \alpha \text{ is a } C^\infty \text{ function of } \tau. \tag{0.2}$$

Conditions (0.1) – (0.2) are satisfied in several situations that have been studied – see below – raising a natural question: *Which functions $\tau \mapsto \alpha$ can be realized in this way?* The present paper provides an answer in the Kähler case, outside of the classes that are already well understood. See Theorems D and E.

There are three well-known classes of examples leading to (0.1) – (0.2).

- (I) Non-Einstein gradient Ricci almost-solitons [**20, 1**], including (non-Einstein) gradient Ricci solitons [**15**]. Here α is a nonzero constant.
- (II) Conformally-Einstein metrics g, with $\tau > 0$ and $\alpha = (n-2)/\tau$, the Einstein metric being $\hat{g} = g/\tau^2$. Cf. [**11**, formula (6.2)].
- (III) Special Kähler-Ricci potentials τ on Kähler manifolds, at points where r is not a multiple of g [**11**, Remark 7.4].

1991 *Mathematics Subject Classification.* Primary 53C55 and Secondary 53C25.

Key words and phrases. Ricci-Hessian equation, Kähler manifold.

Both authors' research was supported in part by a FAPESP-OSU 2015 Regular Research Award (FAPESP grant: 2015/50265-6).

A *special Kähler-Ricci potential* [**11**, Sect. 7] on a Kähler manifold (M,g) with the complex-structure tensor J is any nonconstant function τ on M having a real-holomorphic gradient $v=\nabla\tau$ for which, at points where $v\neq 0$, all nonzero vectors orthogonal to v and Jv are eigenvectors of both $\nabla d\tau$ and r. Such triples (M,g,τ) are completely understood, both locally [**11**] and in the compact case [**12**].

The classes (I) – (III) are far from disjoint: for instance [**11**, Corollary 9.3], in the Kähler category, if $n>4$, (II) is a special case of (III).

We are interested in M,g,τ,α,σ satisfying (0.1) – (0.2) along with

$$2\mathrm{r}(v,\,\cdot\,)=-dY, \tag{0.3}$$

where, throughout the paper, the notational conventions

$$v=\nabla\tau,\qquad Q=g(v,v),\qquad Y=\Delta\tau,\qquad n=\dim M \tag{0.4}$$

are used whenever $\tau:M\to\mathrm{I\!R}$ for a Riemannian manifold (M,g). As we point out near at the end of Section 1, with J denoting the complex-structure tensor,

$$\begin{array}{l}\text{for Kähler metrics } g,\text{ conditions (0.1) – (0.2) imply (0.3),}\\ \text{and the gradient } v=\nabla\tau \text{ is a real-holomorphic vector field}\\ \text{or, equivalently, } Jv \text{ is a real-holomorphic } g\text{-Killing field.}\end{array} \tag{0.5}$$

Assuming (0.1) – (0.2), we may treat the derivatives $\alpha'=d\alpha/d\tau$ and α'' both as functions of the real variable τ and as functions $M\to\mathrm{I\!R}$. In Sections 2 and 3 we prove the following two results, as well as Theorem D, stated below.

Theorem A. *Under the hypotheses* (0.1) – (0.3)*, at points where* $\alpha''+\alpha\alpha'\neq 0$ *and* $d\tau\neq 0$*, both* $Q=g(\nabla\tau,\nabla\tau)$ *and* $Y=\Delta\tau$ *are, locally, functions of* τ*.*

Theorem B. *Let functions* τ,α,σ *satisfy a special Ricci-Hessian equation* (0.1)*, with* (0.2)*, on a Kähler manifold* (M,g) *of real dimension* $n\geq 4$*. If* $\alpha\,d\alpha$ *and* $d\tau$ *are nonzero at all points of an open submanifold* U *of* M*, and*

(i) $n>4$*, or*
(ii) $n=4$ *and* $d\sigma\wedge d\tau=0$ *identically in* U *or, finally,*
(iii) $dQ\wedge d\tau=0$ *everywhere in* U*, where* $Q=g(\nabla\tau,\nabla\tau)$*,*

then $\tau:U\to\mathrm{I\!R}$ *is a special Kähler-Ricci potential on the Kähler manifold* (U,g)*.*

With v,Q,Y as in (0.4), a function τ on a Riemannian manifold (M,g) has $dQ\wedge d\tau=0$ if and only if Q is locally, at points where $d\tau\neq 0$, a function of τ. This amounts to requiring the integral curves of v to be reparametrized geodesics (since, due to formula (1.2) below, the latter condition means that $\nabla_{\!v}v$ is a functional multiple of v). Such functions τ, called *transnormal*, have been studied extensively [**21, 18, 3**], and are referred to as *isoparametric* when, in addition, $dY\wedge d\tau=0$.

Theorem B renders the transnormal case $dQ\wedge d\tau=0$, as well as real dimensions $n>4$, rather uninteresting in the context of special Ricci-Hessian equations (0.1) – (0.2) on Kähler manifolds, since at $d\alpha$-generic points (see the end of Section 1) one then ends up with examples (I) or (III) above, cf. Remark 3.3, of which the former is the subject of a large existing literature, and the latter, as mentioned earlier, has been completely described. This is why our next two results focus exclusively on the real dimension four and functions τ with $dQ\wedge d\tau$ not identically zero.

Remark C. Equation (0.1), with (0.2), remains satisfied after τ and the function $\tau\mapsto\alpha=\alpha(\tau)$ have been subjected to an *affine modification* in the sense of being replaced with $\hat\tau$ and $\hat\tau\mapsto\hat\alpha(\hat\tau)$ given by $\hat\tau=p+\tau/c$ and $\hat\alpha(\hat\tau)=c\alpha(c\hat\tau-cp)$ for real constants $c\neq 0$ and p.

Theorem D. *If the special Ricci-Hessian equation* (0.1) *and* (0.2) *both hold for functions* τ, α, σ *on a Kähler manifold* (M, g) *of real dimension four, while* $dQ \wedge d\tau \neq 0$ *everywhere in an open connected set* $U \subseteq M$*, then the function* α *of the variable* τ *and its derivative* $\alpha' = d\alpha/d\tau$ *satisfy, on* U*, the equation*

$$\alpha'' + \alpha\alpha' = 0, \text{ that is, } 2\alpha' + \alpha^2 = 4\varepsilon \text{ with a constant } \varepsilon \in \mathrm{I\!R}. \tag{0.6}$$

In addition, for Q *and* Y *as in* (0.4)*, the functions*

$$2\theta = \alpha \mathrm{s} + 4\varepsilon Y \quad\text{and}\quad \kappa = \theta\psi + \alpha^{-1}Y - Q \quad\text{are both constant,} \tag{0.7}$$

ψ *being given by* $4\varepsilon\psi = \tau - 2/\alpha$*, if* $\varepsilon \neq 0$*, or* $3\psi = 2/\alpha^3$*, when* $\varepsilon = 0$*. Furthermore,* σ *in* (0.1) *and the function* F *of the variable* τ *characterized by*

$$4\varepsilon F = \theta(2 - \tau\alpha) + 4\varepsilon\kappa\alpha \text{ for } \varepsilon \neq 0, \text{ and } F = \kappa\alpha - 2\theta/(3\alpha^2) \text{ if } \varepsilon = 0, \tag{0.8}$$

and thus depending on the real constants θ, κ*, satisfy the conditions*

$$\text{a) } Y - Q\alpha = F, \quad \text{b) } 2\sigma = -(Q\alpha' + F'), \quad \text{c) } \Delta\alpha = F\alpha' = -F''. \tag{0.9}$$

Finally, up to affine modifications – see Remark C *– the pair* $(\alpha(\tau), \varepsilon)$ *is one of the following five*: $(2, 1)$, $(2/\tau, 0)$, $(2\tanh\tau, 1)$, $(2\coth\tau, 1)$, $(2\cot\tau, -1)$.

Theorem E. *Each of the five options listed in Theorem* D*, namely,*

$$(2, 1), \quad (2/\tau, 0), \quad (2\tanh\tau, 1), \quad (2\coth\tau, 1), \quad (2\cot\tau, -1),$$

is realized by a special Ricci-Hessian equation (0.1) – (0.2) *on a real-analytic Kähler manifold* (M, g) *of real dimension four such that, with* $v = \nabla\tau$ *and* $Q = g(v, v)$*, one has* $dQ \wedge d\tau \neq 0$ *somewhere in* M *and* Jv *lies in a two-dimensional Abelian Lie algebra of Killing fields.*

For $(2, 1)$ *and* $(2/\tau, 0)$ *one can choose* (M, g) *to be compact and biholomorphic to the two-point blow-up of* $\mathbb{C}\mathrm{P}^2$*, with* g *which is the Wang-Zhu toric Kähler-Ricci soliton* [**22**, Theorem 1.1] *or, respectively, the Chen-LeBrun-Weber conformally-Einstein Kähler metric* [**6**, Theorem A].

In contrast with the final clause of Theorem E, we do not know whether the remaining three options, $(2\tanh\tau, 1)$, $(2\coth\tau, 1)$ and $(2\cot\tau, -1)$, may be realized on a *compact* Kähler surface. An analytic-continuation phenomenon described below (Section 11) suggests that it might make sense to try obtaining such compact examples via small deformations of the Wang-Zhu or Chen-LeBrun-Weber metric, combined with suitable affine modifications.

For the pairs $(2, 1)$ and $(2/\tau, 0)$ in Theorem D, the constancy conclusions of (0.7) are well known [**7**, p. 201], [**9**, p. 417, Prop. 3(i) and p. 419, formula (40)].

The paper is organized as follows. Section 1 contains the preliminaries. Consequences of special Ricci-Hessian equations, leading to proofs of Theorems A, B and D, are presented in the next two sections. Sections 4 through 10 are devoted to proving Theorem E: we rephrase it as solvability of the system (5.1) – (5.2) of quasi-linear first-order partial differential equations, which allows us to derive our claim from the Cartan-Kähler theorem for exterior differential systems.

