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SOME THEOREMS ON CONFORMALLY SYMMETRIC MANIFOLDS.

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§1. **Introduction.** The present paper concerns with n-dimensional ($n \geq 4$) Riemannian manifolds (not necessarily of definite metrics) which are conformally symmetric [1],[1] i.e. satisfy the relation

$$(1) \qquad C^h{}_{ijk,l} = 0 ,$$

where

$$C^h{}_{ijk} = R^h{}_{ijk} - (g_{ij}R^h{}_k - g_{ik}R^h{}_j + \delta^h_k R_{ij} - \delta^h_j R_{ik})/(n-2)$$
$$+ R(\delta^h_k g_{ij} - \delta^h_j g_{ik})/(n-1)(n-2)$$

denotes the Weyl's conformal curvature tensor of the manifold. More precisely, we consider manifolds which are essentially conformally symmetric, that is, satisfy (1) but are neither conformally flat nor locally symmetric. To investigate these manifolds we examine the behaviour of the parallel tensor field $C_{mijk}C^{mrst}$. Section 3 of this paper is devoted to essentially conformally symmetric manifolds which satisfy the condition

$$(2) \qquad C_{mijk}C^{mrst} = 0 .$$

In Section 4 we consider the remaining case

$$(3) \qquad C_{mijk}C^{mrst} \neq 0 ,$$

i.e. the case where $C_{mijk}C^{mrst}$ does not vanish identically. We prove there (Theorem 6) that every essentially conformally symmetric manifold satisfying (3) is Ricci-recurrent. As a consequence, we obtain the following result (Corollary 1): Every essentially conformally symmetric manifold with a metric of index one is Ricci-recurrent. Finally, in Section 5 we state some general theorems on conformally symmetric manifolds. We prove there that every essentially conformally symmetric manifold admits a non-trivial null parallel distribution and satisfies the relations $R = 0$, $R_{hijk,lm} - R_{hijk,ml} = 0$, $R_i{}^r R_{rj} = 0$, $R_i{}^r R_{rj,k} = 0$ and $R_{ir}C^r{}_{jkl} = 0$.

Throughout this paper, by a manifold we shall mean a connected Hausdorff manifold of class C^∞. All manifolds considered below are of dimension $n \geq 4$.

§2. **Preliminaries.** The Weyl's tensor satisfies the well-known relations which we list here for convenience:

$$(4) \quad \begin{aligned} &\text{a)} \quad R^h{}_{ijk} = C^h{}_{ijk} + (g_{ij}R^h{}_k - g_{ik}R^h{}_j + \delta^h_k R_{ij} - \delta^h_j R_{ik})/(n-2) \\ &\qquad\qquad - R(\delta^h_k g_{ij} - \delta^h_j g_{ik})/(n-1)(n-2) , \\ &\text{b)} \quad C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi} = C_{kjih} , \\ &\text{c)} \quad C^h{}_{ijk} + C^h{}_{jki} + C^h{}_{kij} = 0 , \qquad \text{d)} \quad C^r{}_{ijr} = C^r{}_{irj} = C^r{}_{rij} = 0 , \\ &\text{e)} \quad C^r{}_{ijk,r} = (n-3)[R_{ij,k} - R_{ik,j} - (R_{,k}g_{ij} - R_{,j}g_{ik})/2(n-1)]/(n-2) . \end{aligned}$$

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1) Numbers in brackets refer to the references at the end of the paper.

Every essentially conformally symmetric manifold satisfies the relations

$$(5) \qquad R = \text{constant}\,, \qquad (\text{see [8], Theorem 6})\,,$$

$$(6) \qquad R_{ij,k} = R_{ik,j}\,,$$

which follows from (5) and (6),

$$(7) \qquad R^h{}_{ijk,l} = (g_{ij}R^h{}_{k,l} - g_{ik}R^h{}_{j,l} + \delta^h_k R_{ij,l} - \delta^h_j R_{ik,l})/(n-2)\,,$$

which is a consequence of (4)a), (1) and (5), and

$$(8) \quad \begin{aligned} &\text{a)} \quad R_{ri,j}R^r{}_{k,l} = 0\,, \qquad \text{b)} \quad R_{ri,j}C^r{}_{klm} = 0\,,\\ &\text{c)} \quad R_{rj}C^r{}_{ikl} + R_{rk}C^r{}_{ilj} + R_{rl}C^r{}_{ijk} = 0\,,\\ &\text{d)} \quad R_{hl,p}C_{mijk} - R_{hm,p}C_{lijk} + R_{il,p}C_{hmjk} - R_{im,p}C_{hljk} + R_{jl,p}C_{himk}\\ &\qquad - R_{jm,p}C_{hilk} + R_{kl,p}C_{hijm} - R_{km,p}C_{hijl} = 0 \end{aligned}$$

(see [2], formulae (19), (18), (7) and (20)).

Lemma 1. *Every conformally symmetric manifold satisfies the relation*

$$(9) \quad \begin{aligned} &R_{hl}C_{mijk} - R_{hm}C_{lijk} + R_{il}C_{hmjk} - R_{im}C_{hljk} + R_{jl}C_{himk} - R_{jm}C_{hilk}\\ &+ R_{kl}C_{hijm} - R_{km}C_{hijl} + g_{hl}R_m{}^rC_{rijk} - g_{hm}R_l{}^rC_{rijk} + g_{il}R_m{}^rC_{hrjk}\\ &- g_{im}R_l{}^rC_{hrjk} + g_{jl}R_m{}^rC_{hirk} - g_{jm}R_l{}^rC_{hirk} + g_{kl}R_m{}^rC_{hijr} - g_{km}R_l{}^rC_{hijr}\\ &- R(g_{hl}C_{mijk} - g_{hm}C_{lijk} + g_{il}C_{hmjk} - g_{im}C_{hljk} + g_{jl}C_{himk} - g_{jm}C_{hilk}\\ &+ g_{kl}C_{hijm} - g_{km}C_{hijl})/(n-1)\\ &= (n-2)(C_{lmh}{}^rC_{rijk} + C_{lmi}{}^rC_{hrjk} + C_{lmj}{}^rC_{hirk} + C_{lmk}{}^rC_{hijr})\,. \end{aligned}$$

