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PROJECTIVELY FLAT SURFACES, NULL PARALLEL DISTRIBUTIONS, AND CONFORMALLY SYMMETRIC MANIFOLDS

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Abstract. We determine the local structure of all pseudo-Riemannian manifolds of dimensions greater than 3 whose Weyl conformal tensor is parallel and has rank 1 when treated as an operator acting on exterior 2-forms at each point. If one fixes three discrete parameters: the dimension, the metric signature (with at least two minuses and at least two pluses), and a sign factor accounting for semidefiniteness of the Weyl tensor, then the local-isometry types of our metrics correspond bijectively to equivalence classes of surfaces with equiaffine projectively flat torsionfree connections; the latter equivalence relation is provided by unimodular affine local diffeomorphisms. The surface just mentioned arises, locally, as the leaf space of a codimension-two parallel distribution on the pseudo-Riemannian manifold in question, naturally associated with its metric. We construct examples showing that the leaves of this distribution may form a fibration with the base which is a closed surface of any prescribed diffeomorphic type.

Our result also completes a local classification of pseudo-Riemannian metrics with parallel Weyl tensor that are neither conformally flat nor locally symmetric: for those among such metrics which are not Ricci-recurrent, the Weyl tensor has rank 1, and so they belong to the class discussed in the previous paragraph; on the other hand, the Ricci-recurrent ones have already been classified by the second author.

Introduction. The main result of the present paper, Theorem 21.1, describes the local structure of pseudo-Riemannian metrics whose Weyl conformal tensor W is parallel and, as an operator acting on exterior 2-forms, has rank 1 at each point. Combined with a theorem of the second author [22], our description leads to a local classification of all essentially conformally symmetric pseudo-Riemannian manifolds. Here are some details.

A pseudo-Riemannian manifold (M, g) with $\dim M \geq 4$ is said to be *conformally symmetric* [2] if its Weyl tensor W is parallel. Obvious examples arise when (M, g) is conformally flat or locally symmetric; conformally symmetric manifolds which are neither have usually been referred to as *essentially conformally symmetric*. For a sample of recent results on conformally symmetric manifolds, see [24, 21, 8, 9, 13].

An essentially conformally symmetric pseudo-Riemannian metric is always indefinite [4, Theorem 2]; if it is Lorentzian, it must also be Ricci-recurrent [5, Corollary 1 on p. 21]. Known examples of essentially conformally symmetric indefinite metrics include both Ricci-recurrent [22] and non-Ricci-recurrent ones [3], in every dimension $n \geq 4$.

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Here we call a pseudo-Riemannian manifold *Ricci-recurrent* if, for every tangent vector field v, the Ricci tensor ρ and the covariant derivative $\nabla_v \rho$ are linearly dependent at each point, or, equivalently, $\nabla\rho = \xi \otimes \rho$ on the open set where $\rho \neq 0$, for some 1-form ξ.

In [22] the second author completely determined the local structure of Ricci-recurrent essentially conformally symmetric metrics at points where ρ and $\nabla\rho$ are nonzero: in suitable coordinates, such metrics have the canonical form of [22, formula (34)].

Our Theorem 21.1, mentioned above, leads to a classification of the remaining essentially conformally symmetric pseudo-Riemannian metrics. The reason is that the Weyl tensor has rank 1 for every such metric which is not Ricci-recurrent [6, Theorem 4 on p. 17]. Unlike the result of [22], Theorem 21.1 requires no general-position hypothesis.

The text consists of three parts, each devoted to one of the three topics named in the title. Part I presents some known classification theorems about projectively flat connections on surfaces, followed by results on solvability of a specific partial differential equation (Sections 10–12), both of which are needed as a reference for Part III.

Part II deals with null parallel distributions $\mathcal{P}$ on pseudo-Riemannian manifolds such that the Levi-Civita connection ∇ is, in a natural sense, $\mathcal{P}^\perp$-projectable onto a torsionfree connection D on the (local) leaf space $\varSigma$ of $\mathcal{P}^\perp$, with $\mathcal{P}^\perp$ always denoting the g-orthogonal complement of $\mathcal{P}$. We observe that $\mathcal{P}^\perp$-projectability of ∇ is characterized by a simple curvature condition and, therefore, it holds for the two-dimensional null parallel distribution $\mathcal{P}$ present on every conformally symmetric manifold with $\operatorname{rank} W = 1$. This is one of the steps needed for the argument in Part III.