1. Preliminaries

All manifolds and Riemannian metrics are assumed to be of class C^∞. By definition, a manifold is connected. We use the symbol δ for divergence.

On a manifold with a torsion-free connection ∇, the Ricci tensor r satisfies the Bochner identity $\mathrm{r}(\,\cdot\,, v) = \delta\nabla v - d[\delta v]$, where v is any vector field. Its coordinate form $R_{jk}v^k = v^k{}_{,jk} - v^k{}_{,kj}$ arises via contraction from the Ricci identity $v^l{}_{,jk} - v^l{}_{,kj} = R_{jkq}{}^l v^q$. (We use the sign convention for R such that $R_{jk} = R_{jqk}{}^q$.) Applied to the gradient v of a function τ on a Riemannian manifold, this yields

$$\delta[\nabla d\tau] \,=\, \mathrm{r}(v,\,\cdot\,) \,+\, dY, \qquad \text{if} \;\; v = \nabla\tau \;\; \text{and} \;\; Y = \Delta\tau. \tag{1.1}$$

On the other hand, given a function τ on a Riemannian manifold,

$$2[\nabla d\tau](v,\,\cdot\,) \,=\, dQ, \qquad \text{where} \;\; v = \nabla\tau \;\; \text{and} \;\; Q = g(v,v), \tag{1.2}$$

as one sees noting that, in local coordinates, $(\tau_{,k}\tau^{,k})_{,j} = 2\tau_{,kj}\tau^{,k}$. We can rewrite relations (1.1) – (1.2) using the interior product $\imath_v$, obtaining

$$\text{a)}\;\; \delta[\nabla d\tau] \,=\, \imath_v\mathrm{r} \,+\, dY, \qquad \text{b)}\;\; 2\imath_v[\nabla d\tau] \,=\, dQ, \qquad \text{with (0.4).} \tag{1.3}$$

Finally, for the Ricci tensor r and scalar curvature s of any Riemannian metric,

$$2\delta\mathrm{r} \,=\, d\mathrm{s}, \tag{1.4}$$

which is known as the Bianchi identity for the Ricci tensor. Its coordinate form $2g^{kl}R_{jk,l} = s_{,j}$ is immediate if one transvects with ("multiplies" by) g^{kl} the equality $R_{jkl}{}^q{}_{,q} = R_{jl,k} - R_{kl,j}$ obtained by contracting the second Bianchi identity.

The *harmonic-flow condition* for a vector field v on a Riemannian manifold (M,g), meaning that the flow of v consists of (local) harmonic diffeomorphisms, is known [**19**] to be equivalent to the equation

$$g(\Delta v,\,\cdot\,) \,=\, -\mathrm{r}(v,\,\cdot\,) \tag{1.5}$$

the vector field Δv having the local components $[\Delta v]^j = v^{j,k}{}_k$. See also [**14**, Theorem 3.1]. When $v = \nabla\tau$ is the gradient of a function $\tau : M \to \mathrm{I\!R}$,

(1.6) the harmonic-flow condition (1.5) amounts to (0.3).

In fact, by (1.1), $2\mathrm{r}(v,\,\cdot\,) + dY = \delta[\nabla d\tau] + \mathrm{r}(v,\,\cdot\,) = g(\Delta v,\,\cdot\,) + \mathrm{r}(v,\,\cdot\,)$, as $[\Delta v]_j = v_{j,k}{}^k = \tau_{,jk}{}^k = \tau_{,kj}{}^k = \tau^{,kj}{}_k = (\delta[\nabla d\tau])_j$.

On the other hand – see, e.g., [**11**, Lemma 5.2] – on a Kähler manifold (M,g),

(1.7) conditions (0.1)–(0.2) imply real-holomorphicity of the gradient $v = \nabla\tau$, while Jv is then a holomorphic Killing field, due to the resulting Hermitian symmetry of $\nabla d\tau$.

Since holomorphic mappings between Kähler manifolds are harmonic, every holomorphic vector field on a Kähler manifold satisfies (1.5), cf. [**14**, Remark 3.2]. Now (0.5) follows from (1.6). Note that, as also also observed by Calabi [**5**], on Kähler manifolds one has

(1.8) equation (0.3), with (0.4), for all real-holomorphic gradients $v = \nabla\tau$.

Given a tensor field Θ on a manifold M, we say that a point $x \in M$ is *Θ-generic* if x has a neighborhood on which either $\Theta = 0$ identically, or $\Theta \neq 0$ everywhere. Such points clearly form a dense open subset of M.

2. Ricci-Hessian equations

As a consequence of (0.1), for the scalar curvature s, with (0.4),

$$n\sigma \,=\, Y\alpha \,+\, \mathrm{s}, \quad \text{where} \quad n = \dim M. \tag{2.1}$$

Applying $2\imath_v$ or 2δ to (0.1), we obtain, from (1.3) – (1.4) and (0.4),

$$\begin{aligned} &\text{i)} \quad \alpha dQ \,+\, 2\mathrm{r}(v,\,\cdot\,) \,=\, 2\sigma d\tau,\\ &\text{ii)} \quad 2[\nabla d\tau](\nabla\alpha,\,\cdot\,) \,+\, 2\alpha[\mathrm{r}(v,\,\cdot\,) \,+\, dY] \,+\, d\mathrm{s} \,=\, 2d\sigma. \end{aligned} \tag{2.2}$$

In the case where (0.1) – (0.2) hold along with (0.3), one may rewrite (2.2) as

$$\begin{aligned} &\text{i)} \quad \alpha dQ \,-\, dY \,=\, 2\sigma d\tau,\\ &\text{ii)} \quad 2[\nabla d\tau](\nabla\alpha,\,\cdot\,) \,+\, \alpha dY \,+\, d\mathrm{s} \,=\, 2d\sigma, \end{aligned} \tag{2.3}$$

which, in view of (1.2) and (2.1), amounts to nothing else than

$$\begin{aligned} &\text{i)} \quad d(Q\alpha \,-\, Y) \,=\, (Q\alpha' \,+\, 2\sigma)d\tau,\\ &\text{ii)} \quad d[Q\alpha' \,+\, (n-2)\sigma] \,=\, (Q\alpha'' \,+\, Y\alpha')d\tau, \end{aligned} \tag{2.4}$$

as the assumption, in (0.2), that α is a C^∞ function of τ allows us to write

$$d\alpha \,=\, \alpha' d\tau, \quad \nabla\alpha \,=\, \alpha' v, \quad 2[\nabla d\tau](\nabla\alpha,\,\cdot\,) = \alpha' dQ, \quad \text{where} \quad \alpha' = d\alpha/d\tau, \tag{2.5}$$

since (1.2) gives $2[\nabla d\tau](\nabla\alpha,\,\cdot\,) = 2\alpha'[\nabla d\tau](v,\,\cdot\,) = \alpha' dQ$. Due to (2.4), conditions (0.1) – (0.3) imply that, locally, at points at which $d\tau \neq 0$,

$$\begin{array}{l} Q\alpha - Y \text{ and } Q\alpha' + (n-2)\sigma \text{ are functions of } \tau, \text{ with}\\ \text{the respective } \tau\text{-derivatives } Q\alpha' + 2\sigma \text{ and } Q\alpha'' + Y\alpha'\\ \text{which, consequently, must themselves be functions of } \tau. \end{array} \tag{2.6}$$

Proof of Theorem A. At the points in question, using (2.6) to equate both $Q\alpha - Y$ and $Q\alpha'' + Y\alpha'$ to some specific functions of τ, we obtain a system of two linear equations with the nonzero determinant $\alpha'' + \alpha\alpha'$, imposed on the unknowns Q, Y, and our assertion follows since $\alpha'' + \alpha\alpha'$ is also a function of τ. □

Assuming only (0.1), for $n = \dim M$, with the aid of (2.1) we rewrite (2.2) as

$$\begin{aligned} &n[\alpha dQ \,+\, 2\mathrm{r}(v,\,\cdot\,)] \,-\, 2(Y\alpha \,+\, \mathrm{s})d\tau \,=\, 0,\\ &2n\{[\nabla d\tau](\nabla\alpha,\,\cdot\,) + \alpha\mathrm{r}(v,\,\cdot\,)\} + 2[(n-1)\alpha dY - Y d\alpha] + (n-2)d\mathrm{s} = 0, \end{aligned}$$

If (0.3) holds as well, replacing $2\mathrm{r}(v,\,\cdot\,)$ here with $-dY$ we obtain $n(\alpha dQ - dY) - 2(Y\alpha + \mathrm{s})d\tau = 0$ and $2n[\nabla d\tau](\nabla\alpha,\,\cdot\,) + (n-2)(\alpha dY + d\mathrm{s}) - 2Y d\alpha = 0$. Thus, when (0.1) – (0.3) are all satisfied, (2.5) gives

$$\begin{aligned} &\text{a)} \quad n(\alpha dQ \,-\, dY) \,-\, 2(Y\alpha \,+\, \mathrm{s})d\tau \,=\, 0,\\ &\text{b)} \quad n\alpha' dQ \,+\, (n-2)(\alpha dY \,+\, d\mathrm{s}) \,-\, 2Y\alpha' d\tau \,=\, 0. \end{aligned} \tag{2.7}$$

3. Ricci-Hessian equations on Kähler manifolds

The goal of this section is proving Theorems B and D.