Proof. From (1) and Ricci identity we obtain

$$0 = C_{hijk,ml} - C_{hijk,lm} = R_{mlh}{}^rC_{rijk} + R_{mli}{}^rC_{hrjk} + R_{mlj}{}^rC_{hirk} + R_{mlk}{}^rC_{hijr}\,,$$

which, in view of (4)a), turns into

$$\begin{aligned} 0 = {}&C_{mlh}{}^rC_{rijk} + C_{mli}{}^rC_{hrjk} + C_{mlj}{}^rC_{hirk} + C_{mlk}{}^rC_{hijr} + [g_{lh}R_m{}^rC_{rijk} - g_{mh}R_l{}^rC_{rijk}\\ &+ R_{hl}C_{mijk} - R_{hm}C_{lijk} + g_{li}R_m{}^rC_{hrjk} - g_{mi}R_l{}^rC_{hrjk} + R_{il}C_{hmjk} - R_{im}C_{hljk}\\ &+ g_{lj}R_m{}^rC_{hirk} - g_{mj}R_l{}^rC_{hirk} + R_{jl}C_{himk} - R_{jm}C_{hilk} + g_{lk}R_m{}^rC_{hijr}\\ &- g_{mk}R_l{}^rC_{hijr} + R_{kl}C_{hijm} - R_{km}C_{hijl} - R(g_{hl}C_{mijk} - g_{hm}C_{lijk} + g_{il}C_{hmjk}\\ &- g_{im}C_{hljk} + g_{jl}C_{himk} - g_{jm}C_{hilk} + g_{kl}C_{hijm} - g_{km}C_{hijl})/(n-1)]/(n-2)\,. \end{aligned}$$

This, together with (4)b), implies immediately our assertion.

Now we state four algebraic lemmas, which will be used in the next sections:

Lemma 2 ([2], Lemma 1). *Let a tensor $A_{lmjs_1\ldots s_N}$ be symmetric in (l, m) and skew-symmetric in (m, j). Then $A_{lmjs_1\ldots s_N} = 0$.*

Lemma 3. *Let e_{ij} and T_{ijk} be tensors, satisfying the conditions*

$$(10) \quad \begin{aligned} &\text{a)} \quad e_{ij} = e_{ji}\,, \qquad T_{ijk} + T_{ikj} = 0\,, \qquad \text{b)} \quad T_{ijk} + T_{jki} + T_{kij} = 0\,,\\ &\text{c)} \quad e_{hl}T_{ijk} + e_{il}T_{hkj} + e_{jl}T_{khi} + e_{kl}T_{jih} = 0\,,\\ &\text{d)} \quad e_{hl}T_{mkj} - e_{hm}T_{lkj} + e_{jl}T_{hkm} - e_{jm}T_{hkl} + e_{kl}T_{hmj} - e_{km}T_{hlj} = 0\,. \end{aligned}$$

Then they also satisfy

(11) $$e_{hl}T_{ijk} = e_{il}T_{hjk} .$$

Proof. Set $q = \operatorname{rank} e_{ij}$. By a suitable choice of coordinates we may check that $e_{11} \neq 0, \ldots, e_{qq} \neq 0$ and all other components of e_{ij} vanish. Since our assertion is obvious when $T_{ijk} = 0$ or $e_{ij} = 0$, we shall assume that $T_{ijk} \neq 0$ and $q \geq 1$. We adopt the following convention about indices: h, i, j, k, l, m will index tensors, while a, b, c, d will be used to denote their components in our coordinate system. Thus e_{ij} is a tensor, but e_{ab} is a real number.

We have

(12) if $T_{abc} \neq 0$ for some particular indices a, b, c, then the set $\{1, \ldots, q\}$ is contained in $\{a, b, c\}$.

In fact, conversely we could choose d with $1 \leq d \leq q$ and $d \neq a$, $d \neq b$, $d \neq c$. Setting in (10)c) $h = l = d$ and $i = a$, $j = b$, $k = c$, we would obtain $e_{dd}T_{abc} = 0$, a contradiction.

Now we assert that $q = 1$. To prove this, suppose $q \geq 2$ and set (10)c) $i = k = l = 1$, $j = 2$ and $h = a > 1$. This yields $e_{11}T_{a12} + e_{11}T_{21a} = 0$, so $T_{212} = 0$ and, by (10)a), $T_{21a} = -T_{a12} = T_{a21}$ for $a > 2$. In view of (10)b), $T_{21a} + T_{a21} + T_{1a2} = 0$, hence $T_{12a} = -T_{1a2} = 2T_{a21} = 2T_{21a}$ for $a > 2$. Therefore, setting in (10)d) $h = k = l = 2$, $m = 1$ and $j = a > 2$, we have $0 = e_{22}T_{12a} + e_{22}T_{21a} = 3e_{22}T_{21a}$, which yields, for $a > 2$, $T_{12a} = T_{21a} = T_{a12} = T_{a21} = T_{1a2} = T_{2a1} = 0$. From the above equalities and (12) it follows that the only components of T_{ijk} which may not vanish are $T_{112} = -T_{121}$. Setting now in (10)d) $h = j = l = 2$, $k = m = 1$, we obtain $e_{22}T_{112} = 0$, so $T_{ijk} = 0$, a contradiction. This shows that $q = 1$.

Setting in (10)c) $h = j = l = 1$, $i = a$, $k = b$, $a, b > 1$, we have $e_{11}T_{a1b} + e_{11}T_{b1a} = 0$, so $T_{a1b} = -T_{b1a}$ for $a, b > 1$. Therefore (10)b) implies, for $a, b > 1$, $T_{a1b} = T_{1ab} + T_{b1a} = T_{1ab} - T_{a1b}$, which yields

(13) $$T_{1ab} = 2T_{a1b} \qquad \text{for} \quad a, b > 1 .$$

Setting in (10)d) $h = j = l = 1$, $k = a$, $m = b$, $a, b > 1$, we obtain $0 = e_{11}T_{ba1} + e_{11}T_{1ab} = -e_{11}(T_{b1a} + T_{1ba})$, so, by (13), $0 = -3e_{11}T_{b1a}$, i.e. $T_{1ab} = T_{a1b} = T_{ab1} = 0$ for $a, b > 1$. From (12) it follows now that

(14) $$T_{abc} = 0 \qquad \text{whenever} \quad a > 1 .$$

We are now in a position to verify (11). Both sides vanish unless $l = 1$. If $h, i > 1$, then they vanish again. In the case $h = i = l = 1$, both sides are equal to $e_{11}T_{1jk}$. The remaining case reduces to $h = l = 1$, $i > 1$. The right-hand side vanishes as $e_{i1} = 0$, the left-hand one does so by (14). This completes the proof.

Lemma 4. *Let A_i be a non-zero covariant vector and $B_{i_1 \ldots i_N}$ a tensor, satisfying the relation*

(15) $$A_i B_{j i_2 \ldots i_N} = A_j B_{i i_2 \ldots i_N} .$$

Then there exists a unique tensor $D_{i_2 \ldots i_N}$ such that

(16) $$B_{i_1 \ldots i_N} = A_{i_1} D_{i_2 \ldots i_N} .$$

Proof. For any contravariant vectors $a_2^i, \ldots, a_N^i$, (15) yields $B_{i i_2 \ldots i_N} a_2^{i_2} \cdots a_N^{i_N} = \lambda A_i$, the (uniquely determined) scalar λ being a multilinear function of our contravariant arguments, say $\lambda = D_{i_2\ldots i_N}a_2^{i_2}\cdots a_N^{i_N}$. Since these arguments are taken arbitrarily, (16) follows from the resulting equality $B_{ii_2\ldots i_N}a_2^{i_2}\cdots a_N^{i_N} = A_i D_{i_2\ldots i_N}a_2^{i_2}\cdots a_N^{i_N}$.