In Part III we establish our main result (Theorem 21.1), first showing that, locally, in dimensions $n \geq 4$, a conformally symmetric metric g with $\operatorname{rank} W = 1$ is a warped product in which the totally geodesic factor is four-dimensional, conformally symmetric and has $\operatorname{rank} W = 1$, while the umbilical factor is flat. The problem is thus reduced to the case $n = 4$. When $n = 4$, we prove that g is conformal to a metric of a kind first classified by Ruse [23]; therefore, g itself is one of Patterson and Walker's Riemann extension metrics [19]. We also establish, for every $n \geq 4$, projective flatness of the connection D arising, as in the last paragraph, on the surface $\varSigma$ which is, locally, the leaf space of $\mathcal{P}^\perp$. The local-isometry type of g then turns out to be parametrized by the dimension n, the metric signature $--\cdots++$, a factor $\varepsilon = \pm 1$ describing semidefiniteness of W, and the (local) equiaffine equivalence class of D. See Section 24.

Finally, in Section 23, we describe examples showing that any prescribed closed surface $\varSigma$ can be realized as the *global* leaf space of $\mathcal{P}^\perp$ for some non-Ricci-recurrent essentially conformally symmetric manifold (M, g) of any given dimension $n \geq 4$. More precisely, the leaves of $\mathcal{P}^\perp$ then are the fibres of a bundle with the total space M and base $\varSigma$.

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1. Preliminaries. Throughout this paper, all manifolds, bundles, their sections and subbundles, as well as connections and mappings, including bundle morphisms, are assumed to be of class C^∞. A manifold is by definition connected; a bundle morphism may operate only between two bundles with the same base manifold, and acts by identity on the base.

'A bundle' always means 'a C^∞ locally trivial bundle' and the same symbol is used both for a given bundle and for its total space. The bundle projection onto the base manifold is denoted by π, and Ker $d\pi$ stands for the vertical distribution.

By a *differential k-form valued in a vector bundle* $\mathcal{X}$ over a manifold Σ we mean, as usual [1, p. 24], any vector-bundle morphism $(T\Sigma)^{\wedge k} \to \mathcal{X}$. This includes the case of ordinary (real-valued) forms, where $\mathcal{X}$ is the product bundle $\Sigma \times \boldsymbol{R}$, the sections of which are functions $\Sigma \to \boldsymbol{R}$.

The symbols ∇ and D will be used for various connections in vector bundles. Our sign convention about the curvature tensor $R = R^\nabla$ of a connection ∇ in a vector bundle $\mathcal{X}$ over a manifold Σ is

$$R(u, v)\psi = \nabla_v \nabla_u \psi - \nabla_u \nabla_v \psi + \nabla_{[u,v]}\psi\,, \tag{1}$$

for sections ψ of $\mathcal{X}$ and vector fields u, v tangent to Σ. Thus,

$$R \text{ is a 2-form on } \Sigma \text{ valued in } \operatorname{Hom}(\mathcal{X}, \mathcal{X})\,. \tag{2}$$

Here $\operatorname{Hom}(\mathcal{X}, \mathcal{Y})$, for real vector bundles $\mathcal{X}, \mathcal{Y}$ over a manifold Σ, is the vector bundle over Σ, the sections of which are vector-bundle morphisms $\mathcal{X} \to \mathcal{Y}$. For instance, $\mathcal{X}^* = \operatorname{Hom}(\mathcal{X}, \Sigma \times \boldsymbol{R})$ is the dual of $\mathcal{X}$. By (1), for connections in the tangent bundle,

$$R_{jkl}{}^m = \partial_k \Gamma^m_{jl} - \partial_j \Gamma^m_{kl} + \Gamma^m_{ks}\Gamma^s_{jl} - \Gamma^m_{js}\Gamma^s_{kl}\,, \tag{3}$$

∂_j and Γ^l_{jk} being the partial derivatives and the connection components.

