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Killing Fields on Compact Pseudo-Kähler Manifolds

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Abstract
We show that a Killing field on a compact pseudo-Kähler ddbar manifold is necessarily (real) holomorphic. Our argument works without the ddbar assumption in real dimension four. The claim about holomorphicity of Killing fields on compact pseudo-Kähler manifolds appears in a 2012 paper by Yamada, and in an appendix we provide a detailed explanation of why we believe that Yamada's argument is incomplete.

Keywords Compact pseudo-Kähler manifold · Killing vector field

Mathematics Subject Classification Primary 53C50 · Secondary 53C56

1 Introduction

By a pseudo-Kähler manifold we mean a pseudo-Riemannian manifold (M, g) endowed with a ∇-parallel almost-complex structure J, for the Levi-Civita connection ∇ of g, such that the operator $J_x : T_x M \to T_x M$ is a linear g_x-isometry (or is, equivalently, g_x-skew-adjoint) at every point $x \in M$. This implies integrability of J (see the comment preceding Lemma 3.1). We then call (M, g) a *pseudo-Kähler* $\partial\bar\partial$ *manifold* if, in addition, the underlying complex manifold M has the following $\partial\bar\partial$ *property*, also referred to as *the* $\partial\bar\partial$ *lemma*:

$$\begin{aligned}&\text{every closed } \partial\text{-exact or } \bar\partial\text{-exact } (p, q) \text{ form}\\&\text{equals } \partial\bar\partial\lambda \text{ for some } (p-1, q-1) \text{ form } \lambda.\end{aligned} \tag{1.1}$$

It is well known that the $\partial\bar\partial$ property follows if M is compact and admits a Riemannian Kähler metric [5, Prop. 6.17 on p. 144].

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Theorem A *Every Killing vector field on a compact pseudo-Kähler* $\partial\bar\partial$ *manifold is real holomorphic.*

We provide two proofs of Theorem A, in Sects. 3 and 4. The former is derived directly from the $\partial\bar\partial$ condition; the latter, shorter, relies on the Hodge decomposition, which is equivalent to the $\partial\bar\partial$ property [2, p. 269, subsect. (5.21)].

The Riemannian-Kähler case of Theorem A is well known, and straightforward [1, the lines following Remark 4.83 on pp. 60–61]. See also Remark 2.2.

For pseudo-Kähler surfaces, our argument yields a stronger conclusion.

Theorem B *In real dimension four, the assertion of Theorem A holds without the* $\partial\bar\partial$ *hypothesis.*

The authors wish to express their gratitude to Kirollos Masood for bringing Yamada's paper [7] to the first author's attention and discussing with him issues involving Theorem B, formula (4.3), and the Appendix. We also thank Fangyang Zheng for very useful suggestions about Lemma 3.1, and Takumi Yamada for a brief but helpful communication.

2 Proof of Theorem B

All manifolds, mappings, tensor fields, and connections are assumed smooth.

Lemma 2.1 *Given a connection* ∇ *on a manifold* M*, let a vector field* v *on* M *be affine in the sense that its local flow preserves* ∇*. Then, for any* ∇*-parallel tensor field* Θ *on* M*, of any type, the Lie derivative* $\pounds_v \Theta$ *is* ∇*-parallel as well. If* Θ *happens to be a closed differential form,* $\pounds_v \Theta = d[\Theta(v, \cdot, \dots, \cdot)]$.

Proof Clearly, $-\pounds_v \Theta$ is the derivative with respect to the real variable t, at $t = 0$, of the push-forwards $[d\phi_t]\Theta$ under the local flow $t \mapsto \phi_t$ of v. All $[d\phi_t]\Theta$ being ∇-parallel, so is $\pounds_v \Theta$. For the final clause, use Cartan's homotopy formula $\pounds_v = \imath_v d + d\imath_v$ for $\pounds_v$ acting on differential forms [4, Thm. 14.35, p. 372]. $\square$

Lemma 2.1 also follows from the Leibniz rule: $\pounds_v(\nabla\Theta) = (\pounds_v \nabla)\Theta + \nabla(\pounds_v \Theta)$.