In any complex manifold, $d\omega = 0$ and $\omega(J\,\cdot\,,\,\cdot\,)$ is symmetric if $\omega = i\partial\overline{\partial}\tau$, that is, if $2\omega = -d[J^*d\tau]$ for a real-valued function τ, with the 1-form $J^*d\tau = (d\tau)J$, which sends any tangent vector field v to $d_{Jv}\tau$. Our exterior-derivative and exterior-product conventions, for 1-forms ξ, ξ' and vector fields u, v, are

$$\begin{aligned} [d\xi](u,v) \,&=\, d_u[\xi(v)] \,-\, d_v[\xi(u)] \,-\, \xi([u,v]),\\ [\xi\wedge\xi'](u,v) \,&=\, \xi(u)\xi'(v) \,-\, \xi(v)\xi'(u). \end{aligned} \tag{3.1}$$

For a torsionfree connection ∇, (3.1) gives $[d\xi](u,v)=[\nabla_u\xi](v)-[\nabla_v\xi](u)$, so that, if in addition $\nabla J=0$, on a complex manifold,

$$2i\partial\overline{\partial}\tau \;=\; [\nabla d\tau](J\,\cdot\,,\,\cdot\,)\;-\;[\nabla d\tau](\,\cdot\,,J\,\cdot\,). \tag{3.2}$$

In the case of a Kähler metric g on a complex manifold M, (0.1) implies that

$$i\alpha\partial\overline{\partial}\tau\;+\;\rho\;=\;\sigma\omega, \tag{3.3}$$

ω,ρ being the Kähler and Ricci forms, with both terms on the right-hand side of (3.2) equal, as Hermitian symmetry of $\nabla d\tau$ follows from those of ρ and ω.

Remark 3.1. As an obvious consequence of the last line in (1.7), if g is a Kähler metric, conditions (0.1) – (0.2) are *equivalent* to (3.3) along with (0.2) and real-holomorphicity of the gradient $v=\nabla\tau$.

Remark 3.2. For the Kähler form ω of a Kähler manifold (M,g) of real dimension $n\ge 4$, the operator $\zeta\mapsto\zeta\wedge\omega$ acting on differential q-forms is injective if $q=2$ and $n>4$, or $q=1$. Namely, the contraction of $\zeta\wedge\omega$ against ω yields a nonzero constant times $(n-4)\zeta+2\langle\omega,\zeta\rangle\omega$ (if $q=2$), or times $(n-2)\zeta$ (if $q=1$). In the former case, ζ with $\zeta\wedge\omega=0$ is thus a multiple of ω, and hence 0.

Remark 3.3. Whenever (3.3) with a constant α holds on a Kähler manifold of real dimension $n\ge 4$, constancy of σ follows (from Remark 3.2, as $d\sigma\wedge\omega=0$).

We have the following result, due to Maschler [**17**, Proposition 3.3].

Lemma 3.4. *Condition* (0.1) *on a Kähler manifold* (M,g) *of real dimension* $n>4$ *implies that* $d\sigma\wedge d\alpha=0$. *Equivalently, wherever* $d\alpha$ *is nonzero,* σ *must, locally, be a function of* α.

Proof. By (3.3), $0=d\rho=d\sigma\wedge\omega-d\alpha\wedge i\,\partial\overline{\partial}\tau$, so that $d\alpha\wedge d\sigma\wedge\omega=0$, and our assertion is immediate from Remark 3.2. $\square$

Proof of Theorem B. In all three cases (i) – (iii), $d\sigma\wedge d\tau=0$. For (i), this follows from Lemma 3.4 while, when $dQ\wedge d\tau=0$ on U, we see that, in view of the equality $\alpha' dQ+\alpha\,dY=d(2\sigma-\mathrm{s})$ arising from (2.3.ii) and (2.5), Y and $2\sigma-\mathrm{s}$ are, locally, functions of τ, and hence so is σ, as a consequence of (2.1) with $n\ge 4$. Now [**11**, Corollary 9.2] yields our claim. $\square$

Proof of Theorem D. As $dQ\wedge d\tau\ne 0$ everywhere in U, Theorem A implies (0.6) and, consequently, also the final clause about the five possible pairs.

Next, in (0.6), $4\,d\theta=2d[\alpha\mathrm{s}+(2\alpha'+\alpha^2)Y]=2[\alpha\,d\mathrm{s}+\mathrm{s}\,d\alpha+(2\alpha'+\alpha^2)dY]$ which, as $n=4$, equals, in view of (2.5),

$$\alpha[n\alpha' dQ+(n-2)(\alpha\,dY+d\mathrm{s})-2Y\alpha' d\tau]\;-\;\alpha'[n(\alpha\,dQ-dY)-2(Y\alpha+\mathrm{s})\,d\tau],$$

and hence vanishes due to (2.7). On the other hand, the function ψ of τ defined in the theorem is an antiderivative of $1/\alpha^2$, meaning that

$$\psi'\;=\;1/\alpha^2. \tag{3.4}$$

Namely, by (0.6). $4\varepsilon\psi'=1+2\alpha'/\alpha^2=(2\alpha'+\alpha^2)/\alpha^2=4\varepsilon/\alpha^2$ if $\varepsilon\ne 0$, and $3\psi'=-6\alpha'/\alpha^4=3/\alpha^2$ when $\varepsilon=0$, as $2\alpha'=-\alpha^2$.

Furthermore, $d(\theta\psi+\alpha^{-1}Y-Q)=0$. In fact, $d\alpha=\alpha' d\tau$ in (2.5), and similarly for ψ, so that, from (3.4), $d(\theta\psi)=\theta\,d\psi=\theta\psi' d\tau=\theta\alpha^{-2}d\tau$, and $\alpha d(\alpha^{-1}Y)=dY-\alpha^{-1}Y\alpha' d\tau$. Also, $2(\theta-Y\alpha')=(Y\alpha+\mathrm{s})\alpha$ from (0.6) – (0.7). These relations yield $-4\alpha\, d(\theta\psi+\alpha^{-1}Y-Q)=4[(\alpha\, dQ-dY)-(\theta-Y\alpha')\alpha^{-1}d\tau]=n(\alpha\, dQ-dY)-2(Y\alpha+\mathrm{s})\,d\tau$, with $n=4$, which equals 0 by (2.7.a).

Finally, (0.8) and the second relation in (0.7) easily give (0.9-a). Thus, by (0.4) and (2.5), $(Q\alpha'+F')Q=(Q\alpha'+F')d_v\tau=Qd_v\alpha+d_vF$ which, due to (0.9-a), equals $d_vY-\alpha d_vQ$. At the same time, $-\imath_v$ applied to (2.3.i) yields $d_vY-\alpha d_vQ=-2Q\sigma$. We thus get (0.9-b). To obtain (0.9-c), note that, from (0.4), $\Delta\alpha=Q\alpha''+Y\alpha'$ which, by (0.6) and (0.9-a), equals $(Y-Q\alpha)\alpha'=F\alpha'=-F''$, where the last equality trivially follows from (0.8) $\square$

Theorem D has a partial converse: if a nonconstant function τ with real-holomorphic gradient $v=\nabla\tau$ on a Kähler surface (M,g) and a function α of the real variable τ satisfy (0.6) and (0.7), then they must also satisfy the Ricci-Hessian equation (0.1) with σ given by (2.1) for $n=4$.

In fact, $b(v,\,\cdot\,)=0$, where b denotes the traceless Hermitian symmetric 2-tensor field $\alpha\nabla d\tau+\mathrm{r}-\sigma g$. Namely, (0.3) – (0.5) and (1.3-b) yield $4b(v,\,\cdot\,)=2\alpha\, dQ-2\,dY-4\sigma\, d\tau$ which, due to (2.1) and (2.5), equals $2\alpha\, dQ-2\,dY-(Y\alpha+\mathrm{s})\,d\tau$, and so $-4\alpha b(v,\,\cdot\,)=2\alpha^2d(\theta\psi+\alpha^{-1}Y-Q)+(\alpha\mathrm{s}+4\varepsilon Y-2\theta)\,d\tau$. (Note that, by (0.6) and (3.4), $4\varepsilon=2\alpha'+\alpha^2$ and $2\alpha^2d(\theta\psi)=2\theta\, d\tau$.) Thus, $b=0$, since b corresponds, via g, to a *complex-linear* bundle morphism $TM\to TM$.

4. The local Kähler potentials

This and the following six sections are devoted to proving Theorem E.

In an open set $M\subseteq\mathbb{R}^4$ with the Cartesian coordinates x,x',u,u' arranged into the complex coordinates $(x+ix',u+iu')$ for the complex plane $\mathbb{C}^2=\mathbb{R}^4$ carrying the standard complex structure J, one has $J^*dx=-dx'$ and $J^*du=-du'$, so that, if a C^∞ function f on M only depends on x and u, the relation $2i\partial\bar\partial f=-d[J^*df]$ yields, with subscripts denoting partial differentiations, $2i\partial\bar\partial f=f_{xx}\,dx\wedge dx'+f_{xu}(dx\wedge du'+du\wedge dx')+f_{uu}\,du\wedge du'$, since $df=f_x\,dx+f_u\,du$. Furthermore, we set

$$v=\partial_x\quad\text{and}\quad w=\partial_u\quad\text{(the real coordinate vector fields).}\tag{4.1}$$

For the Kähler metric g on M having the Kähler form $\omega=2i\partial\bar\partial\phi$, where the function $\phi:M\to\mathbb{R}$ is assumed to depend on x and u only, 2ϕ is a Kähler potential [**2**, p. 85] of g, and the above formula for $2i\partial\bar\partial f$ becomes

$$\omega=\phi_{xx}\,dx\wedge dx'+\phi_{xu}(dx\wedge du'+du\wedge dx')+\phi_{uu}\,du\wedge du'.\tag{4.2}$$

Generally, for a skew-Hermitian 2-form $\zeta=Q\,dx\wedge dx'+S\,(dx\wedge du'+du\wedge dx')+B\,du\wedge du'$ and the Hermitian symmetric 2-tensor field a with $\zeta=a(J\,\cdot\,,\,\cdot\,)$ one has $a=Q(dx\otimes dx+dx'\otimes dx')+S\,(dx\otimes du+du\otimes dx+dx'\otimes du'+du'\otimes dx')+B(du\otimes du+du'\otimes du')$, due to (3.1), and so the components of a relative to the coordinates (x,x',u,u') form the matrix

$$\begin{bmatrix}Q&0&S&0\\0&Q&0&S\\S&0&B&0\\0&S&0&B\end{bmatrix},\quad\text{with the determinant }(QB-S^2)^2.\tag{4.3}$$