Lemma 5. *Let $A_{i_1\ldots i_K}$ and $B_{j_1\ldots j_N}$ be non-zero tensors satisfying the relation*

$$(17) \qquad A_{ii_2\ldots i_K}B_{jj_2\ldots j_N} = A_{ji_2\ldots i_K}B_{ij_2\ldots j_N}\ .$$

Then (i) *the space of covariant vectors contains a unique line L such that any vector v_i of the form*

$$(18) \qquad \text{a)}\ \ v_i = A_{ii_2\ldots i_K}u_2^{i_2}\cdots u_K^{i_K} \qquad or \qquad \text{b)}\ \ v_i = B_{ii_2\ldots i_N}w_2^{i_2}\cdots w_N^{i_N}$$
$$(v_i = B_i\ \ if\ \ N=1)\ ,$$

where $u_2^i, w_2^i, \ldots$ etc. are contravariant vectors, lies in L; (ii) *if, moreover, $A_{i_1\ldots i_K}$ is symmetric in all indices, then $A_{i_1\ldots i_K} = \lambda v_{i_1}\cdots v_{i_K}$, v_i being any non-zero vector of L and λ a certain scalar.*

Proof. In view of (17) any two vectors of types (18), respectively, are collinear. Since our tensors are both non-zero, (i) is clear. To prove (ii) we may proceed by induction on K. The inductive step in based on the equality $A_{i_1\ldots i_K} = D_{i_1\ldots i_{K-1}}v_{i_K}$ for some symmetric tensor $D_{i_1\ldots i_{K-1}}$, which can be obtained as follows: By (i) and the assumption of symmetry we have, for arbitrary covariant vectors $u_1^i, \ldots, u_{K-1}^i$, $A_{i_1\ldots i_K}u_1^{i_1}\cdots u_{K-1}^{i_{K-1}} = \lambda v_{i_K}$, where λ depends on these vectors multilinearly, say $\lambda = D_{i_1\ldots i_{K-1}}u_1^{i_1}\cdots u_{K-1}^{i_{K-1}}$. The inductive hypothesis, applied to $D_{i_1\ldots i_{K-1}}$, completes the proof.

A Riemannian manifold M is said to be Ricci-recurrent provided that for each $x\in M$ such that $R_{ij}(x)\neq 0$, there exists a tangent vector A_k at x, which satisfies the condition

$$(19) \qquad R_{ij,k}(x) = A_k R_{ij}(x)\ .$$

The existence of essentially conformally symmetric Ricci-recurrent manifolds can be established (see [6], Theorem 3 and Lemma 6, and [7], Theorem 1) as follows:

Theorem 1. *Let M denote the Euclidean n-space ($n\geq 4$) endowed with the metric g_{ij} given by*

$$(20) \qquad g_{ij}dx^i dx^j = \varphi(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1 dx^n\ , \qquad \varphi = (Ak_{\lambda\mu} + a_{\lambda\mu})x^\lambda x^\mu\ ,$$

where $i, j, \ldots = 1, \ldots, n$ and $\lambda, \mu, \alpha, \beta, \ldots = 2, \ldots, n-1$ and A is a non-constant function of x^1 only, $[k_{\lambda\mu}]$ and $[a_{\lambda\mu}]$ are non-zero symmetric matrices such that $[k_{\lambda\mu}]$ is non-singular and $k^{\lambda\mu}a_{\lambda\mu} = 0$, $[k^{\lambda\mu}]$ being the reciprocal of $[k_{\lambda\mu}]$. Then M is an essentially conformally symmetric Ricci-recurrent Riemannian manifold.

Remark 1. For a metric g_{ij} of the form (20) we have

$$(21) \qquad \text{index of } [g_{ij}] = \text{index of } [k_{\lambda\mu}] + 1\ ,$$

the index of a symmetric matrix being understood as the number of negative entries in its diagonal form. In fact, it is sufficient to verify (21) at the point x with coordinates $x^1 = \cdots = x^n = 0$. Denoting by $X_1, \ldots, X_n$ the basis of T_xM determined by our coordinate system and by $\langle\ldots, \ldots\rangle$ the inner product induced by $g_{ij}(x)$, we have $\langle X_1, X_1\rangle = \langle X_1, X_\lambda\rangle = \langle X_n, X_\lambda\rangle = \langle X_n, X_n\rangle = 0$, $\langle X_1, X_n\rangle = 1$, $\langle X_\lambda, X_\mu\rangle = k_{\lambda\mu}$. Setting now $Y_1 = 2^{-1/2}(X_1 + X_n)$, $Y_n = 2^{-1/2}(X_1 - X_n)$, we obtain a new basis Y_1,

$X_2, \ldots, X_{n-1}, Y_n$ for $T_x M$ and (21) follows immediately from the relations $\langle Y_1, Y_1\rangle = -\langle Y_n, Y_n\rangle = 1$, $\langle Y_1, Y_n\rangle = \langle Y_1, Y_\lambda\rangle = \langle Y_n, Y_\lambda\rangle = 0$.

Remark 2. In view of (21), essentially conformally symmetric Riemannian metrics may assume all indices from the range $\{1, 2, \ldots, n-1\}$, n being the dimension of the underlying manifold. On the other hand, such metrics are never definite ([2], Theorem 2).

Lemma 6. *A Riemannian metric g_{ij} of the form* (20) *satisfies* (2) *if and only if*

$$k^{\lambda\mu}a_{\lambda\alpha}a_{\mu\beta} = 0 \,. \tag{22}$$

Proof. It is easy to verify (see [6], p. 93) that the only components of C_{hijk}, which may not vanish, are those related to

$$C_{1\lambda\mu1} = \tfrac{1}{2}\partial_\lambda\partial_\mu\varphi - k_{\lambda\mu}(k^{\alpha\beta}\partial_\alpha\partial_\beta\varphi)/2(n-2) = a_{\lambda\mu}\,,$$

and that the reciprocal tensor g^{ij} of g_{ij} is given by $g^{1n} = 1$, $g^{11} = g^{1\lambda} = g^{n\lambda} = 0$, $g^{\lambda\mu} = k^{\lambda\mu}$, $g^{nn} = -\varphi$. Our assertion can now be obtained by an explicit computation of $C_{mijk}C^{mrst} = g^{lm}C_{lijk}C_{mrst}$.

In the sequel, we shall use the notion of distribution in the sense explained in ([4], p. 10). A distribution on a Riemannian manifold will be called null if any vector u_i of the distribution satisfies the relation $u_iu^i = 0$.