Let (M, g) now be a fixed pseudo-Kähler manifold. If v is any vector field on M then, with J and ∇v treated as bundle morphisms $TM \to TM$,

$$\text{for } B = \nabla v \text{ and } A = \pounds_v J \text{ one has } A = [J, B] \text{ and } JA = -AJ, \tag{2.1}$$

which is immediate from the Leibniz rule. For the Kähler form $\omega = g(J\cdot, \cdot)$ of (M, g) and any g-Killing vector field v, it follows from (2.1) and Lemma 2.1 that

$$\begin{array}{l} \text{i) } A = \pounds_v J \text{ and } \alpha = \pounds_v \omega \text{ are related by } \alpha = g(A\cdot, \cdot), \text{ while} \\ \text{ii) } A^* = -A, \quad JA = -AJ, \quad \nabla A = 0, \quad \nabla\alpha = 0, \text{ and } \alpha \text{ is exact.} \end{array} \tag{2.2}$$

Given an exact p-form α on a compact pseudo-Riemannian manifold (M, g),

$$\alpha \text{ is } L^2\text{-orthogonal to all parallel } p \text{ times covariant tensor fields } \theta \text{ on } M. \tag{2.3}$$

Namely, $(\theta, \alpha) = (\mu, \alpha) = (\mu, d\beta) = (d^*\mu, \beta)$ for β with $\alpha = d\beta$ and the skew-symmetric part μ of θ, while $d^*\mu = 0$, as $\nabla\mu = 0$. Here, $(,)$ is the L^2 inner product, assigning to two tensor fields of the same type the integral over M of their g-inner product, and d^* denotes the g-divergence.

Remark 2.2 By (2.2-ii) and (2.3), for a Killing field v on a compact *Riemannian* Kähler manifold, $\pounds_v\, \omega$ is L^2-orthogonal to itself, and so, as a consequence of (2.2-i), v must be real holomorphic.

Let (M, g) be, again, a pseudo-Kähler manifold. The vector bundle morphisms $C : TM \to TM$ having $C^* = -C$ (that is, g_x-skew-adjoint at every point $x \in M$) constitute the sections of

$$\text{the vector subbundle } \mathfrak{so}(TM) \text{ of } \mathrm{End}_{\mathbb{R}}(TM) = \mathrm{Hom}_{\mathbb{R}}(TM, TM). \tag{2.4}$$

We denote by $\mathcal{E}$ the vector subbundle of $\mathfrak{so}(TM)$, the sections C of which are also complex-antilinear (so that $JC = -CJ$, in addition to $C^* = -C$). Then,

$$\begin{aligned}&\mathcal{E} \text{ is a complex vector bundle of rank } m(m-1)/2, \text{ where } m = \dim_{\mathbb{C}} M,\\ &\text{with a pseudo-Hermitian fiber metric having the real part induced by } g.\end{aligned} \tag{2.5}$$