When $a=g$, (4.2) gives $(Q,S,B)=(\phi_{xx},\phi_{xu},\phi_{uu})$. Thus,

$$\phi_{xx}>0\ \text{ and }\ \gamma>0,\ \text{ for }\ \gamma=\phi_{xx}\phi_{uu}-\phi_{xu}^2,\tag{4.4}$$

which amounts to Sylvester's criterion for positive definiteness of g, namely, positivity of the upper left subdeterminants of (4.3). From now on we set

$$\begin{array}{l}(\tau,\lambda,Q,S,B)=(\phi_x,\phi_u,\phi_{xx},\phi_{xu},\phi_{uu}),\text{ so that}\\ Q>0\text{ and }\gamma=QB-S^2>0\text{ due to (4.4).}\end{array}\tag{4.5}$$

With div,Δ denoting the g-divergence and g-Laplacian, for τ,λ,Q in (4.5),

(a) the functions τ,λ have the holomorphic g-gradients $v=\partial_x$ and $w=\partial_u$,
(b) the other coordinate fields Jv and Jw are holomorphic g-Killing fields,
(c) $Q=g(v,v)$ and $\Delta\tau=\mathrm{div}\,v=(\log\gamma)_x$, while $\Delta\lambda=\mathrm{div}\,w=(\log\gamma)_u$.

Namely, (4.1) and (4.3) yield (a). Also, (b) follows since ϕ only depends on x and u. Finally, (4.3) has the determinant γ^2, and so $\gamma\,dx\wedge dx'\wedge du\wedge du'$ is the volume form of g, on which $\pounds_v$ and $\pounds_w$ act via partial differentiations ∂_x, ∂_u of the γ factor. Thus, $\mathrm{div}\,v=(\log\gamma)_x$ and $\mathrm{div}\,w=(\log\gamma)_u$, cf. [**16**, p. 281]. On the other hand, by (a), (4.1) and (4.5), $g(v,v)=d_v\tau=\partial_x\tau=\partial_x\phi_x=\phi_{xx}=Q$.

For our $(\tau,\lambda)=(\phi_x,\phi_u)$, the mapping $(x,u)\mapsto(\tau,\lambda)$ is locally diffeomorphic due to (4.4), which makes $(Q,S,B)=(\phi_{xx},\phi_{xu},\phi_{uu})$, locally, a triple of real-valued functions of the new variables τ,λ. With subscripts still denoting partial differentiations, the integrability conditions $Q_u-S_x=S_u-B_x=0$ and (4.5) give

$$SQ_\tau+BQ_\lambda=QS_\tau+SS_\lambda,\quad SS_\tau+BS_\lambda=QB_\tau+SB_\lambda,\quad Q>0,\quad QB>S^2.\tag{4.6}$$

Conversely, if functions Q,S,B of the variables τ,λ satisfy (4.6), then, locally,

(d) $(Q,S,B)=(\phi_{xx},\phi_{xu},\phi_{uu})$ for a function ϕ, with (4.4), of the variables x,u related to τ,λ via $(\tau,\lambda)=(\phi_x,\phi_u)$, and Q,S,B determine each of ϕ,x,u uniquely up to additive constants.

In fact, the equalities in (4.6) state precisely that the vector fields $Q\partial_\tau+S\partial_\lambda$ and $S\partial_\tau+B\partial_\lambda$ commute or, equivalently, the 1-forms $(QB-S^2)^{-1}(B\,d\tau-S\,d\lambda)$ and $(QB-S^2)^{-1}(Q\,d\lambda-S\,d\tau)$, dual to them, are closed, and we may declare these vector fields (or, 1-forms) to be ∂_x, ∂_u or, respectively, dx, du. Now that, locally, x,u are defined, up to additive constants, we obtain ϕ by solving the system $(\phi_x,\phi_u)=(\tau,\lambda)$, where τ,λ are treated as functions of x,u via the resulting locally diffeomorphic coordinate change $(\tau,\lambda)\mapsto(x,u)$. Closedness of $\tau\,dx+\lambda\,du$ and the equality $(Q,S,B)=(\phi_{xx},\phi_{xu},\phi_{uu})$ are obvious: our choice of dx and du trivially gives $(d\tau,d\lambda)=(Q\,dx+S\,du,S\,dx+B\,du)$ and $d\tau\wedge dx+d\lambda\wedge du=0$.

The g-Laplacians of τ and λ can also be expressed as

(e) $\Delta\tau=Q_\tau+S_\lambda$ and $\Delta\lambda=S_\tau+B_\lambda$.

To see this, first note that, from the chain and Leibniz rules, one gets $(QB-S^2)_x=Q(QB-S^2)_\tau+S(QB-S^2)_\lambda=Q(QB_\tau+SB_\lambda-SS_\tau)-S(QS_\tau+SS_\lambda-BQ_\lambda)-S^2S_\lambda+QBQ_\tau$. Using (4.6) to replace the two three-term sums in parentheses by BS_λ and SQ_τ, we thus obtain $(QB-S^2)_x=(Q_\tau+S_\lambda)(QB-S^2)$. Similarly, $(QB-S^2)_u=S(QB-S^2)_\tau+B(QB-S^2)_\lambda$ is rewritten as $S(QB_\tau-BS_\lambda-SS_\tau)-S^2S_\tau+B(BQ_\lambda+SQ_\tau-SS_\lambda)+QBB_\lambda$, and analogous three-term replacements give $(QB-S^2)_u=(S_\tau+B_\lambda)(QB-S^2)$. Now (e) follows from (c) and (4.5).

Theorem 4.1. *In $\mathbb{C}^2$ with the complex coordinates $(x+ix',u+iu')$, given an open subset M and a function $\phi:M\to\mathrm{I\!R}$ of the real variables x,u, having the property* (4.4)*, let g be the Kähler metric on M with the Kähler potential 2ϕ. The following two conditions are equivalent.*

(i) *The special Ricci-Hessian equation* (0.1) – (0.2), *holds on* M *for* $\tau = \phi_x$, *and* $dQ \wedge d\tau \neq 0$ *everywhere in* M, *with* $Q = g(\nabla\tau, \nabla\tau)$. *Thus, by Theorem* D, *one has* (0.6) *and* (0.9-a), *where* $Y = \Delta\tau$ *and* F *is the function of* τ *characterized by* (0.8), *for the constants* θ, κ *in* (0.7).

(ii) *The triple* $(Q, S, B) = (\phi_{xx}, \phi_{xu}, \phi_{uu})$ *of functions of the new variables* $(\tau, \lambda) = (\phi_x, \phi_u)$ *satisfies* (4.6) *with* $Q_\lambda \neq 0$ *everywhere, as well as the equations* $S_\tau + B_\lambda = S\alpha - G$, $Q_\tau + S_\lambda = Q\alpha + F$, $G_\tau = S\alpha'$, $G_\lambda = -(Q\alpha' + F')$ *for some function* G *and* $(\ )' = d/d\tau$.

Proof. By (0.9-b), equation (0.1) is, in case (i), equivalent to

$$2\alpha\nabla d\tau + 2\mathrm{r} = -(Q\alpha' + F')g, \tag{4.7}$$

where all the terms are Hermitian symmetric 2-tensor fields, and hence correspond, via g, to *complex-linear* bundle morphisms $TM \to TM$. Thus, (4.7) amounts to

(f) equalities of the images of both sides in (4.7) under $\imath_v$ and $\imath_w$.

The equality of the $\imath_v$-images is, by (1.3-b) and (0.3), the result of applying d, via (2.5), to the relation (0.9-a) in (i): $Y - Q\alpha = F$. This last relation and (e), with $Y = \Delta\tau$ due to (0.4), show that (i) implies the equality $Q_\tau + S_\lambda = Q\alpha + F$ in (ii). Defining G to be $S\alpha - S_\tau - B_\lambda$ we get $S_\tau + B_\lambda = S\alpha - G$. On the other hand, the equality of the $\imath_w$-images in (4.7) reads

$$\alpha\, dS - d\Delta\lambda = -(Q\alpha' + F')\, d\lambda. \tag{4.8}$$

In fact, the first term equals $\alpha\, dS$ since, for two commuting gradients $v = \nabla\tau$ and $w = \nabla\lambda$, one has $2\nabla_w d\tau = d[g(v, w)]$ or, in local coordinates, $2w^k v_{,jk} = w^k v_{,jk} + v^k w_{,jk} = (v^k w_k)_{,j}$, and $S = \phi_{xu} = g(v, w)$ by (4.3). The second term is $-d\Delta\lambda$ due to (a) and (1.8). By (e), $G = S\alpha - S_\tau - B_\lambda = S\alpha - \Delta\lambda$, and so (4.8) becomes $\alpha\, dS + d(G - S\alpha) = -(Q\alpha' + F')\, d\lambda$, that is, according to (2.5),

$$dG = S\, d\alpha - (Q\alpha' + F')\, d\lambda = S\alpha' d\tau - (Q\alpha' + F')\, d\lambda$$

or, in other words, $G_\tau = S\alpha'$ and $G_\lambda = -(Q\alpha' + F')$. Consequently, (i) implies (ii), since (4.6) arises as the integrability conditions $Q_u - S_x = S_u - B_x = 0$ combined with (4.4), and the equality $dQ = Q_\tau\, d\tau + Q_\lambda\, d\lambda$ yields $dQ \wedge d\tau = -Q_\lambda\, d\tau \wedge d\lambda$.