The Ricci tensor $\rho = \rho^{\mathrm{D}}$ of a connection D on a manifold Σ (that is, in the tangent bundle $T\Sigma$) is given by $\rho(u, w) = \operatorname{Trace}[v \mapsto R(u, v)w]$, where $R = R^{\mathrm{D}}$ and u, v, w are vectors tangent to Σ at any point. It is well-known that, if D is torsionfree and $\dim \Sigma = r$, the following four conditions are equivalent:

(a) the connection induced by D in the line bundle $(T\Sigma)^{\wedge r}$ is flat,

(b) a nonzero D-parallel differential r-form exists on every simply connected open subset of Σ,

(c) the operator $R^{\mathrm{D}}(u, v) : T_y\Sigma \to T_y\Sigma$ given by $w \mapsto R^{\mathrm{D}}(u, v)w$ is traceless for every $y \in \Sigma$ and all $u, v \in T_y\Sigma$,

(d) ρ^{D} is symmetric.

Indeed, the curvature tensor of the connection that D induces in $(T\Sigma)^{\wedge r}$, viewed as a real-valued 2-form (by (2) with the identification $\operatorname{Hom}((T\Sigma)^{\wedge r}, (T\Sigma)^{\wedge r}) = \Sigma \times \boldsymbol{R}$), sends u, v in $T_y\Sigma$ to $\operatorname{Trace}[R^{\mathrm{D}}(u, v)]$, while, by the first Bianchi identity, $\operatorname{Trace}[R^{\mathrm{D}}(u, v)] = \rho^{\mathrm{D}}(u, v) - \rho^{\mathrm{D}}(v, u)$. Thus, (a) is equivalent both to (c) and to (d). Finally, the connections that D induces in $(T\Sigma)^{\wedge r}$ and $(T^*\Sigma)^{\wedge r} = [(T\Sigma)^{\wedge r}]^*$ are each other's duals, so that (a) holds if and only if (b) does.

A fixed connection ∇ in a vector bundle $\mathcal{X}$ over a manifold Σ gives rise to the operator of *exterior derivative* d^∇ acting on $\mathcal{X}$-valued differential forms on Σ which, for the standard flat connection in the product bundle $\Sigma \times \boldsymbol{R}$, is the ordinary exterior derivative d for real-valued forms [1, p. 24]. Explicitly, for an $\mathcal{X}$-valued 1-form Ψ,

$$(d^{\nabla}\Psi)(u, v) = \nabla_u[\Psi(v)] - \nabla_v[\Psi(u)] - \Psi([u, v])\,. \tag{4}$$

We will establish the local solvability of various systems of linear partial differential equations (homogeneous or not) on a simply connected manifold Σ by introducing a flat connection ∇ in a suitable vector bundle $\mathcal{X}$ over Σ and showing that

(i) our (homogeneous) system amounts to requiring an unknown section of $\mathcal{X}$ to be ∇-parallel, or,

(ii) the (nonhomogeneous) system can be rewritten so as to be imposed on $\mathcal{X}$-valued differential forms, and then its solvability is equivalent to d^{∇}-exactness of the right-hand side (which, locally, is the same as its easily-verified d^{∇}-closedness).

Note that the Poincaré lemma can be applied, in (ii), since flatness of ∇ and simple connectivity of Σ allow us to treat $\mathcal{X}$-valued differential forms as forms taking values in a fixed finite-dimensional vector space, in such a way that d^{∇} becomes the ordinary d.

REMARK 1.1. Twice-covariant tensor fields τ on a manifold Σ will also be treated as $T^*\Sigma$-valued 1-forms. Namely, we choose $\tau(w, \cdot)$ to be the section of $T^*\Sigma$ to which the 1-form τ sends a vector field w on Σ. Here τ is not assumed symmetric; the use of $\tau(\cdot, w)$ instead of $\tau(w, \cdot)$ would amount to replacing τ by its *transpose* τ^*, which is the twice-covariant tensor field assigning $\tau(w', w)$ to vector fields w, w'. For instance, if $\tau = \mathrm{D}\xi$ for a connection D and a 1-form ξ on Σ, we have $\tau(w, \cdot) = \mathrm{D}_w\xi$, and, in local coordinates, the components of $\tau = \mathrm{D}\xi$ and $\lambda = (\mathrm{D}\xi)^*$ are $\tau_{jk} = \xi_{k,j}$ and $\lambda_{jk} = \xi_{j,k}$.