In fact, $C \mapsto JC$ provides the complex structure for $\mathcal{E}$. Nondegeneracy of g restricted to $\mathcal{E}$ follows from g-orthogonality of the decomposition $\mathrm{End}_{\mathbb{R}}(TM) = \mathrm{End}_{\mathbb{C}}(TM) \oplus \mathcal{E} \oplus \mathcal{D}$, the sections C of the subbundle $\mathcal{D}$ being characterized by $JC = -CJ$ and $C^* = C$, with $\mathrm{End}_{\mathbb{C}}(TM)$ orthogonal to $\mathcal{E} \oplus \mathcal{D}$ since any antilinear morphism $C : TM \to TM$ is conjugate, via J, to $-C$, and so $\mathrm{tr}_{\mathbb{R}}\, C = 0$. The pseudo-Hermitian fiber metric in $\mathcal{E}$ arises by restricting $\langle\cdot,\cdot\rangle - i\langle J\cdot,\cdot\rangle$ to $\mathcal{E}$, for the pseudo-Riemannian fiber metric $\langle\cdot,\cdot\rangle$ in $\mathrm{End}_{\mathbb{R}}(TM)$ induced by g. The rank $m(m-1)/2$ follows since $\mathfrak{so}(TM) = \mathfrak{u}(TM) \oplus \mathcal{E}$, with $\mathfrak{u}(TM) \subseteq \mathfrak{so}(TM)$ characterized by having sections $C : TM \to TM$ that commute with J (which, due to their g-skew-adjointness, makes them also $g^{\mathbf{c}}$-skew-adjoint, for $g^{\mathbf{c}} = g - i\omega$): $\mathfrak{so}(TM)$ and $\mathfrak{u}(TM)$ have the real ranks $m(2m-1)$ and m^2.

Proof of Theorem B By (2.5), with $m = 2$, the pseudo-Hermitian fiber metric in the *line* bundle $\mathcal{E}$ must be positive or negative definite. Hence, so is its g-induced real part. For any Killing field v, (2.2-ii) implies that $A = \pounds_v\, J$ is a section of $\mathcal{E}$ which, due to (2.2)–(2.3), is L^2-orthogonal to itself, and so $\pounds_v\, J = 0$. □

The above proof does not extend to compact pseudo-Kähler manifolds (M, g) of complex dimensions $m > 2$ with indefinite metrics. Namely, if the pair (j, k) represents the metric signature of g, with j minuses and k pluses (both j, k even, $j + k = 2m$), then the analogous signature of the real part (induced by g) of the pseudo-Hermitian fiber metric in $\mathcal{E}$ is $(jk/2,\ [j^2 + k^2 - 2(j + k)]/4)$, with both components (indices) positive unless $jk = 0$ or $j = k = 2$.

One easily verifies this last claim, about the signature, by using a J_x-invariant timelike-spacelike orthogonal decomposition of $T_x M$, at any $x \in M$, to obtain obvious three-summand orthogonal decompositions of both $\mathfrak{so}(TM)$ and $\mathfrak{u}(TM)$ at x, two summands being spacelike, and one timelike.

3 Proof of Theorem A

We denote by $\Omega^{p,q}M$ the space of complex-valued differential (p,q) forms on a complex manifold M. On such M, as $\bar\partial\zeta = 0$ whenever $d\zeta = 0$,

$$\text{closedness of a } (p,0) \text{ form } \zeta \text{ implies its holomorphicity.} \tag{3.1}$$

Conversely, according to [2, p. 269, subsect. (5.21)] and [6, p. 101, Corollary 9.5], on a compact complex $\partial\bar\partial$ manifold,

$$\text{all holomorphic differential forms are closed.} \tag{3.2}$$

Since many expositions do not state what happens when, in the $\partial\bar\partial$ property (1.1), p or q equals 0, we note that, as Fangyang Zheng pointed out to us, (1.1) for $(p,0)$ forms easily follows from the case where p and q are positive.

Lemma 3.1 *On a compact complex manifold M with the "positive (p,q) version" of the $\partial\bar\partial$ property, if $\xi\in\Omega^{p,0}M$, for $p\ge 1$, and $\partial\xi$ is closed, then $\partial\xi = 0$.*

Proof As $0 = d\partial\xi = \bar\partial\partial\xi = -\partial\bar\partial\xi$, the "positive" $\partial\bar\partial$ lemma applied to the closed $\bar\partial$-exact $(p,1)$ form $\bar\partial\xi$ gives $\bar\partial\xi = \bar\partial\partial\eta$ for some $\eta\in\Omega^{p-1,0}M$. Being thus holomorphic, $\xi-\partial\eta\in\Omega^{p,0}M$ is closed by (3.2), and $0=\partial(\xi-\partial\eta)=\partial\xi$. ◻