Conversely, assuming (ii), we get (i) from (f). Namely, as we saw above, the equality of the $\imath_v$-images in (4.7) arises by applying d to $Y - Q\alpha = F$, that is – cf. (e) – to $Q_\tau + S_\lambda = Q\alpha + F$. Also, (4.8) follows from (ii) and (e): $\alpha\, dS - d\Delta\lambda = \alpha\, dS - d(S_\tau + B_\lambda) = \alpha\, dS - d(S\alpha - G) = dG - S\, d\alpha = dG - S\alpha' d\tau = dG - G_\tau\, d\tau = G_\lambda\, d\lambda = -(Q\alpha' + F')\, d\lambda$. □

5. Reduction of order

We now proceed to discuss the first-order system equivalent, as we saw in the last section, to the Kähler-potential problem, the solution of which amounts to proving Theorem E. The main result established here, Theorem 5.3, will lead – in Section 7 – to a unique-extension property of integral lines, which results in applicability of the Cartan-Kähler theorem to our situation.

Theorem 4.1 reduces constructing local examples of special Ricci-Hessian equations (0.1) – (0.2) with $dQ \wedge d\tau \neq 0$, on Kähler surfaces, which is a fourth-order problem in the Kähler potential 2ϕ, to the following system of quasi-linear first-order partial differential equations:

$$\begin{array}{l} QS_\tau + SS_\lambda = SQ_\tau + BQ_\lambda, \qquad SS_\tau + BS_\lambda = QB_\tau + SB_\lambda, \\ S_\tau + B_\lambda = S\alpha - G, \ \ Q_\tau + S_\lambda = Q\alpha + F, \ \ G_\tau = S\alpha', \ \ G_\lambda = -(Q\alpha' + F'), \end{array} \tag{5.1}$$

imposed on real-valued functions Q, S, B, G of the real variables τ, λ, with

$$QB > S^2, \qquad Q > 0, \qquad Q_\lambda \neq 0 \quad \text{everywhere.} \tag{5.2}$$

Subscripts denote here partial differentiations, α is a function of the variable τ such that $2\alpha' + \alpha^2 = 4\varepsilon$ for a constant $\varepsilon \in \mathrm{I\!R}$, where $(\ )' = d/d\tau$, and F is the function of τ, characterized by (0.8):

$$4\varepsilon F = \theta(2 - \tau\alpha) + 4\varepsilon\kappa\alpha \ \text{ if } \ \varepsilon \neq 0, \ \text{ or } \ F = \kappa\alpha - 2\theta/(3\alpha^2) \ \text{ when } \ \varepsilon = 0,$$

which depends on two further real constants θ and κ. Consequently,

$$\alpha'' + \alpha\alpha' = 0, \qquad F'' = -F\alpha'. \tag{5.3}$$

We obviously also assume that

$$\tau \text{ ranges over the intersection of the domains of } \alpha \text{ and } F. \tag{5.4}$$

THEOREM 5.1. *Real-analytic solutions* $\boldsymbol{Z} = (Q, S, B, G)$ *to* (5.1) *with* (5.2) – (5.3)*, exist, locally, on a neighborhood of any* $(\tau, \lambda) \in \mathrm{I\!R}^2$ *as in* (5.4)*.*

More precisely, one obtains a locally-unique such solution $\boldsymbol{Z}$ *by prescribing* $\boldsymbol{Z}$ *and the partial derivatives* $\boldsymbol{Z}_\tau, \boldsymbol{Z}_\lambda$ *real-analytically along an arbitrary real-analytic embedding* $t \mapsto (\tau, \lambda) \in \mathrm{I\!R}^2$ *of an interval, so as to satisfy* (5.1)*,* (5.2)*,* (5.4)*, and the condition* $\dot{\boldsymbol{Z}} = \dot\tau\boldsymbol{Z}_\tau + \dot\lambda\boldsymbol{Z}_\lambda$*, where* $(\ )^\cdot = d/dt$*.*

We prove Theorem 5.1 at the end of Section 8. Note that it establishes the existence of a solution defined on a neighborhood of the embedded interval, with the image of $t \mapsto (\tau, \lambda, \boldsymbol{Z}, \boldsymbol{Z}_\tau, \boldsymbol{Z}_\lambda) \in \mathcal{N}$ serving as initial data in a Cauchy initial-value problem, to which the Cauchy-Kovalevskaya theorem is applied [**4**, p. 83].

Later we will also think of $Q, S, B, G, Q_\tau, S_\tau, B_\tau, G_\tau, Q_\lambda, S_\lambda, B_\lambda, G_\lambda$ as new variables, rather than functions and their partial derivatives. Treating $K = Q_\tau$ and $L = Q_\lambda$ as parameters, we may solve (5.1) for the eight "subscripted" symbols:

$$\begin{array}{l} Q_\tau = K, \qquad Q_\lambda = L, \qquad S_\lambda = Q\alpha + F - K, \\ QS_\tau = (2K - Q\alpha - F)S + BL, \\ Q^2 B_\tau = [2(SK + BL) + (G - S\alpha)Q]S \\ \qquad + \ (Q\alpha + F - K)(QB - 2S^2), \\ QB_\lambda = (2Q\alpha + F - 2K)S - BL - QG, \\ G_\tau = S\alpha', \qquad G_\lambda = -(Q\alpha' + F'). \end{array} \tag{5.5}$$

The next lemma, although completely trivial, is phrased in a rather convoluted way. For reasons that will become clear in the next section, it is absolutely crucial not to assume that the "subscripted" letters Z_τ, Z_λ stand for the partial derivatives of $Z = Q, S, B, G$. At the same time, we use the symbol $\equiv$, as if to pretend that, nevertheless, $dZ = Z_\tau d\tau + Z_\lambda d\lambda$.

Lemma 5.2. *Let functions* $Q,S,B,G,Q_\tau,S_\tau,B_\tau,G_\tau,Q_\lambda,S_\lambda,B_\lambda,G_\lambda,K,L$ *of the real variables* τ,λ, *and* α,F *depending only on* τ, *satisfy* (5.5). *Then*

$$\begin{aligned}
&dQ_\tau\wedge d\tau = dK\wedge d\tau,\quad dQ_\lambda\wedge d\lambda = dL\wedge d\lambda,\\
&dS_\lambda\wedge d\lambda \equiv [(\alpha Q_\tau + Q\alpha' + F')\,d\tau - dK]\wedge d\lambda,\\
&Q\,dS_\tau\wedge d\tau \equiv [2S\,dK + B\,dL - (S_\tau + S\alpha)Q_\lambda\,d\lambda]\wedge d\tau,\\
&\qquad\qquad + [LB_\lambda + (2K - Q\alpha - F)S_\lambda]\,d\lambda\wedge d\tau,\\
&Q^2dB_\tau\wedge d\tau \equiv [(4S^2 - QB)\,dK + 2SB\,dL]\wedge d\tau + QSG_\lambda\,d\lambda\wedge d\tau\\
&\qquad\qquad + [2BL + 4(2K - Q\alpha - F)S + (G - 2S\alpha)Q]S_\lambda\,d\lambda\wedge d\tau\\
&\qquad\qquad + [(2Q\alpha + F - K)B + (G - 3S\alpha)S - 2QB_\tau]Q_\lambda\,d\lambda\wedge d\tau\\
&\qquad\qquad + [2SL + (Q\alpha + F - K)Q]B_\lambda\,d\lambda\wedge d\tau,\\
&Q\,dB_\lambda\wedge d\lambda \equiv [(2Q\alpha + F - 2K)S_\tau + (2S\alpha - G - B_\lambda)Q_\tau]\,d\tau\wedge d\lambda\\
&\qquad\qquad + [(2Q\alpha' + F')S - QG_\tau - LB_\tau]\,d\tau\wedge d\lambda\\
&\qquad\qquad - (2S\,dK + B\,dL)\wedge d\lambda,
\end{aligned}\tag{5.6}$$

where $(\ )' = d/d\tau$ *and* $\equiv$ *means that all occurrences of* dZ, *for* $Z = Q,S,B,G$, *arising when* d *is applied to the right-hand sides in* (5.5) *– often via differentiation by parts – have been replaced by* $Z_\tau d\tau + Z_\lambda d\lambda$.

Theorem 5.3. *If one replaces all occurrences of* Q_τ *and* Q_λ *on the right-hand sides in* (5.6) *by* K *and* L, *the combination of those six right-hand sides with the respective coefficients* $QB, QB, -2QS, -2S, 1, Q$ *equals* 0.

Proof. In our combination, the occurrences of dK and dL undergo a total cancellation, and the sum of the remaining terms equals $d\tau\wedge d\lambda$ times

$$\begin{aligned}
&BKL + [(S\alpha - G)S - (2Q\alpha + F)B]L - (SF' + GK)Q\\
&\quad + [(2Q\alpha + F - 2K)Q - 2SL]S_\tau\\
&\quad + [2(2Q\alpha + F - 2K)S - QG - 2BL]S_\lambda\\
&\quad + [LB_\tau - (Q\alpha + F)B_\lambda - QG_\tau - SG_\lambda]Q.
\end{aligned}\tag{5.7}$$

Multiplying (5.7) by Q and replacing $QS_\tau, Q^2B_\tau, G_\tau, S_\lambda, QB_\lambda, G_\lambda$ with the right-hand sides in (5.5), we get 0 via a tedious but straightforward calculation.

Unless one uses a symbolic computation software, such a calculation, done by hand, can be considerably simplified if one proceeds by the following six steps. In each step the terms containing a specific factor should be marked and then crossed out (after one sees that they add up to zero). The six factors are, in this order, α', F', SL, BL, Q^2G and, finally, $(2K - 2Q\alpha - F)QS$. □

6. The relevant exterior differential systems

As a next step, we now present a first-prolongation version of the exterior differential system associated with equations (5.1).