§ 3. **The case $C_{mijk}C^{mrst} = 0$.** This section is devoted to essentially conformally symmetric manifolds which satisfy (2). First we establish their existence:

Theorem 2. *For each $n \geq 4$ and each $q \in \{2, \ldots, n-2\}$ there exists an n-dimensional essentially conformally symmetric Riemannian manifold with a metric g_{ij} of index q, which satisfies* (2).

Proof. We define the metric g_{ij} by (20), setting $k_{22} = \cdots = k_{qq} = -1$, $k_{q+1,q+1} = \cdots = k_{n-1,n-1} = 1$, $k_{\lambda\mu} = 0$ for $\lambda \neq \mu$ and $a_{22} = a_{2,n-1} = a_{n-1,2} = a_{n-1,n-1} = 1$ and $a_{\lambda\mu} = 0$ for other values of λ and μ. It is easy to see that $k^{\lambda\mu}a_{\lambda\mu} = 0$ and to verify (22). Our assertion follows now from Theorem 1, Lemma 6 and (21).

Theorem 3. *Let M be an n-dimensional essentially conformally symmetric Riemannian manifold with a metric of index q. If M satisfies the relation $C_{mijk}C^{mrst} = 0$, then it admits a null parallel distribution Δ such that*

$$2 \leq \dim \Delta \leq \min(q, n-q) \leq \tfrac{1}{2}n \,. \tag{23}$$

Proof. Given $x \in M$, define D_x to be the set of all vectors w_i of the form $w_i = C_{ijkl}a^jb^kc^l$, where a^i, b^i, c^i run through T_xM. In view of (2) and two vectors of D_x are isotropic and mutually orthogonal. From (1) it now follows easily that by assigning to each $x \in M$ the linear span Δ_x of D_x, we define a null parallel distribution Δ on M.

Now fix $x \in M$ and choose vectors a^i, b^i tangent at x so that $C_{ijkl}a^kb^l = F_{ij} \neq 0$. Since F_{ij} is an exterior 2-form, we have $\operatorname{rank} F_{ij} \geq 2$. It is clear that Δ_x contains all vectors d_i of the form $d_i = F_{ij}c^j$, where c^i runs through T_xM, and that these vectors form a subspace of T_xM of dimension $\operatorname{rank} F_{ij}$. This shows that $\dim \Delta \geq 2$. The remaining inequality follows, in a purely algebraic manner (see [5], p. 362, Corollary 2 and [3], p. 229, Problem 3) from the fact that Δ is null. This completes the proof.

Lemma 7. *Let M be an essentially conformally symmetric manifold satisfying* (2). *Then M satisfies the relations*

$$\text{a)}\quad R=0\,,\qquad \text{b)}\quad R_i{}^rR_{rj}=0\,,\qquad \text{c)}\quad R_i{}^rR_{rj,k}=0\,,\qquad \text{d)}\quad R_{ir}C^r{}_{jkl}=0\,,$$

$$\text{(24)}\qquad \text{e)}\quad R_{kl}C_{mijk}-R_{hm}C_{lijk}+R_{il}C_{hmjk}-R_{im}C_{hljk}+R_{jl}C_{himk}-R_{jm}C_{hilk}$$
$$+R_{kl}C_{hijm}-R_{km}C_{hijl}=0\,.$$

Proof. By (2), the right-hand side of (9) vanishes. Contracting (9) with g^{hl} and using (4)b), d) and the obvious formula $R^{rs}C_{rsij}=0$, we obtain

$$\text{(25)}\qquad RC_{mijk}+(n-1)R_{mr}C^r{}_{ijk}+R_{ir}C^r{}_{mjk}+g_{jm}R^{rs}C_{riks}-g_{km}R^{rs}C_{rijs}$$
$$+R_{jr}C^r{}_{imk}+R_{kr}C^r{}_{ijm}+R_{mr}C^r{}_{ikj}+R_{mr}(C^r{}_{ikj}+C^r{}_{kji}+C^r{}_{jik})$$
$$-R[(n-1)C_{mijk}+C_{imkj}+C_{ijkm}+C_{ikmj}]/(n-1)=0\,.$$

In view of (4)c) and (8)c) we have $R_{jr}C^r{}_{imk}+R_{kr}C^r{}_{ijm}+R_{mr}C^r{}_{ikj}=0$, $C^r{}_{ikj}+C^r{}_{kji}+C^r{}_{jik}=0$ and $C_{imjk}+C_{ijkm}+C_{ikmj}=0$, so (25) takes the form

$$\text{(26)}\qquad (n-1)R_{mr}C^r{}_{ijk}+R_{ir}C^r{}_{mjk}+g_{jm}R^{rs}C_{riks}-g_{km}R^{rs}C_{rijs}=0\,.$$

Transvecting (26) with $C^k{}_{hlp}$ and using (2) we obtain $C_{mhlp}R^{rs}C_{rijs}=0$, which yields $R^{rs}C_{rijs}=0$. Now (26) turns into $(n-1)R_{mr}C^r{}_{ijk}+R_{ir}C^r{}_{mjk}=0$, i.e. $R_{ir}C^r{}_{mjk}=(1-n)R_{mr}C^r{}_{ijk}=(1-n)^2R_{ir}C^r{}_{mjk}$, which clearly implies (24)d).

Transvecting now (9) with $C^k{}_{pst}$ and using (2) and (24)d), we obtain $R(C_{lpst}C_{hijm}-C_{mpst}C_{hijl})=0$. Therefore the tensor $A_{lmjpsthi}=RC_{lpst}C_{hijm}$ satisfies the hypothesis of Lemma 2. Hence $RC_{lpst}C_{hijm}=0$, which implies (24)a), since M is not conformally flat.

In virtue of (2), (24)a) and (24)d), equality (9) takes the form (24)c).

Transvecting (24)c) with $R_p{}^i$ and using (24)d), we see that $R_p{}^iR_{il}C_{hmjk}=R_p{}^iR_{im}C_{hljk}$. Hence, applying Lemma 2 to $A_{lmhpjk}=R_p{}^iR_{il}C_{hmjk}$, we conclude that $R_p{}^iR_{il}C_{hmjk}=0$, which obviously implies (24)b). To prove (24)c), we use the same procedure, i.e. transvect (24)e) with $R_p{}^i{}_{,r}$ taking into account (8)b), and then apply Lemma 2 to $A_{lmhprjk}=R_p{}^i{}_{,r}R_{il}C_{hmjk}$. This completes the proof.

§ 4. **The case $C_{mijk}C^{mrst}\neq 0$.** In this section we consider essentially conformally symmetric manifolds which satisfy (3). First we prove that this class contains all essentially conformally symmetric Riemannian metrics of index 1 (and $n-1$) and some metrics of other indices.