Lemma 3.1 implies, via complex conjugation, its analog for $(0,q)$ forms. Also by Lemma 3.1, on a compact complex manifold M with the $\partial\bar\partial$ property,

$$\text{the only exact } (p,0) \text{ form } \zeta \text{ on } M \text{ is } \zeta = 0, \tag{3.3}$$

since exactness of $\zeta\in\Omega^{p,0}M$ amounts to its ∂-exactness and implies its closedness.

For a pseudo-Kähler manifold (M,g), a bundle morphism $A: TM\to TM$, and the corresponding twice-covariant tensor field $\alpha = g(A\cdot,\,\cdot)$, one clearly has

$$\alpha(J\cdot, J\cdot) = \pm\alpha \text{ if and only if } JA = \pm AJ, \quad \text{with either sign } \pm\,. \tag{3.4}$$

Given a pseudo-Kähler manifold (M,g), vector fields u, v on M and sections A, C of $\mathfrak{so}(TM)$, cf. (2.4), may be used to represent a complex-valued 1-form ξ and 2-form ζ on M, as follows,

$$\xi = u + iv, \quad \zeta = A + iC, \tag{3.5}$$

meaning that $\xi = g(u,\,\cdot) + ig(v,\,\cdot)$ and $\zeta = g(A\cdot,\,\cdot) + ig(C\cdot,\,\cdot)$. We prefer not to think of (3.5) as sections of the complexifications of TM or $\mathfrak{so}(TM)$. For a vector field v treated via (3.5) as a real 1-form, and $B=\nabla v$, our factor convention for the exterior derivative gives

$$dv = B - B^*, \quad \text{and so } d(Jv) = \nabla(Jv) - [\nabla(Jv)]^* = JB + B^*J. \tag{3.6}$$

Remark 3.2 On a complex manifold, a real-valued 2-form α is the real part of a complex-bilinear complex-valued 2-form ζ if and only if $\alpha(J\cdot, J\cdot) = -\alpha$, and then necessarily $\zeta = \alpha - i\alpha(J\cdot, \cdot)$. (This clearly remains valid for arbitrary twice-covariant tensor fields, without skew-symmetry.)

Remark 3.3 For a complex-valued 2-form ζ on a complex manifold M, having bidegree $(2, 0)$, or $(0, 2)$, or $(1, 1)$ clearly amounts to its being complex-bilinear, or bi-antilinear or, respectively, J-invariant: $\zeta(J\cdot, J\cdot) = \zeta$. Sums ζ of $(2, 0)$ and $(0, 2)$ forms are similarly characterized by J-anti-invariance: $\zeta(J\cdot, J\cdot) = -\zeta$. Thus, by (3.4), in the pseudo-Kähler case, $\zeta = A + iC$ in (3.5) is a $(1, 1)$ form if and only if A and C commute with J.

Lemma 3.4 *For a Killing vector field v on a pseudo-Kähler manifold (M, g), using the notation of* (3.5)*, we have*

$$\begin{aligned} &\xi \in \Omega^{1,0}M, \quad \zeta \in \Omega^{2,0}M, \quad \partial\xi = \zeta, \quad \bar\partial\xi = i(JBJ - B), \quad \text{where} \\ &\xi = Jv - iv, \quad \zeta = A - iAJ, \quad \text{with} \quad A = [J, B] \quad \text{for} \quad B = \nabla v. \end{aligned} \tag{3.7}$$

Proof First, $JBJ - B$, as well as $A = [J, B]$ and AJ, are g_x-skew-adjoint at every point $x \in M$, since so is $B = \nabla v$, and A anticommutes with J, cf. (2.1). Thus, ξ, ζ and $\gamma = i(JBJ - B)$ are indeed differential forms of degrees 1, 2, 2.