REMARK 1.2. Let D be a torsionfree connection on a manifold Σ. A twice-covariant symmetric tensor field τ on Σ is said to satisfy the *Codazzi equation* if $d^{\mathrm{D}}\tau = 0$ for τ treated as a $T^*\Sigma$-valued 1-form (Remark 1.1), which in coordinates reads $\tau_{jl,k} = \tau_{kl,j}$, cf. (4). Suppose now that, in addition, the Ricci tensor ρ^{D} is symmetric. In view of the second Bianchi identity, the curvature tensor R^{D} has zero divergence ($\mathrm{div}^{\mathrm{D}}R^{\mathrm{D}} = 0$) if and only if the Codazzi equation $d^{\mathrm{D}}\rho^{\mathrm{D}} = 0$ holds for ρ^{D}. (In coordinates, this means that the condition $R_{jkl}{}^{s}{}_{,s} = 0$ is equivalent to $R_{jl,k} = R_{kl,j}$.) Riemannian manifolds with $\mathrm{div}^{\nabla}R^{\nabla} = 0$ for the Levi-Civita connection ∇ are said to have *harmonic curvature* [1].

REMARK 1.3. We always treat 2-forms on a manifold Σ valued in ordinary 1-forms (that is, $T^*\Sigma$-valued) as 1-forms on Σ valued in ordinary 2-forms (that is, $(T^*\Sigma)^{\wedge 2}$-valued), via the obvious identification.

REMARK 1.4. Aside from differentials of functions, we will apply the exterior derivative operators d, d^{D} and d^{∇} *only to* 1*-forms* valued in various vector bundles. For instance, given a twice-covariant tensor field τ and a torsionfree connection D on a manifold Σ, the exterior derivative $d^{\mathrm{D}}\tau$ (of τ viewed as a $T^*\Sigma$-valued 1-form, cf. Remark 1.1), is itself treated, according to Remark 1.3, as a 1-form valued in 2-forms. If, in addition, $\dim \Sigma = 2$ and ρ^{D} is symmetric, we use this last interpretation to define the exterior derivative $dd^{\mathrm{D}}\tau$, writing d

instead of d^{D} to reflect the fact that the existence, locally in Σ, of a D-parallel area form (see (a)–(d) above), allows us to regard $(T^*\Sigma)^{\wedge 2}$-valued forms, locally, as real-valued.

The $(T^*\Sigma)^{\wedge 2}$-valued 2-form $A = dd^{\mathrm{D}}\tau$ assigns to vector fields u, v the real-valued 2-form $A(u, v)$ with the local-coordinate expression $[A(u, v)]_{lm} = u^j v^k A_{jklm}$, where $A_{jklm} = \tau_{mk,lj} - \tau_{lk,mj} - \tau_{mj,lk} + \tau_{lj,mk}$.

2. Exterior products. If D is a torsionfree connection on a surface and the Ricci tensor ρ^{D} is symmetric, the curvature tensor R^{D} is given by

$$R^{\mathrm{D}} = \rho^{\mathrm{D}} \wedge \mathrm{Id}\,, \quad \text{that is, in coordinates,} \quad R_{jkl}{}^m = R_{jl}\delta_k^m - R_{kl}\delta_j^m\,. \tag{5}$$

Indeed, for $R = R^{\mathrm{D}}$ and $\rho = \rho^{\mathrm{D}}$ treated as algebraic objects, at any fixed point, ρ is the Ricci contraction of $R = \rho \wedge \mathrm{Id}$, and so the operator $\rho \mapsto R = \rho \wedge \mathrm{Id}$, being injective, is also surjective. (Both R, ρ range over 3-dimensional spaces, cf. (c) in Section 1.)

Aside from formula (5), we will use the symbol $\wedge$ only in two situations, which involve a real-valued 1-form ξ and twice-covariant tensor fields τ, λ on any given manifold Σ. Namely, $\xi \wedge \lambda$ stands for the 1-form valued in 2-forms, sending a vector field w to the 2-form that assigns to vector fields u, v the function $\xi(u)\lambda(v, w) - \xi(v)\lambda(u, w)$, while $\tau \wedge \lambda$ is a 2-form valued in 2-forms: it sends vector fields w, w' to $\tau(w, \cdot) \wedge \lambda(w', \cdot) - \tau(w', \cdot) \wedge \lambda(w, \cdot)$ (which in turn is the 2-form associating with u, v the function $\tau(w, u)\lambda(w', v) - \tau(w, v)\lambda(w', u) - \tau(w', u)\lambda(w, v) + \tau(w', v)\lambda(w, u)$). Note that ξ, τ and λ may be viewed as differential forms on Σ valued in differential forms (Remark 1.1), and then $\wedge$ becomes the usual exterior product (with the identification described in Remark 1.3). For twice-covariant tensor fields τ, λ on a *surface* Σ,

$$\lambda \wedge \tau = dd^{\mathrm{D}}\lambda = 0 \quad \text{if } \tau \text{ is symmetric and } \lambda \text{ is skew-symmetric}\,. \tag{6}$$