Theorem 4. (i) *Every essentially conformally symmetric Riemannian manifold with a metric of index* 1 (*or* $n-1$) *satisfies the relation* $C_{mijk}C^{mrst}\neq 0$. (ii) *For each* $n\geq 4$ *and any* $q\in\{2,\ldots,n-2\}$ *there exists an n-dimensional essentially conformally symmetric Riemannian manifold with a metric g_{ij} of index q, which satisfies* (3).

Proof. (i) Let M be an essentially conformally symmetric manifold with a metric of index q such that $C_{mijk}C^{mrst}=0$. By (23) we have $2\leq q\leq n-2$, which proves our assertion. (ii) Define the metric g_{ij} by (20) with matrices $[k_{\lambda\mu}]$ and $[a_{\lambda\mu}]$ given by $k_{22}=\cdots=k_{qq}=-1$, $k_{q+1,q+1}=\cdots=k_{n-1,n-1}=1$, $k_{\lambda\mu}=0$ for $\lambda\neq\mu$ and $a_{22}=a_{n-1,n-1}=1$ and $a_{\lambda\mu}=0$ for other values of λ and μ. Clearly, $k^{\lambda\mu}a_{\lambda\mu}=0$, so in view of Theorem 1 and (21), g_{ij} is an essentially conformally symmetric Riemannian metric of index q. We have $k^{\lambda\mu}a_{\lambda 2}a_{\mu 2}=k^{22}a_{22}a_{22}=-1$, hence our assertion follows from Lemma 6. This completes the proof.

Theorem 5. *Let M be an essentially conformally symmetric Riemannian manifold such that the parallel tensor field $C_{mijk}C^{mrst}$ does not vanish identically on M. For $x \in M$, denote by L_x the set of all vectors v_i at x of the form $v_i = C_{mijk}C^{mrst}a^j b^k c_r d_s e_t$, where a^i, b^i, c^i, d^i, e^i run through T_xM. Then the assignment $x \mapsto L_x$ defines a null parallel 1-dimensional distribution L on M. Distribution L has the following property: If $x \in M$ and N is a positive integer, then L_x contains all vectors w_i of the type*

$$w_i = R_{ij,p_1\ldots p_N} u^j u_1^{p_1} \cdots u_N^{p_N}, \tag{27}$$

where u^i, u_1^i, $\ldots$, u_N^i are vectors tangent at x.

Proof. Transvecting (8)d) with C^{mrst} and with C^{irst}, respectively, and using (8)b), we obtain

$$R_{kl,p}C_{mijk}C^{mrst} + R_{il,p}C_{mhkj}C^{mrst} + R_{jl,p}C_{mkhi}C^{mrst} + R_{kl,p}C_{mjih}C^{mrst} = 0\,,$$

$$R_{hl,p}C_{imkj}C^{irst} - R_{hm,p}C_{ilkj}C^{irst} + R_{jl,p}C_{ihkm}C^{irst} - R_{jm,p}C_{ihkl}C^{irst}$$
$$+ R_{kl,p}C_{ihmj}C^{irst} - R_{km,p}C_{ihlj}C^{irst} = 0\,.$$

Now fix p, r, s, t and set $e_{ij} = R_{ij,p}$, $T_{ijk} = C_{mijk}C^{mrst}$. From the above equalities it follows immediately that e_{ij} and T_{ijk} satisfy (10). By Lemma 3, they satisfy (11), i.e.

$$R_{kl,p}C_{mijk}C^{mrst} = R_{il,p}C_{mhjk}C^{mrst}\,. \tag{28}$$

Now choose $x \in M$ such that $R_{ij,k}(x) \neq 0$. From (28) and (i) of Lemma 5 it follows that L_x is a line in T_xM. Moreover, L_x is null in view of Lemma 5 and (8)a). By (1), the same holds for each $x \in M$, which proves that L is a null parallel 1-dimensional distribution on M.

Differentiating (28) covariantly, we obtain $R_{kl,p_1\ldots p_N}C_{mijk}C^{mrst} = R_{il,p_1\ldots p_N}C_{mhjk} \times C^{mrst}$. It follows now from Lemma 5 that L contains all vectors of the form (27), which completes the proof.

Remark 3. By (i) of Theorem 4, Theorem 5 is a generalization of Theorem 4 of [2] (the distribution, constructed there, clearly coincides with L).

Using Theorem 5, we shall prove

Lemma 8. *Let M be an essentially conformally symmetric manifold satisfying* (3). *Then M satisfies the relation*

$$C_{lmk}{}^{r}C_{rijk} + C_{lmi}{}^{r}C_{hrjk} + C_{lmj}{}^{r}C_{hirk} + C_{lmk}{}^{r}C_{hijr} = 0\,. \tag{29}$$

Proof. We restrict our consideration to a fixed point x of M. Choose a non-zero vector w_i of L_x. By Theorem 5 we have $C_{mijk}C^{m}{}_{rst}a^j b^k c^r d^s e^t = \lambda w_i$ for any vectors a^i, b^i, c^i, d^i, e^i of T_xM, λ being a 5-linear real function of these vectors, say $\lambda = G_{jkrst}a^j b^k c^r d^s e^t$. Thus we have $C_{mijk}C^{m}{}_{rst} = w_i G_{jkrst}$. The obvious equality $C_{mijk}C^{m}{}_{rst} = C_{mrst}C^{m}{}_{ijk}$ can now be written as

$$w_i G_{jkrst} = w_r G_{stijk}\,. \tag{30}$$

Set $H_{jkrst} = G_{jkrst} - G_{strjk}$. By (30) we have

$$w_i H_{jkrst} = -w_r H_{jkist}\,. \tag{31}$$

Now choose a vector c^i at x such that $c^i w_i = 1$. Transvecting (31) with c^i, we obtain

$$H_{jkrst} = -w_r c^h H_{jkhst}\,, \tag{32}$$

so (31) yields $-w_i w_r c^h H_{jkhst} = w_i H_{jkrst} = -w_r H_{jkist} = w_r w_i c^h H_{jkhst}$. Hence $c^h H_{jkhst} = 0$ and, by (32), $H_{jkrst} = 0$. Thus we have $G_{jkrst} = G_{strjk}$, so (30) takes the form $w_i G_{jkrst} = w_r G_{jkist}$. From Lemma 4 we obtain $G_{jkrst} = w_r K_{jkst}$ for some tensor K_{jkst}, i.e. $C_{mijk} C^m{}_{rst} = w_i w_r K_{jkst}$. Using (4)b), we now compute

$$C_{lmh}{}^r C_{rijk} + C_{lmi}{}^r C_{hrjk} + C_{lmj}{}^r C_{hirk} + C_{lmk}{}^r C_{hijr} = C_{rijk} C^r{}_{hml} - C_{rhjk} C^r{}_{iml}$$
$$+ C_{rkhi} C^r{}_{jml} - C_{rjhi} C^r{}_{kml} = w_i w_h K_{jkml} - w_h w_i K_{jkml} + w_k w_j K_{himl} - w_j w_k K_{himl} = 0\,,$$

which completes the proof.