Furthermore, ξ is complex-linear, and ζ complex-bilinear. This is immediate for ξ. For ζ, note that $\zeta = \alpha - i\alpha(J\cdot, \cdot)$, where $\alpha = g(A\cdot, \cdot)$, while (2.1) and (3.4) give $\alpha(J\cdot, J\cdot) = -\alpha$. Now we can use Remark 3.2.

Thus, $\xi \in \Omega^{1,0}M$. Also, according to Remark 3.3, $\zeta \in \Omega^{2,0}M$ and $\gamma \in \Omega^{1,1}M$, since $JBJ - B$ obviously commutes with J. Finally, for $A = [J, B]$, (3.6) with $B^* = -B$ gives $d\xi = A - 2iB = [A - i(JBJ + B)] + i(JBJ - B)$, while the summands $A - i(JBJ + B) = A - iAJ = \zeta$ and $i(JBJ - B) = \gamma$ lie in $\Omega^{2,0}M$ and $\Omega^{1,1}M$, which completes the proof. □

Proof of Theorem A By (2.2) and (3.4), the ∂-exact $(2, 0)$ form $\zeta = \partial\xi$ in (3.7) is parallel, and hence closed. Lemma 3.1 now gives $\zeta = 0$, so that $\pounds_v J = A = 0$ due to (2.1) and (3.7). □

4 Another Proof of Theorem A

On a compact complex manifold M with the $\partial\bar\partial$ property, every cohomology space $H^k(M, \mathbb{C})$ has the Hodge decomposition [2, p. 269, subsect. (5.21)]:

$$H^k(M, \mathbb{C}) = H^{k,0}M \oplus H^{k-1,1}M \oplus \ldots \oplus H^{1,k-1}M \oplus H^{0,k}M, \tag{4.1}$$

with each $H^{p,q}M$ consisting of cohomology classes of closed (p, q) forms. The complex conjugation of differential forms descends to a real-linear involution of $H^k(M, \mathbb{C})$, the fixed points of which obviously are the real cohomology classes (those containing real closed differential forms). In terms of the decomposition (4.1), a complex cohomology class

$$\begin{aligned} &\text{is real if and only if, for all } p \text{ and } q, \text{ its } H^{q,p} \text{ com-} \\ &\text{ponent equals the conjugate of its } H^{p,q} \text{ component.} \end{aligned} \tag{4.2}$$

The standard formula $N(u, v) = [u, v] + J[Ju, v] + J[u, Jv] - [Ju, Jv]$, for the Nijenhuis tensor N of an almost-complex structure J on a manifold M and any vector fields u, v, clearly becomes

$$N(u, v) = [\nabla_{Jv} J]u - [\nabla_{Ju} J]v + J[\nabla_u J]v - J[\nabla_v J]u \tag{4.3}$$

when one uses any fixed torsionfree connection ∇ on M. We call ∇ a *Kähler connection* for the given almost-complex structure J if it is torsionfree and $\nabla J = 0$. By (4.3), J then must be integrable. This implies *integrability of* J *in any* pseudo-*Kähler manifold*, as one then has $\nabla J = 0$ for the Levi-Civita connection ∇.

Lemma 4.1 *For any ∇-parallel real 2-form α on a complex manifold M with a Kähler connection ∇, such that $\alpha(J\cdot, J\cdot) = -\alpha$, the complex-valued 2-form $\zeta = \alpha - i\alpha(J\cdot, \cdot)$ is holomorphic. If, in addition, M is also compact and has the $\partial\bar\partial$ property, while α is exact, then $\alpha = 0$.*

Proof The relation $\alpha(J\cdot, J\cdot) = -\alpha$ amounts to complex-bilinearity of ζ, and so $\zeta \in \Omega^{2,0}M$ (Remarks 3.2 – 3.3). Being ∇-parallel, ζ is closed, and hence holomorphic due to (3.1). The final clause: exactness of α makes $[i\zeta] \in H^{2,0}M$ a real cohomology class, so that, by (4.2), ζ is exact, and (3.3) gives $\zeta = 0$. □