Indeed, in local coordinates $A = \tau \wedge \lambda$ is given by $A_{jklm} = \tau_{jl}\lambda_{km} - \tau_{jm}\lambda_{kl} - \tau_{kl}\lambda_{jm} + \tau_{km}\lambda_{jl}$, with the only essential component $A_{1212} = 0$. On the other hand, locally, $\lambda = f\alpha$, where α is a fixed D-parallel area form, and so $d^{\mathrm{D}}\lambda = df \wedge \alpha = -df \otimes \alpha$ (equality of 1-forms valued in 2-forms; the sum $df \otimes \alpha + df \wedge \alpha$ must vanish, as it is a real-valued differential 3-form, while $\dim \Sigma = 2$). Thus, $dd^{\mathrm{D}}\lambda = -ddf \otimes \alpha = 0$.

For a torsionfree connection D and a 1-form ξ on a manifold Σ,

$$\text{(i)} \quad d^{\mathrm{D}}\mathrm{D}\xi = \xi R^{\mathrm{D}}\,, \qquad \text{(ii)} \quad d^{\mathrm{D}}(\mathrm{D}\xi)^* = \mathrm{D}d\xi + \xi R^{\mathrm{D}}\,. \tag{7}$$

Here (i) is the *Ricci identity* (in coordinates, $\xi_{j,kl} - \xi_{j,lk} = R_{lkj}{}^s\xi_s$), R^{D} denotes, as usual, the curvature tensor of D, and ξR^{D} is the 2-form valued in 1-forms that assigns to vector fields w, w' on Σ the composite vector-bundle morphism $T\Sigma \to T\Sigma \to \Sigma \times \boldsymbol{R}$ in which $R^{\mathrm{D}}(w, w')$ is followed by ξ (cf. (2)). To establish (ii), note that the coordinate version of (i) gives $\xi_{k,lj} - \xi_{j,lk} = (\xi_{k,j} - \xi_{j,k})_{,l} + R_{jlk}{}^s\xi_s - R_{klj}{}^s\xi_s$, the last two terms of which may be replaced by $R_{jkl}{}^s\xi_s$ in view of the first Bianchi identity.

If, in addition, $\dim \Sigma = 2$ and the Ricci tensor ρ^{D} is symmetric,

$$\text{(i)} \quad \xi R^{\mathrm{D}} = -\xi \wedge \rho^{\mathrm{D}}\,, \qquad \text{(ii)} \quad d^{\mathrm{D}}\mathrm{D}\xi = -\xi \wedge \rho^{\mathrm{D}}\,, \tag{8}$$

by (5) and (7-i). The coordinate form of (8-ii) is $\xi_{j,kl} - \xi_{j,lk} = \xi_k R_{lj} - \xi_l R_{kj}$.

The Ricci identity for twice-covariant tensor fields λ (sections of $(T^*M)^{\otimes 2}$) reads

$$d^\nabla \nabla\lambda = \lambda \cdot R\,, \quad \text{that is}\,, \quad \lambda_{lm,kj} - \lambda_{lm,jk} = R_{jkl}{}^s\lambda_{sm} + R_{jkm}{}^s\lambda_{ls}\,. \tag{9}$$

Here M is any manifold with a fixed torsionfree connection (this time denoted by ∇) and $R = R^\nabla$, while the covariant derivative $\nabla\lambda$, treated as a $(T^*M)^{\otimes 2}$-valued 1-form on M, associates with a vector field w the section $\nabla_w\lambda$ of $(T^*M)^{\otimes 2}$. Finally, $\lambda \cdot R$ denotes the $(T^*M)^{\otimes 2}$-valued 2-form, sending vector fields w, w' to the section of $(T^*M)^{\otimes 2}$ which assigns to vector fields u, v the function $\lambda(R(w, w')u, v) + \lambda(u, R(w, w')v)$.