Let the open subset $\mathcal{Y}$ of $\mathbb{R}^6$, with the coordinates τ,λ,Q,S,B,G, consist of all $\mathbf{y} = (\tau,\lambda,Q,S,B,G)\in\mathbb{R}^6$ such that Q and $QB - S^2$ are both positive, while τ lies in the domains of α and F. The set $\mathcal{N}$ of all 14-tuples

$$(\tau,\lambda,Q,S,B,G,Q_\tau,S_\tau,B_\tau,G_\tau,Q_\lambda,S_\lambda,B_\lambda,G_\lambda)\in\mathcal{Y}\times\mathbb{R}^8\subseteq\mathbb{R}^{14}\tag{6.1}$$

satisfying (5.1) is an eight-dimensional submanifold of $\mathcal{Y}\times\mathbb{R}^8$, diffeomorphic to $\mathcal{Y}\times\mathbb{R}^2$ via the diffeomorphic embedding $\mathcal{Y}\times\mathbb{R}^2\to\mathcal{N}\subseteq\mathcal{Y}\times\mathbb{R}^8$ that sends $(\tau,\lambda,Q,S,B,G,K,L)$ to (6.1) with the last eight components given by (5.5); the inverse diffeomorphism is the restriction to $\mathcal{N}$ of the projection assigning to the 14-tuple (6.1) the octuple $(\tau, \lambda, Q, S, B, G, Q_\tau, Q_\lambda)$. Thus,

$$(\tau, \lambda, Q, S, B, G, K, L) \text{ is a global coordinate system for } \mathcal{N}. \tag{6.2}$$

By an *exterior differential system* on a manifold M one means an ideal $\mathcal{I}$ in the graded algebra $\Omega^* M$, closed under exterior differentiation; its *integral manifolds* (or, *integral elements*) are those submanifolds of M (or, subspaces of its tangent spaces) on which every form in $\mathcal{I}$ vanishes [**4**, pp. 16, 65]. When such objects have dimension 1 or 2, we call them *integral curves/surfaces* or *lines/planes.*

If $E \subseteq T_z M$ is a p-dimensional integral element of $\mathcal{I}$, one sets [**4**, pp. 67-68]:

$$\begin{aligned} &H(E) = \{v \in T_z M : \zeta(v, e_1, \ldots, e_p) = 0 \text{ for all } \zeta \in \mathcal{I} \cap \Omega^{p+1} M\}, \\ &\text{using a basis } e_1, \ldots, e_p \text{ of } T_z M, \text{ and } r(E) = \dim H(E) - (p+1), \end{aligned} \tag{6.3}$$

so that $H(E)$ is a vector subspace of $T_z M$, not depending on $e_1, \ldots, e_p$ since

$$H(E) = \{v \in T_z M : \operatorname{span}(v, E) \text{ is an integral element of } \mathcal{I}\}. \tag{6.4}$$

Letting the variable Z assume as "values" the symbols Q, S, B, G, we introduce

$$\begin{aligned} &\text{the exterior differential system on } \mathcal{Y} \times \mathrm{I\!R}^8 \text{ generated by the} \\ &\text{four 1-forms } Z_\tau d\tau + Z_\lambda d\lambda - dZ \text{ and their exterior deriva-} \\ &\text{tives } dZ_\tau \wedge d\tau + dZ_\lambda \wedge d\lambda, \text{ where } Z \text{ ranges over } Q, S, B, G. \end{aligned} \tag{6.5}$$

Restricting (6.5) to $\mathcal{N}$, we obtain an exterior differential system $\mathcal{I}$ on $\mathcal{N}$. (For some context, see Remark 6.1.) As we show below, $\mathcal{I}$ is generated by

$$\begin{aligned} &\text{the restrictions to } \mathcal{N} \text{ of the four 1-forms in (6.5),} \\ &dQ_\tau \wedge d\tau + dQ_\lambda \wedge d\lambda, \text{ and } dS_\tau \wedge d\tau + dS_\lambda \wedge d\lambda. \end{aligned} \tag{6.6}$$

In other words, $dG_\tau \wedge d\tau + dG_\lambda \wedge d\lambda$ and $dB_\tau \wedge d\tau + dB_\lambda \wedge d\lambda$ are redundant for defining $\mathcal{I}$, that is, they lie in the ideal $\mathcal{I}' \subseteq \Omega^* \mathcal{N}$ generated by the four 1-forms $Z_\tau d\tau + Z_\lambda d\lambda - dZ$, where $Z = Q, S, B, G$, along with $dQ_\tau \wedge d\tau + dQ_\lambda \wedge d\lambda$ and $dS_\tau \wedge d\tau + dS_\lambda \wedge d\lambda$ (all of them restricted to $\mathcal{N}$).

To verify redundancy of $dG_\tau \wedge d\tau + dG_\lambda \wedge d\lambda$, it suffices to note that it coincides, on $\mathcal{N}$, with the 2-form $\alpha'[(Q_\tau d\tau + Q_\lambda d\lambda - dQ) \wedge d\lambda - (S_\tau d\tau + S_\lambda d\lambda - dS) \wedge d\tau]$, since the equalities $G_\tau = S\alpha'$ and $G_\lambda = -(Q\alpha' + F')$ in (5.1) make the former, on $\mathcal{N}$, equal to the latter minus $[Q\alpha'' + F'' + (Q_\tau + S_\lambda)\alpha']\, d\tau \wedge d\lambda$, while by (5.3), the last expression in square brackets is nothing else than $[Q_\tau + S_\lambda - (Q\alpha + F)]\alpha'$, and so it vanishes on $\mathcal{N}$ due to (5.1).

For $dB_\tau \wedge d\tau + dB_\lambda \wedge d\lambda$ the redundancy claim follows since, by Theorem 5.3,

$$\begin{aligned} Q^2(dB_\tau \wedge d\tau + dB_\lambda \wedge d\lambda) \equiv{}& 2QS(dS_\tau \wedge d\tau + dS_\lambda \wedge d\lambda) \\ &- QB(dQ_\tau \wedge d\tau + dQ_\lambda \wedge d\lambda), \end{aligned} \tag{6.7}$$

on $\mathcal{N}$, with $\equiv$ now denoting congruence modulo $\mathcal{I}'$. (Note that $Q > 0$ in (5.2), while $\equiv$ in Theorem 5.3 implies $\equiv$ as defined here.)

Remark 6.1. The following comment will not be used in our argument, and may be ignored by the reader not interested in a broader context of the preceding discussion. (We provide it just to point out where our approach fits within the standard theory.) The exterior differential system naturally associated with (5.1) is the one on our open set $\mathcal{Y} \subseteq \mathrm{I\!R}^6$, generated by the 1-form $dG - S\alpha' d\tau + (Q\alpha' + F')\, d\lambda$, the 2-form $dS \wedge d\lambda + d\tau \wedge dB + (G - S\alpha)\, d\tau \wedge d\lambda$, their exterior derivatives, and the exterior derivatives of $(QB - S^2)^{-1}(B\, d\tau - S\, d\lambda)$ and $(QB - S^2)^{-1}(Q\, d\lambda - S\, d\tau)$.

Our exterior differential system $\mathcal{I}$, on $\mathcal{N}$, defined in the lines following (6.5), is its first prolongation [**4**, p. 147], using the independence condition [**4**, p. 103] for which our $d\tau \wedge d\lambda$ serves as the form denoted by Ω in [**4**]. We refer to this independence condition as *horizontality* (see (7.1) below). The need for the first prolongation arises since the original system fails to satisfy the assumptions of the Cartan-Kähler theorem. In agreement with [**4**, p. 147], our $\mathcal{N}$ may be identified with an open subset of the Grassmann manifold $\mathrm{Gr}_2\mathcal{Y}$ of two-planes tangent to $\mathcal{Y}$. The identification in question associates with a horizontal plane at a point (τ,λ,Q,S,B,G) of $\mathcal{Y}$ the 14-tuple (6.1) such that the vectors $(1,0,Q_\tau,S_\tau,B_\tau,G_\tau)$ and $(0,1,Q_\lambda,S_\lambda,B_\lambda,G_\lambda)$ form a basis of the plane.

7. The unique-extension theorem

In this section we prove the result announced at the beginning of Section 5. With the assumptions and notation of Section 6, for a subspace $E \subseteq \mathcal{Y}\times \mathrm{I\!R}^8$,

$$\text{we call } E \text{ horizontal when } (d\tau, d\lambda) : E \to \mathrm{I\!R}^2 \text{ is injective.} \tag{7.1}$$

Due to (6.2), the four 1-forms in (6.5), restricted to $\mathcal{N}$, remain linearly independent at every point. Their simultaneous kernel in $\mathcal{N}$ thus constitutes a four-dimensional distribution $\mathcal{D}$ on $\mathcal{N}$.

It is convenient to rewrite (6.1) as $(\tau,\lambda,\boldsymbol{Z},\boldsymbol{Z}_\tau,\boldsymbol{Z}_\lambda)$ with $\boldsymbol{Z},\boldsymbol{Z}_\tau,\boldsymbol{Z}_\lambda \in \mathrm{I\!R}^4$. Now, obviously, a vector $(\dot\tau,\dot\lambda,\dot{\boldsymbol{Z}},\dot{\boldsymbol{Z}}_\tau,\dot{\boldsymbol{Z}}_\lambda)$ tangent to $\mathcal{Y}\times\mathrm{I\!R}^8$ at $(\tau,\lambda,\boldsymbol{Z},\boldsymbol{Z}_\tau,\boldsymbol{Z}_\lambda)$ lies in the simultaneous kernel of the four 1-forms in (6.5) if and only if

$$\dot{\boldsymbol{Z}} \,=\, \dot\tau\boldsymbol{Z}_\tau \,+\, \dot\lambda\boldsymbol{Z}_\lambda. \tag{7.2}$$

Our diffeomorphic embedding $\mathcal{Y}\times\mathrm{I\!R}^2 \to \mathcal{N} \subseteq \mathcal{Y}\times\mathrm{I\!R}^8$ now becomes

$$(\tau,\lambda,\boldsymbol{Z},K,L) \mapsto (\tau,\lambda,\boldsymbol{Z},\boldsymbol{Z}_\tau,\boldsymbol{Z}_\lambda) = (\tau,\lambda,\boldsymbol{Z},\,\boldsymbol{W}(\tau,\boldsymbol{Z},K,L)), \tag{7.3}$$

with the $\mathrm{I\!R}^8$-valued function $\boldsymbol{W}$ representing the right-hand sides of (5.5), so that $\boldsymbol{W}$ does not depend on λ, and its only dependence on τ is through α and F.