Lemma 9. *Let M be an essentially conformally symmetric manifold satisfying* (3). *Denote by U the open subset of M consisting of points at which $R_{ij,k} \neq 0$. Then there exists a unique C^∞ vector field v_i on U such that*

$$R_{ij,k} = v_i v_j v_k\,. \tag{33}$$

Moreover, v_i spans the distribution L (defined in Theorem 5) at each point of U and we have

$$\text{a)}\quad R_{ir} v^r = R v_i / 2(n-1)\,, \qquad \text{b)}\quad R_{ij,klm} = v_i v_j v_k v_l P_m \tag{34}$$

for some (uniquely determined) vector field P_i on U.

Proof. From (28), (6) and (ii) of Lemma 5 we obtain (33). Clearly, v_i spans L on U. Since L is parallel and null, we have

$$v_{i,j} = v_i S_j \tag{35}$$

for a certain vector field S_i on U and

$$v_i v^i = 0\,, \qquad v_i \neq 0\,. \tag{36}$$

From (33) we obtain

$$R_{ij,kl} = 3 v_i v_j v_k S_l\,, \tag{37}$$

which yields, in view of Ricci identity and (4)a),

$$\begin{aligned} 3v_i v_j (v_k S_l - v_l S_k) &= R_{ij,kl} - R_{ij,lk} \\ &= R_{kli}{}^r R_{rj} + R_{klj}{}^r R_{ir} = C_{kli}{}^r R_{rj} + C_{klj}{}^r R_{ir} \\ &\quad + (g_{il} R_k{}^r R_{rj} - g_{ik} R_l{}^r R_{rj} + g_{jl} R_k{}^r R_{ir} - g_{jk} R_l{}^r R_{ir})/(n-2) \\ &\quad - R(g_{il} R_{jk} + g_{jl} R_{ik} - g_{ik} R_{jl} - g_{jk} R_{il})/(n-1)(n-2)\,. \end{aligned} \tag{38}$$

Now set

$$u_i = R_{ir} v^r\,. \tag{39}$$

Differentiating (38) covariantly and using (35), (39) and (8)b), we obtain

$$\begin{aligned} 3v_i v_j (3 v_k S_l S_m - 3 v_l S_k S_m + v_k S_{l,m} - v_l S_{k,m}) &= R_{ij,klm} - R_{ij,lkm} \\ &= (g_{il} u_k v_j v_m + g_{il} v_k u_j v_m - g_{ik} u_l v_j v_m - g_{ik} v_l u_j v_m + g_{jl} u_k v_i v_m \\ &\quad + g_{jl} v_k u_i v_m - g_{jk} u_l v_i v_m - g_{jk} v_l u_i v_m)/(n-2) \\ &\quad - R(g_{il} v_j v_k v_m - g_{ik} v_j v_l v_m + g_{jl} v_i v_k v_m - g_{jk} v_i v_l v_m)/(n-1)(n-2)\,. \end{aligned} \tag{40}$$

Transvecting (40) with v^i and taking into account (36), we have

$$u_k v_l v_j v_m + v^i u_i g_{jl} v_k v_m - u_l v_j v_k v_m - v^i u_i g_{jk} v_l v_m = 0\,. \tag{41}$$

Contracting this with g^{jk}, we obtain $(2-n) v^i u_i v_l v_m = 0$, i.e. $v^i u_i = 0$. Therefore

(41) takes the form $(u_k v_l - u_l v_k)v_j v_m = 0$, so that we have

$$u_i = uv_i \tag{42}$$

for some C^∞ function u on U. Now (40) turns into

$$\begin{aligned} 3v_i v_j(3v_k S_l S_m - 3v_l S_k S_m + v_k S_{l,m} - v_l S_{k,m}) &= R_{ij,klm} - R_{ij,lkm} \\ &= (2u - R/(n-1))(g_{il}v_j v_k v_m - g_{ik}v_j v_l v_m + g_{jl}v_i v_k v_m - g_{jk}v_i v_l v_m)/(n-2)\,. \end{aligned} \tag{43}$$

We assert that

$$u = R/2(n-1)\,. \tag{44}$$

In fact, suppose that $(2u - R/(n-1))/(n-2) = Q \neq 0$. Fix a vector b^i such that $v_i b^i = 1$. For any vector c^i orthogonal to v^i we obtain, by transvecting (43) with $b^j b^k b^m c^l$, $Q(c_i + c_j b^j v_i) = \lambda v_i$ for some scalar λ, so v_i and c_i are collinear. Thus any vector orthogonal to v_i is collinear to v_i, a contradiction, which proves (44). Now (34)a) is an immediate consequence of (39), (42) and (44). From (43) and (44) we obtain $v_k(3S_l S_m + S_{l,m}) = v_l(3S_k S_m + S_{k,m})$, which yields, by Lemma 4,

$$3S_l S_m + S_{l,m} = \tfrac{1}{3}v_l P_m \tag{45}$$

for a certain vector field P_i on U. Differentiating (37) covariantly and using (35), we obtain (34)b), which completes the proof.

Lemma 10. *Let M be an essentially conformally symmetric manifold satisfying* (3). *Then M satisfies the relations*

$$\text{a)}\quad R = 0\,, \qquad \text{b)}\quad R_i{}^r R_{rj,k} = 0\,. \tag{46}$$

Proof. We use the notations of Lemma 9. First, we assert that M satisfies the relation

$$R_i{}^r R_{rj,k} = RR_{ij,k}/2(n-1)\,. \tag{47}$$

In fact, (47) holds on the open subset U in view of (34)a) and (33). On the other hand, it is trivially satisfied outside of U, i.e. at points where $R_{ij,k}$ vanishes.

From (34)b) and Ricci identity we obtain, using (4)a),

$$\begin{aligned} v_i v_j v_k(v_l P_m - v_m P_l) &= R_{ij,klm} - R_{ij,kml} \\ &= R_{lmi}{}^r R_{rj,k} + R_{lmj}{}^r R_{ir,k} + R_{lmk}{}^r R_{ij,r} \\ &= -R(g_{im}R_{jk,l} - g_{il}R_{jk,m} + g_{jm}R_{ik,l} - g_{jl}R_{ik,m} + g_{km}R_{ij,l} \\ &\quad - g_{kl}R_{ij,m})/2(n-1)(n-2) + (R_{im}R_{jk,l} - R_{il}R_{jk,m} \\ &\quad + R_{jm}R_{ik,l} - R_{jl}R_{ik,m} + R_{km}R_{ij,l} - R_{kl}R_{ij,m})/(n-2)\,. \end{aligned} \tag{48}$$

Contracting (48) with g^{im} and making use of (36) and (47) we have $v^i P_i v_j v_k v_l = Rv_j v_k v_l/2(n-1)$, i.e.

$$v^i P_i = R/2(n-1)\,. \tag{49}$$

Transvecting now (48) with v^m, using (33), (36), (34)a) and (49), we obtain $Rv_i v_j v_k v_l \div 2(n-1) = 0$, so R vanishes on U. From (5) we obtain (46)a), which in turn implies (46)b) in view of (47). This completes the proof.