Given a twice-covariant tensor field τ and a vector field u on a pseudo-Riemannian manifold (M, g), we define τu to be the vector field on M which is the image of u under the vector-bundle morphism $TM \to TM$ obtained from τ by index raising; thus,

$$g(\tau u, v) = \tau(u, v) \ \text{ for all vector fields } \ u \ \text{ and } \ v\,. \tag{10}$$

In a pseudo-Riemannian manifold (M, g) of any dimension n, we denote by ∇ its Levi-Civita connection, and by R, ρ (rather than R^∇, ρ^∇) its curvature and Ricci tensors. The same symbol R is used for the four-times covariant curvature tensor (a 2-form valued in 2-forms), with $R(u, v, w, w') = g(R(u, v)w, w')$. If $n \geq 4$, the Weyl conformal tensor of (M, g) is defined by $W = R - (n-2)^{-1} g \wedge \sigma$, where $\sigma = \rho - (2n-2)^{-1}\mathrm{s}g$ is the *Schouten tensor*, with $\mathrm{s} = \mathrm{Trace}_g\,\rho$ standing for the scalar curvature, and $\wedge$ as above.

For vector fields u, v, a differential 2-form ω, the Levi-Civita connections $\nabla, \tilde\nabla$ and Ricci tensors $\rho, \tilde\rho$ of conformally related metrics g and $\tilde g = f^{-2}g = e^{2\phi}g$ on a manifold M, with functions $f > 0$ and $\phi = -\log f$, their g-gradients $\nabla\phi, \nabla f$, and the g-Laplacian $\Delta f = g^{jk}f_{,jk}$ of f, we have (cf. [10, formulae (16.8), (16.9), (16.13) on pp. 528–529])

(a) $\tilde\nabla_u v = \nabla_u v + g(u, \nabla\phi)v + g(v, \nabla\phi)u - g(u, v)\nabla\phi$,

(b) $\tilde\nabla_u \omega = \nabla_u\omega - 2(d_u\phi)\omega + \omega(u, \cdot) \wedge d\phi + g(u, \cdot) \wedge \omega(\nabla\phi, \cdot)$,

(c) $\tilde\rho = \rho + (n-2)f^{-1}\nabla df + [f^{-1}\Delta f - (n-1)f^{-2}g(\nabla f, \nabla f)]g$, where $n = \dim M$,

d_u in (b) being the directional derivative, so that $d_u\phi = g(u, \nabla\phi)$.

PART I. PROJECTIVELY FLAT SURFACES

Except for Sections 10 through 12, the material in Part I is known, and consists of classification results about projectively flat equiaffine torsionfree connections on surfaces. A self-contained presentation of those results is provided here for the reader's convenience: such a connection serves as the single non-discrete parameter in our classification of conformally symmetric manifolds with $\mathrm{rank}\, W = 1$ (see Section 21). For more on projectively flat surfaces, see [20] and Simon's article on affine differential geometry [10, pp. 905–961].

3. Projective flatness in dimension 2. A connection D on a manifold $\varSigma$ is called *projectively flat* if $\varSigma$ is covered by coordinate systems in which the geodesics of D appear as (re-parametrized) straight-line segments.

We begin with a well-known lemma, going back to Weyl [29, p. 100]:

LEMMA 3.1. *Two torsionfree connections* D *and* $\tilde{\mathrm{D}}$ *on a manifold* Σ *have the same re-parametrized geodesics if and only if* $B = 2\xi \odot \mathrm{Id}$ *for the tensor field* $B = \tilde{\mathrm{D}} - \mathrm{D}$ *and a real-valued* 1*-form* ξ *on* Σ*, or, in coordinates,* $B^l_{jk} = \xi_j \delta^l_k + \xi_k \delta^l_j$*. Then also*

(i) *the Ricci tensors* ρ^{D} *of* D *and* $\tilde{\rho}$ *of* $\tilde{\mathrm{D}}$ *are related by* $\tilde{\rho} = \rho^{\mathrm{D}} + (r-1)\xi \otimes \xi - r\mathrm{D}\xi + (\mathrm{D}\xi)^*$ *for* $r = \dim \Sigma$*, with* $(\mathrm{D}\xi)^*$ *as in Remark* 1.1,

(ii) $d\xi = 0$ *if both* D *and* $\tilde{\mathrm{D}}$ *have symmetric Ricci tensors.*