Lemma 7.1. *The linear operator sending $(\dot\tau,\dot\lambda,\dot K,\dot L)$ to the vector*

$$(\dot\tau,\dot\lambda,\dot\tau\boldsymbol{Z}_\tau+\dot\lambda\boldsymbol{Z}_\lambda, d\boldsymbol{W}_{(\tau,\boldsymbol{Z},K,L)}(\dot\tau,\dot\tau\boldsymbol{Z}_\tau+\dot\lambda\boldsymbol{Z}_\lambda,\dot K,\dot L)) \tag{7.4}$$

is an isomorphism of $\mathrm{I\!R}^4$ onto the fibre $\mathcal{D}_{\boldsymbol{C}}$ of $\mathcal{D}$ at the image $\boldsymbol{C}$ of $(\tau,\lambda,\boldsymbol{Z},K,L)$ under the diffeomorphic embedding (7.3)*. Every horizontal plane at $\boldsymbol{C}$, contained in $\mathcal{D}_{\boldsymbol{C}}$, has a unique basis having, for some $(\dot K_1,\dot L_1,\dot K_2,\dot L_2)\in\mathrm{I\!R}^4$, the form*

$$(1,0,\boldsymbol{Z}_\tau,d\boldsymbol{W}_{(\tau,\boldsymbol{Z},K,L)}(1,\boldsymbol{Z}_\tau,\dot K_1,\dot L_1)),\ (0,1,\boldsymbol{Z}_\lambda,d\boldsymbol{W}_{(\tau,\boldsymbol{Z},K,L)}(0,\boldsymbol{Z}_\lambda,\dot K_2,\dot L_2)). \tag{7.5}$$

The span of (7.5) *is an integral plane of $\mathcal{I}$ if and only if*

$$\begin{aligned}&\dot K_2=\dot L_1 \text{ and } Q\dot K_1+2S\dot K_2+B\dot L_2=C \text{ for the expression } C \text{ given by}\\ &C=(Q_\tau\alpha+Q\alpha'+F')Q\,+\,(S_\tau+S\alpha)Q_\lambda\,+\,(Q\alpha+F-2K)S_\lambda-LB_\lambda,\end{aligned} \tag{7.6}$$

where the letter symbols come from $(\tau,\lambda,\boldsymbol{Z},\boldsymbol{Z}_\tau,\boldsymbol{Z}_\lambda)$ in (7.3) *rewritten via* (6.1).

Proof. The claims preceding and including (7.5) are immediate consequences of (7.1) – (7.3). Next, as the vectors (7.5) have the τ and λ components $1,0$ or, respectively, $0,1$, evaluating $dZ_\tau\wedge d\tau+dZ_\lambda\wedge d\lambda$ on them, for $Z=Q,S,B,G$, yields, by (3.1), the difference between the Z_λ component of $d\boldsymbol{W}_{(\tau,\boldsymbol{Z},K,L)}(1,\boldsymbol{Z}_\tau,\dot K_1,\dot L_1)$ and the Z_τ component of $d\boldsymbol{W}_{(\tau,\boldsymbol{Z},K,L)}(0,\boldsymbol{Z}_\lambda,\dot K_2,\dot L_2)$. When $Z=Q$ or $Z=S$, these differences are $\dot L_1-\dot K_2$ and Q^{-1} times $C-(Q\dot K_1+2S\dot K_2+B\dot L_2)$, with C as in (7.6).

(By (5.2), $Q > 0$.) To see this, we apply $(\,)^{\cdot}$ formally to the first four equalities in (5.5), using differentiation by parts (for QS_τ only) along with the Leibniz rule. We then replace α, F and each $\dot{Z}$, where $Z = Q, S, B, G$, by α', F', Z_τ for the first vector of (7.5), and by $0, 0, Z_\lambda$ for the second one, cf. (7.2). Combined with (6.6), this completes the proof. □

As a consequence of the first claim in Lemma 7.1, at any $(\tau, \mathbf{Z}, K, L)$,

(7.7) the operator $\mathrm{I\!R}^2 \ni (\dot{K}, \dot{L}) \mapsto d\mathbf{W}_{(\tau,\mathbf{Z},K,L)}(0, 0, \dot{K}, \dot{L}) \in \mathrm{I\!R}^8$ is injective,

Remark 7.2. If an integral line E of $\mathcal{I}$ lies within in a unique horizontal integral plane E', then E' is the only integral plane containing E. Namely, another such plane E'', being nonhorizontal, would intersect the kernel of $(d\tau, d\lambda)$ along a line. As $D \subseteq H(E)$ for the vector subspace $H(E)$ in (6.3) – (6.4) and the three-dimensional span D of E' and E'', all planes in D containing the line E, other than E'', would be horizontal integral planes, contrary to the uniqueness of E'.

Theorem 7.3. *Every horizontal integral line of $\mathcal{I}$ is contained in a unique integral plane, and this unique plane is also horizontal.*

Proof. Due to Remark 7.2, it suffices to show that, at each $(\tau, \lambda, \mathbf{Z}, K, L)$, with the corresponding $(\tau, \lambda, \mathbf{Z}, \mathbf{Z}_\tau, \mathbf{Z}_\lambda)$ in (7.3), a vector (7.4) having $(\dot{\tau}, \dot{\lambda}) \neq (0, 0)$ lies in in a unique *horizontal* integral plane or, in other words, (7.4) is a linear combination of a unique pair (7.5) satisfying (7.6). Looking at the first two components we see that the coefficients of the combination must be $\dot{\tau}$ and $\dot{\lambda}$. By (7.2), this reduces our problem to the existence and uniqueness of $(\dot{K}_1, \dot{L}_1, \dot{K}_2, \dot{L}_2)$ in $\mathrm{I\!R}^4$ with $d\mathbf{W}_{(\tau,\mathbf{Z},K,L)}(0, 0, \dot{K} - \dot{\tau}\dot{K}_1 - \dot{\lambda}\dot{K}_2, \dot{L} - \dot{\tau}\dot{L}_1 - \dot{\lambda}\dot{L}_2) = 0$, that is, cf. (7.7), $(\dot{K}, \dot{L}) = \dot{\tau}(\dot{K}_1, \dot{L}_1) + \dot{\lambda}(\dot{K}_2, \dot{L}_2)$. Using (7.6) to eliminate $\dot{K}_1$ and $\dot{K}_2$, we rewrite the last condition, with the first component multiplied by Q, as

$$\begin{bmatrix} Q\dot{\lambda} - 2S\dot{\tau} & -B\dot{\tau} \\ \dot{\tau} & \dot{\lambda} \end{bmatrix} \begin{bmatrix} \dot{L}_1 \\ \dot{L}_2 \end{bmatrix} = \begin{bmatrix} Q\dot{K} - C\dot{\tau} \\ \dot{L} \end{bmatrix},$$

which has a unique solution $(\dot{L}_1, \dot{L}_2)$, the determinant $B\dot{\tau}^2 - 2S\dot{\tau}\dot{\lambda} + Q\dot{\lambda}^2$ being, by (5.2), a positive-definite quadratic form in $(\dot{\tau}, \dot{\lambda})$. □

8. Existence of integral surfaces

The next fact – used below to derive our Theorem 5.1 – is a special case of the celebrated Cartan-Kähler theorem [**4**, pp. 81–82]. Since our phrasing differs from that of [**4**], we devote the next section to clarifying how our version amounts to adapting the one in [**4**] to our particular case.

The symbols $\mathcal{N}, \mathcal{D}$ and $\mathcal{I}$ stand here for more general objects that those in Section 6. The definition (7.1) of horizontality, for integral elements, is used more generally, as well as extended, in an obvious fashion, to integral manifolds.

Theorem 8.1. *Let real-analytic functions τ, λ and 1-forms $\xi_1, \dots, \xi_q$ on a manifold $\mathcal{N}$, where $0 < q < \dim \mathcal{N}$, have the property that $d\tau, d\lambda, \xi_1, \dots, \xi_q$ are linearly independent at every point. Denoting by $\mathcal{D}$ and $\mathcal{I}$ the distribution on $\mathcal{N}$ arising as the simultaneous kernel of the 1-forms $\xi_1, \dots, \xi_q$ and, respectively, the exterior differential system on $\mathcal{N}$ generated by $\xi_1, \dots, \xi_q$ and, possibly, some* *higher-degree forms, along with their exterior derivatives, let us suppose that*

(8.1) every horizontal integral line of $\mathcal{I}$, at any point of $\mathcal{N}$, is contained in a unique integral plane of $\mathcal{I}$.

Then every horizontal real-analytic integral curve of $\mathcal{I}$ is contained, locally, in a locally-unique horizontal real-analytic integral surface. Examples of such curves are provided by unparametrized integral curves of any horizontal real-analytic local section, without zeros, of the vector bundle $\mathcal{D}$ over $\mathcal{N}$. Furthermore,

(8.2) integral lines of $\mathcal{I}$ are the same as lines tangent to $\mathcal{D}$.

Proof of Theorem 5.1. Due to (6.6), (6.2) and Theorem 7.3, our $\mathcal{N},\mathcal{D}$ and $\mathcal{I}$, introduced in Section 6, satisfy the hypotheses of Theorem 8.1, with $q=4$, the coordinate functions τ,λ in (6.2), and the four 1-forms $Z_\tau d\tau + Z_\lambda d\lambda - dZ$, where Z ranges over $Z=Q,S,B,G$. As an obvious consequence, one has (8.2).