Lemma 11. *Let M be an essentially conformally symmetric manifold satisfying* (3). *Then M satisfies the relations*

(50) a) $R_i{}^r R_{rj} = 0$, b) $R^{rs} C_{rijs} = 0$.

Proof. In view of (46)b) and (8)b), $R_i{}^r R_{rj}$ and $R^{rs} C_{rijs}$ are both symmetric, parallel tensor fields on M. Therefore we may restrict our reasoning to the open subset U, using the notations of Lemma 9. By (34)a) and (46)a), we have $R_{ir} v^r = 0$, so by transvecting (48) with $R_s{}^k$ we obtain, in virtue of Lemma 10, $0 = (R_s{}^k R_{km} R_{ij,l} - R_s{}^k R_{kl} R_{ij,m})/(n-2)$, which yields, in view of (ii) of Lemma 5,

(51) $$R_i{}^r R_{rj} = f v_i v_j$$

for some function f.

Now suppose $R_i{}^r R_{rj} \neq 0$. Hence, by (52),

(52) $$0 = f v_i v_{j,k} = f_{,k} v_i v_j + 2f S_k v_i v_j \;, \quad \text{i.e.} \quad f_{,k} + 2f S_k = 0 \;, \quad f \neq 0 \;.$$

Therefore S_k is a local gradient, so $S_{l,m} = S_{m,l}$. In view of (44) we have

(53) $$v_l P_m = v_m P_l \;,$$

which yields, in view of (48) and (46)a),

(54) $$0 = R_{im} R_{jk,l} - R_{il} R_{jk,m} + R_{jm} R_{ik,l} - R_{jl} R_{ik,m} + R_{km} R_{ij,l} - R_{kl} R_{ij,m} \;.$$

Transvecting (54) with $R_s{}^m$ and using (46)b), we obtain $0 = R_s{}^m R_{im} R_{jk,l} + R_s{}^m R_{jm} R_{ik,l} + R_s{}^m R_{km} R_{ij,l}$, which, in view of (51) and (33), turns into $0 = 3 f v_i v_j v_k v_l v_s$. Therefore $f = 0$, which contradicts (52). This proves (50)a).

In view of (46)a) and (50)a), formula (38) yields

(55) $$3 v_i v_j (v_k S_l - v_l S_k) = C_{kli}{}^r R_{rj} + C_{klj}{}^r R_{ir} \;.$$

Contracting (55) with g^{jk} and using (36), we obtain

(56) $$C_{rlis} R^{rs} = -3 v_r S^r v_i v_l \;.$$

Now suppose $R^{rs} C_{rijs} \neq 0$. Setting $f = -3 v_r S^r$ we have, by (56), $R^{rs} C_{rijs} = f v_i v_j$. As in the preceding part of the proof, we obtain (52), (53) and (54). Transvecting (54) with $C^i{}_{rs}{}^m$ and using (8)b), we obtain $C^i{}_{rs}{}^m R_{im} R_{jk,l} = 0$, which yields $R^{im} C_{irsm} = 0$, a contradiction. Thus we obtain (50)b), which completes the proof.

Lemma 12. *Let M be an essentially conformally symmetric manifold satisfying* (3). *Then M satisfies the relations*

(57) $$R_{ir} C^r{}_{jkl} = 0 \;,$$

(58) $$R_{hl} C_{mijk} - R_{hm} C_{lijk} + R_{il} C_{hmjk} - R_{im} C_{hljk} + R_{jl} C_{himk} - R_{jm} C_{hilk} + R_{kl} C_{hijm} - R_{km} C_{hijl} = 0 \;.$$

Proof. In virtue of (29) the right-hand side of (9) vanishes. Contracting (9) with g^{hl} and using (46)a), (4)b), d), (50)b) and the obvious equality $R^{rs} C_{rsij} = 0$, we obtain

$$(n-1) R_{mr} C^r{}_{ijk} + R_{ir} C^r{}_{mjk} + R_{mr} C^r{}_{ikj} + R_{kr} C^r{}_{ijm} + R_{jr} C^r{}_{imk} + R_{mr}(C^r{}_{ikj} + C^r{}_{kji} + C^r{}_{jik}) = 0 \;,$$

which, in view of (4)c) and (8)c), reduces to $(n-1) R_{mr} C^r{}_{ijk} + R_{ir} C^r{}_{mjk} = 0$. Therefore $R_{ir} C^r{}_{mjk} = (1-n) R_{mr} C^r{}_{ijk} = (1-n)^2 R_{ir} C^r{}_{mjk}$, which yields (57). Formula (58) is an immediate consequence of (9), (57), (52) and (29). This completes the proof.

We are now in a position to prove the main result of this section.

Theorem 6. *Let M be an essentially conformally symmetric Riemannian manifold. If the parallel tensor field $C_{mijk}C^{mrst}$ does not vanish identically on M, then M is Ricci-recurrent.*

Proof. Transvecting (58) with C^{mrst} and with C^{irst}, respectively, and using (57), we obtain

$$R_{hl}C_{mijk}C^{mrst}+R_{il}C_{mhkj}C^{mrst}+R_{jl}C_{mkhi}C^{mrst}+R_{kl}C_{mjih}C^{mrst}=0\,,$$
$$R_{hl}C_{imkj}C^{irst}-R_{hm}C_{ilkj}C^{irst}+R_{jl}C_{ihkm}C^{irst}-R_{jm}C_{ihkl}C^{irst}$$
$$+R_{kl}C_{ihmj}C^{irst}-R_{km}C_{ihlj}C^{irst}=0\,.$$

For a moment, fix r, s, t and set $e_{ij}=R_{ij}$, $T_{ijk}=C_{mijk}C^{mrst}$. From the above formulae it follows immediately that e_{ij} and T_{ijk} satisfy (10). By Lemma 3, they satisfy (11), i.e.

$$R_{hl}C_{mijk}C^{mrst}=R_{il}C_{mhjk}C^{mrst}\,. \tag{59}$$

Since the vector field v_i spans distribution L (see Lemma 9), it follows now from (ii) of Lemma 5 that

$$R_{ij}=fv_iv_j \tag{60}$$

for some function f on U.

Let $x\in M$ and $R_{ij}(x)\neq 0$. If $x\in U$, then relation (19), with $A_k=v_k(x)/f(x)$, follows immediately from (60) and (33). If $x\notin U$, then (19) is satisfied by $A_k=0$, which completes the proof.