PROOF. In view of the local-coordinate form of the geodesic equation, the two connections have the same geodesics if and only if $B_v v = cv$ for every $y \in \Sigma$ and $v \in T_y\Sigma$, with some $c \in \boldsymbol{R}$ depending on v. The latter condition gives $B = 2\xi \odot \mathrm{Id}$ (by polarization in a fixed basis of $T_y\Sigma$, for any $y \in \Sigma$). Next, if $\tilde{\mathrm{D}} - \mathrm{D} = 2\xi \odot \mathrm{Id}$, the curvature tensors R^{D} of D and $\tilde{R}$ of $\tilde{\mathrm{D}}$ are related by $\tilde{R} = R^{\mathrm{D}} - d\xi \otimes \mathrm{Id} - \mathrm{D}\xi \wedge \mathrm{Id} + \xi \otimes (\xi \wedge \mathrm{Id})$ (that is, $\tilde{R}_{jkl}{}^m = R_{jkl}{}^m + (\xi_{j,k} - \xi_{k,j})\delta^m_l + \xi_{l,k}\delta^m_j - \xi_{l,j}\delta^m_k + \xi_l(\xi_j\delta^m_k - \xi_k\delta^m_j)$), as one easily verifies using (3) in coordinates in which the components of D at the given point vanish. This implies (i). Now (ii) follows: if ρ^{D} and $\tilde{\rho}$ are symmetric, so must be $\mathrm{D}\xi$. □

REMARK 3.2. For a torsionfree connection D on a surface Σ such that the Ricci tensor ρ^{D} is symmetric, a function $f : \Sigma \to (0, \infty)$, and the 1-form $\xi = -d \log f$, the condition $\mathrm{D}df = -f\rho^{\mathrm{D}}$ is equivalent to flatness of the connection $\tilde{\mathrm{D}} = \mathrm{D} + 2\xi \odot \mathrm{Id}$. In fact, by Lemma 3.1, $\tilde{\rho} = 0$ if and only if $\mathrm{D}df = -f\rho^{\mathrm{D}}$, while $\tilde{\rho}$ determines $\tilde{R}$, as in (5).

The next result is the 2-dimensional case of a theorem of Weyl [29, p. 105]:

THEOREM 3.3. *A torsionfree connection* D *on a surface* Σ *with a symmetric Ricci tensor* ρ^{D} *is projectively flat if and only if* ρ^{D} *satisfies the Codazzi equation* $d^{\mathrm{D}}\rho^{\mathrm{D}} = 0$.

PROOF. In the vector bundle $T^*\Sigma \oplus (\Sigma \times \boldsymbol{R})$, whose sections (ξ, f) consist of a 1-form ξ and a function f on Σ, we define a connection ∇ by $\nabla(\xi, f) = (\mathrm{D}\xi + f\rho^{\mathrm{D}}, df - \xi)$, that is, $\nabla_u(\xi, f) = (\mathrm{D}_u\xi + f\rho^{\mathrm{D}}(u, \cdot), d_u f - \xi(u))$ for any vector field u. From (1) with $\psi = (\xi, f)$ we get $R^\nabla(u, v)\psi = (\xi'(u, v), 0)$, where $\xi' = -d^{\mathrm{D}}\mathrm{D}\xi - \xi \wedge \rho^{\mathrm{D}} - f d^{\mathrm{D}}\rho^{\mathrm{D}}$, with $d^{\mathrm{D}}\mathrm{D}\xi$ as in (7-i). Thus, by (8-ii), $d^{\mathrm{D}}\rho^{\mathrm{D}} = 0$ if and only if ∇ is flat.

If ∇ is flat, we may choose, locally, a ∇-parallel section (ξ, f) of $T^*\Sigma \oplus (\Sigma \times \boldsymbol{R})$ with $f > 0$. Now D is projectively flat by Remark 3.2 and Lemma 3.1, as $\mathrm{D}df = -f\rho^{\mathrm{D}}$.