The image of the embedding $t\mapsto(\tau,\lambda,\boldsymbol{Z},\boldsymbol{Z}_\tau,\boldsymbol{Z}_\lambda)\in\mathcal{N}$ arising under the hypotheses of Theorem 5.1 is a horizontal real-analytic integral curve of $\mathcal{I}$. In fact, horizontality follows since $t\mapsto(\tau,\lambda)$ is an embedding, while the resulting tangent directions are integral lines in view of (8.2), the definition of our $\mathcal{D}$, and the relation $\dot{\boldsymbol{Z}}=\dot\tau\boldsymbol{Z}_\tau+\dot\lambda\boldsymbol{Z}_\lambda$ assumed in Theorem 5.1.

The integral surface arising in Theorem 8.1, being horizontal (Theorem 7.3), forms, locally, the graph of a function $(\tau,\lambda)\mapsto(\boldsymbol{Z},\boldsymbol{Z}_\tau,\boldsymbol{Z}_\lambda)\in\mathrm{I\!R}^{12}$. Its $\boldsymbol{Z}$ component $(\tau,\lambda)\mapsto\boldsymbol{Z}=(Q,S,B,G)$ is a solution to (5.1): $t\mapsto(\tau,\lambda,\boldsymbol{Z},\boldsymbol{Z}_\tau,\boldsymbol{Z}_\lambda)$ takes values in the manifold $\mathcal{N}$ *defined* by (5.1) while, as the graph is tangent to the simultaneous kernel of the four 1-forms $Z_\tau d\tau+Z_\lambda d\lambda-dZ$ in (6.5), each of these Z_τ, Z_λ coincides with the respective partial derivative of Z. $\square$

Under the assumptions of Theorem 8.1, let $k=\dim\mathcal{N}$. For all p-dimensional horizontal integral elements $E=E_p$ of $\mathcal{I}$, with $p\in\{0,1\}$, the integer $r(E)=\dim H(E)-(p+1)$ in (6.3) has a fixed nonnegative value, namely,

(8.3)
$$\begin{aligned}&\dim H(E_0)=k-q \text{ and } r(E_0)=k-q-1 \text{ if } p=0,\\ &\dim H(E_1)=2 \text{ and } r(E_1)=0 \text{ in the case where } p=1.\end{aligned}$$

This is obvious from (8.2) or, respectively, (8.1).

9. Where Theorem 8.1 comes from

Here is the Cartan-Kähler theorem, cited *verbatim* from [**4**, pp. 81–82]:

Let $\mathcal{I}\subset\Omega^*(M)$ be a real analytic differential ideal. Let $P\subset M$ be a connected, p-dimensional, real analytic, Kähler-regular integral manifold of $\mathcal{I}$.

Suppose that $r=r(P)$ is a non-negative integer. Let $R\subset M$ be a real analytic submanifold of M which is of codimension r, which contains P, and which satisfies the condition that T_xR and $H(T_xR)$ are transverse in T_xM or all $x\in P$.

Then there exists a real analytic integral manifold of $\mathcal{I}$, X, which is connected and $(p+1)$-dimensional and which satisfies $P\subset X\subset R$. This manifold is unique in the sense that any other real analytic integral manifold of $\mathcal{I}$ with these properties agrees with X on an open neighborhood of P.

As we verify in the following paragraphs, the hypotheses of our Theorem 8.1 imply those listed above, for $(p,r)=(1,0)$, the manifolds M,R above which are both replaced by our $\mathcal{N}$, and the same ideal $\mathcal{I}$ as ours. By our $\mathcal{N}$ and $\mathcal{I}$ we mean the "general" ones (see the three lines preceding Theorem 8.1), rather than the very special choices of $\mathcal{N}$ and $\mathcal{I}$ made in Section 6.

Furthermore, P mentioned above is our (arbitrary) horizontal real-analytic integral curve of $\mathcal{I}$. The resulting manifold X corresponds to the horizontal real-analytic integral surface of $\mathcal{I}$ claimed to exist in Theorem 8.1.

We now proceed to explain why our horizontal integral curve must automatically be Kähler-regular [**4**, p. 81], meaning that its tangent lines are all Kähler-regular in the sense of [**4**, p. 68, Definition 1.7]. To verify this last claim, we first apply Cartan's test [**4**, p. 74, Theorem 1.11]. Namely, in the notation of [**4**, p. 74, Theorem 1.11], $n = 1$ (as we are dealing with tangent *lines*). Due to the relation $\dim H(E_0) = k - q$ in (8.3), and (8.2), $H(E_0)$ is of codimension q in the tangent space of $\mathcal{N}$ containing it, the same as the codimension, in the Grassmann manifold $\mathrm{Gr}_1\mathcal{N}$ of lines tangent to $\mathcal{N}$, of the submanifold $V_1(\mathcal{I})$ formed by all integral lines of $\mathcal{I}$. Cartan's test thus shows that every line E_1 tangent to our horizontal integral curve is *ordinary* [**4**, p. 73, Definition 1.9]. The Kähler-regularity of E_1 now trivially follows, r in [**4**, pp. 67-68] having the constant value 0 according to (8.3). This is also the value $r = r(P)$ in the italicized statement cited above from [**4**]. Cf. [**4**, pp. 81–82, the lines preceding Theorem 2.2].

10. Proof of Theorem E

Let ∇ (or, g) be a connection (or, a pseudo-Riemannian metric) on a C^∞ manifold M. We call ∇ or g *real-analytic* if, in a suitable coordinate system around every point of M, its components Γ_{jk}^l (or, g_{jk}) are real-analytic functions of the coordinates. The C^∞ structure of M then contains a unique real-analytic structure (maximal atlas) making ∇ or, g real-analytic. (The atlas consists of all coordinate systems just mentioned; their mutual transition mappings are real-analytic due to real-analyticity of affine mappings, or isometries, between manifolds with real-analytic connections/metrics, which follows since such mappings appear linear in geodesic coordinates.) Real-analyticity of a metric g obviously implies that of its Levi-Civita connection ∇ (and vice versa, since $\nabla g = 0$).

For a real-analytic (Riemannian) Kähler metric g on a complex manifold M, the unique real-analytic structure described above coincides with the one induced by the complex structure of M. In fact, local holomorphic coordinate functions, being g-harmonic, must be real-analytic relative to the former structure, as a consequence of the standard regularity theory of elliptic partial differential equations applied to the g-Laplacian Δ.

Proof of Theorem E. Combining Theorems 5.1 and 4.1, we obtain the first assertion of Theorem E.

For the second one we invoke the existence results of [**22**] and [**6**]. In both cases, $dQ \wedge d\tau \neq 0$ somewhere, and the metric is real-analytic. The former claim follows, for instance, since a compact Kähler surface with a nontrivial holomorphic gradient $\nabla\tau$ having $dQ \wedge d\tau = 0$ identically for $Q = g(\nabla\tau, \nabla\tau)$ must necessarily [**10**, Sect. 1] be biholomorphic to $\mathbb{C}\mathrm{P}^2$ or a $\mathbb{C}\mathrm{P}^1$ bundle over $\mathbb{C}\mathrm{P}^1$ (rather than the two-point blow-up of $\mathbb{C}\mathrm{P}^2$). The latter, in the case of [**22**], is due to a general reason: all Ricci solitons are real-analytic [**8**, Lemma 3.2]. So are, however, all Riemannian Einstein metrics [**13**, Theorem 5.2], and the Chen-LeBrun-Weber metric of [**6**] is conformal to an Einstein metric $\hat{g}$, while again, for a general reason [**9**, p. 417, Prop. 3(ii)], the conformal change leading from $\hat{g}$ to g has a canonical form (up to a constant factor, it is the multiplication by the cubic root of the norm-squared of the self-dual Weyl tensor). This causes g to be real-analytic as well. □

11. The analytic-continuation phenomenon

We elaborate here on the plausibility of small deformations mentioned in the lines following Theorem E, beginning with the coth-cot analytic continuation. The real-analytic function $\mathbb{R} \ni y \mapsto y^{-1}\tanh y$, with the value 1 at $y = 0$, being even, has the form $\varSigma(y^2)$ for some real-analytic function $\varSigma$. Now $(\varepsilon, \tau) \mapsto \beta_\varepsilon(\tau) = \tau\varSigma(\varepsilon\tau^2)$ is a real-analytic function on an open subset of $\mathbb{R}^2$ and $\beta_\varepsilon(\tau)$ equals $\varepsilon^{-1/2}\tanh(\varepsilon^{1/2}\tau)$, or τ, or $|\varepsilon|^{-1/2}\tan(|\varepsilon|^{1/2}\tau)$, depending on whether $\varepsilon > 0$, or $\varepsilon = 0$, or $\varepsilon < 0$. For $\alpha_\varepsilon(\tau) = 2/\beta_\varepsilon(\tau)$ the analogous expressions are

$$2\varepsilon^{1/2}\coth(\varepsilon^{1/2}\tau) \quad (\text{if } \varepsilon > 0), \qquad 2/\tau \quad (\text{if } \varepsilon = 0), \qquad 2|\varepsilon|^{1/2}\cot(|\varepsilon|^{1/2}\tau) \quad (\text{if } \varepsilon < 0).$$

All α_ε with $\varepsilon > 0$, as well as those with $\varepsilon < 0$, are thus affine (in fact, linear) modifications – see Remark C – of α_1 or, respectively, α_{-1}, and $\alpha_0(\tau) = 2/\tau$.

For a tanh-coth analytic-continuation argument we define $(t, \tau) \mapsto \alpha_t(\tau)$ by $\alpha_t(\tau) = 2(e^\tau - te^{-\tau})/(e^\tau + te^{-\tau})$. Thus, with q such that $2q = \log|t|$, if $t > 0$ (or, $t < 0$), $\alpha_t(\tau) = 2\tanh(\tau - q)$ or, respectively, $\alpha_t(\tau) = 2\coth(\tau - q)$. Again, all α_t for $t > 0$, or those with $t < 0$, are affine (this time, translational) modifications of α_1, or of α_{-1}, while $\alpha_0(\tau) = 2$.

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