Another proof of Theorem A Given a Killing field v, the differential 2-form $\alpha = \pounds_v \omega$ is parallel and exact by (2.2), while (2.2) gives $JA = -AJ$ for $A = \pounds_v J$, related to α via $\alpha = g(A\cdot, \cdot)$, and so $\alpha(J\cdot, J\cdot) = -\alpha$ due to (3.4). Lemma 4.1 and (2.2-i) now yield $\pounds_v \omega = \alpha = 0$ and $\pounds_v J = 0$. □

We do not know whether—aside from Theorem B and the Riemannian case—Theorem A remains valid without the $\partial\bar\partial$ hypothesis. For possible future reference, let us note that, as shown above, one has the following conclusions about a Killing field v on a compact pseudo-Kähler manifold, whether or not the $\partial\bar\partial$ property is assumed. First, for $\alpha = \pounds_v \omega$, the complex-valued 2-form $\zeta = \alpha - i\alpha(J\cdot, \cdot)$ is parallel and holomorphic (see the preceding proof and Lemma 4.1). Also, by (2.2), α is exact, while $A = \pounds_v J : TM \to TM$ is parallel and complex-antilinear, as well as nilpotent at every point. This last conclusion follows since the constant function $\mathrm{tr}_{\mathbb{R}} A^k$, with any integer $k \geq 1$, has zero integral as a consequence of (2.3) applied to $\alpha = g(A\cdot, \cdot)$ and $\theta = g(A^{k-1}\cdot, \cdot)$.

Appendix: Yamada's argument

Yamada's claim [7, Proposition 3.1] that on a compact pseudo-Kähler manifold, Killing fields are real holomorphic, has a proof which reads, *verbatim*, Let X be a Killing vector field. From Propositions 1.2 and 2.12, $Z = X - \sqrt{-1}\,JX$ is holomorphic. Because the real part of a holomorphic vector field is an infinitesimal automorphism of the complex structure, we have our proposition. (A.1)

Proposition 1.2 of [7], cited from Kobayashi's book [3], amounts to the well-known *harmonic-flow condition* satisfied by Killing fields v on pseudo-Riemannian manifolds. Thus, 2.12 in (A.1) should read 2.14, since Propositions 1.2 and 2.14 refer to the Ricci tensor quite prominently, while 2.12 does not mention it at all; also, Proposition 2.14 contains, in its second part, a holomorphicity conclusion.

In the ninth line of the proof of the second part of Proposition 2.14, it is established—correctly—that, for every $(1,0)$ vector field Y, and Z in (A.1), $\nabla''Z$ is L^2-orthogonal to $\nabla''Y$. Then, an attempt is made to conclude that $\nabla''Z = 0$, arguing by contradiction: if $\nabla''Z \neq 0$ at some point z_0, one can—again correctly—find Y having $g(\nabla''Z, \nabla''Y) \neq 0$ everywhere in some neighborhood of z_0. As a next step, it is claimed that a contradiction arises: cited *verbatim*,

By considering a cut-off function, we see that there exists a complex vector field Y such that $\int_M g(\nabla''Z, \nabla''Y)\,dv \neq 0$. (A.2)

It is here that the argument seems incomplete: such a cut-off function φ equals 1 on some small "open ball" B centered at z_0, and vanishes outside a larger "concentric ball" B', and after the original choice of Y has been replaced by φY, there is no way to control the integral of $g(\nabla''Z, \nabla''(\varphi Y))$ over $B' \smallsetminus B$ (while the integrals over B and $M \smallsetminus B'$ have fixed values). More precisely, the sum of the three integrals must be zero, $\nabla''Z$ being L^2-orthogonal to all $\nabla''Y$.

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