From Theorem 6 and (i) of Theorem 4 we conclude immediately

Corollary 1. *Every essentially conformally symmetric Riemannian manifold with a metric of index one is Ricci-recurrent.*

Remark 4. It has been shown in [6] (Theorem 3) that for any point x of an essentially conformally symmetric Ricci-recurrent manifold M such that

$$R_{ij}(x)\neq 0\,,\qquad R_{ij,k}(x)\neq 0\,, \tag{61}$$

a coordinate system in a neighbourhood of x may be chosen so that the metric takes the form (20) (note that the definition of Ricci-recurrency in [6] differs slightly from the ours). Therefore Corollary 1 implies that for an essentially conformally symmetric manifold M with a metric of index one, the metric of M is of type (20) in a large (determined by (61)) open subset of M (e.g. if M is analytic, then this subset is dense).

§5. The general case. Combining the results of Sections 3 and 4, we shall obtain in this section some general statements on essentially conformally symmetric Riemannian manifolds. First we strengthen a result of Tanno ([8], Theorem 6):

Theorem 7. *Every essentially conformally symmetric Riemannian manifold M satisfies the relation*

$$R=0\,, \tag{62}$$

i.e. the scalar curvature of M vanishes identically.

Proof. According as M satisfies (2) or (3), our assertion follows from (24)a) or (46)a), respectively.

Theorem 8. *Every essentially conformally symmetric Riemannian manifold M satisfies the relations*

$$\text{(63)}\quad \begin{aligned} &\text{a)}\ R_i{}^r R_{rj} = 0\,, \qquad \text{b)}\ R_i{}^r R_{rj,k} = 0\,, \qquad \text{c)}\ R_{ir} C^r{}_{jkl} = 0\,, \\ &\text{d)}\ R_{hl} C_{mijk} - R_{hm} C_{lijk} + R_{il} C_{hmjk} - R_{im} C_{hljk} + R_{jl} C_{himk} - R_{jm} C_{hilk} \\ &\qquad + R_{kl} C_{hijm} - R_{km} C_{hijl} = 0\,, \\ &\text{e)}\ R_{hl} C_{mijk} C^{mrst} = R_{il} C_{mhjk} C^{mrst}\,. \end{aligned}$$

Proof. If M satisfies (2), then (63)a)-d) follow from Lemma 7 and (63)e) is obvious. Now assume (3). In this case (63) follow from (50)a), (46)b), (57), (58) and (59), respectively.

Theorem 9. *Every essentially conformally symmetric Riemannian manifold M satisfies the relation*

$$\text{(64)}\qquad R_{hijk,lm} - R_{hijk,ml} = 0\,.$$

Proof. In view of Ricci identity we have $R_{ij,lm} - R_{ij,ml} = R_{lmi}{}^r R_{rj} + R_{lmj}{}^r R_{ri}$. From (4)a), (63)c), (62), (63)a) we obtain $R_{lmi}{}^r R_{rj} = (R_{im} R_{jl} - R_{jm} R_{il})/(n-2)$, which yields $R_{ij,lm} - R_{ij,ml} = 0$. Equality (64) follows now directly from (7).

Remark 5. Relation (64) often occurs in the literature in the case of definite metrics. It is usually written in the form $R(X, Y)R = 0$.

Theorem 10. *Let M be an essentially conformally symmetric n-dimensional Riemannian manifold with a metric of index q. Then M admits a null parallel distribution Δ such that $1 \le \dim \Delta \le \min(q, n-q) \le n/2$.*

Proof. In the case (2) our assertion follows from Theorem 3. Now assume (3). Then we may set $\Delta = L$ (see Theorem 5). This completes the proof.

We do not know whether or not every essentially conformally symmetric manifold is Ricci-recurrent. Certain sufficient conditions for Ricci-recurrency are given by Theorem 6 and Corollary 1. Here we give a condition which is both sufficient and necessary.

Theorem 11. *Let M be an essentially conformally symmetric Riemannian manifold. Then the following two conditions are equivalent:* (i) *M is Ricci-recurrent,* (ii) *For each point x of M which satisfies* (61), *there exists a non-zero parallel vector field on some neighbourhood of x.*

Proof. If M is Ricci-recurrent, then (ii) follows immediately from Theorem 2 of [6]. Now assume (ii) and fix $x \in M$ such that $R_{ij}(x) \neq 0$. If $R_{ij,k}(x) = 0$, then (19) is satisfied by $A_k = 0$. In the case $R_{ij,k}(x) \neq 0$ choose a neighbourhood W of x and a non-zero parallel vector field u_i on W. From Ricci identity we obtain $R_{jki}{}^r u_r = u_{i,jk} - u_{i,kj} = 0$, which yields

$$\text{(65)}\qquad R_{ir} u^r = 0\,.$$

Therefore, in view of the expression of $C^h{}_{ijk}$ and (62), we have

$$\text{(66)}\qquad u^r C_{rijk} = (u_j R_{ki} - u_k R_{ji})/(n-2)\,.$$

We consider two a priori possible cases: the parallel tensor field $T_{ijk} = u^r C_{rijk}$ does or does not vanish. If $T_{ijk} = 0$, then (66) yields $u_h R_{ij} - u_i R_{hj} = 0$. Differentiating this covariantly, we obtain $u_h R_{ij,k} - u_i R_{hj,k} = 0$. From (6) and Lemma 5 it follows now that $R_{ij}(x) = \lambda u_i(x) u_j(x)$ and $R_{ij,k}(x) = \mu u_i(x) u_j(x) u_k(x)$ for some scalars λ and μ, which implies (19) with $A_k = (\mu/\lambda) u_k(x)$.

In the remaining case $T_{ijk} \neq 0$, we obtain, transvecting (63)d) with u^m and with u^i, respectively, and using (65),

$$R_{hl}T_{ijk} + R_{il}T_{hkj} + R_{jl}T_{khi} + R_{kl}T_{jih} = 0 \,,$$
$$R_{hl}T_{mkj} - R_{hm}T_{lkj} + R_{jl}T_{hkm} - R_{jm}T_{hkl} + R_{kl}T_{hmj} - R_{km}T_{hlj} = 0 \,.$$

Applying Lemma 3 to $e_{ij} = R_{ij}$ and to T_{ijk}, we have $R_{hl}T_{ijk} = R_{il}T_{hjk}$ and, since $T_{ijk,p} = 0$, $R_{kl,p}T_{ijk} = R_{il,p}T_{hjk}$. Now choose vectors a^i, b^i at x such that $T_{ijk}a^j b^k = w_i \neq 0$. In view of Lemma 5 and (6), the above equalities yield $R_{ij}(x) = \lambda w_i w_j$ and $R_{ij,k}(x) = \mu w_i w_j w_k$ for some scalars λ and μ, so (19) is satisfied by $A_k = (\mu/\lambda) w_k$. This completes the proof.

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