Conversely, let D be projectively flat. By Lemma 3.1, we can find, locally, a positive function f such that the torsionfree connection $\tilde{\mathrm{D}} = \mathrm{D} + 2\xi \odot \mathrm{Id}$, with $\xi = -d \log f$, is flat. Remark 3.2 then gives $\mathrm{D}df = -f\rho^{\mathrm{D}}$ and, applying d^{D} to both sides (treated as $T^*\Sigma$-valued 1-forms) we get $d^{\mathrm{D}}\mathrm{D}df = -df \wedge \rho^{\mathrm{D}} - f d^{\mathrm{D}}\rho^{\mathrm{D}}$, while $d^{\mathrm{D}}\mathrm{D}df = -df \wedge \rho^{\mathrm{D}}$ in view of (8-ii) with ξ replaced by df. Thus, $d^{\mathrm{D}}\rho^{\mathrm{D}} = 0$. □

Suppose that Σ is an r-dimensional manifold, $\mathcal{V}$ is a real vector space of dimension $r+1$, and $\Phi : \Sigma \to \mathcal{V}$ is an immersion transverse to all lines in $\mathcal{V}$ passing through 0. By the *centroaffine connection* corresponding to Φ we mean the connection D on Σ obtained by first pulling back the standard flat connection of $\mathcal{V}$ to the pullback bundle $\Phi^*T\mathcal{V} = \Sigma \times \mathcal{V}$, and then projecting it onto the summand $T\Sigma$ in $\Phi^*T\mathcal{V} = T\Sigma \oplus \mathcal{N}$. Here $\mathcal{N}$ is the *normal bundle* over Σ whose fibre $\mathcal{N}_y$, for any $y \in \Sigma$, is the line in $\mathcal{V} = T_{\Phi(y)}\mathcal{V}$ spanned by $\Phi(y)$. See [10, pp. 927, 934].

REMARK 3.4. Let D be the centroaffine connection on Σ, for Σ, $\mathcal{V}$, r and Φ as above.

(a) We may assume, locally, that Σ is a submanifold of $\mathcal{V}$ and Φ is the inclusion mapping. The geodesics of D then have the form $\Sigma \cap \Pi$ for planes Π through 0 in $\mathcal{V}$, since for any parametrization of such a curve $\Sigma \cap \Pi$, the acceleration lies in Π, and so its component tangent to Σ is tangent to $\Sigma \cap \Pi$. Cf. [10, p. 927].

(b) The connection D is torsionfree and projectively flat.

(c) The Ricci tensor ρ^{D} is symmetric and $\rho^{\mathrm{D}} = (1-r)b$, where b is the second fundamental form of the immersion Φ, obtained in the usual fashion from the trivialization of the normal bundle provided by the radial (identity) vector field.

Indeed, projective flatness of D is immediate from (a): central projections of Σ into hyperplanes not containing 0 send geodesics of D into lines. To verify the other claims in (b) and (c) we may use the traditional notation in which the inclusion mapping $\Sigma \to \mathcal{V}$, here serving also as a trivializing section of the normal bundle, is represented by the symbol $\mathbf{r}$ and treated as a $\mathcal{V}$-valued function on Σ, while its partial differentiations of all orders, relative to fixed coordinates in Σ, are represented by successive subscripts; for instance, $\mathbf{r}_j$ are the coordinate vector fields. In the expansion

$$\mathbf{r}_{jk} = \Gamma_{jk}^{s}\mathbf{r}_s + b_{jk}\mathbf{r} \tag{11}$$

the coefficients Γ_{jk}^{s} and b_{jk} are the component functions of D and, respectively, of the second fundamental form b mentioned in (c). Thus, D is torsionfree and b is symmetric, since $\mathbf{r}_{jk} = \mathbf{r}_{kj}$. Also, differentiating (11), we now get, from (3), $R_{jkl}{}^{m} = b_{kl}\delta_j^m - b_{jl}\delta_k^m$ (and $b_{kl,j} = b_{jl,k}$), as $\mathbf{r}_{jlk} = \mathbf{r}_{klj}$. Hence $\rho^{\mathrm{D}} = (1-r)b$, as required.

4. Flatness of an associated connection. For a fixed torsionfree connection D on a manifold Σ and a 1-form ξ on Σ, let

$$\mathcal{B}\xi = \mathrm{D}\xi + (\mathrm{D}\xi)^* \quad \text{or, in coordinates,} \quad (\mathcal{B}\xi)_{jk} = \xi_{k,j} + \xi_{j,k}\,. \tag{12}$$

This defines a first-order linear differential operator $\mathcal{B}$ sending 1-forms on Σ to twice-covariant symmetric tensor fields. We use the symbol $\operatorname{Ker}\mathcal{B}$ for its kernel:

$$\operatorname{Ker}\mathcal{B} = \{\xi;\ \xi \text{ is a 1-form on } \Sigma \text{ and } \mathcal{B}\xi = 0\}\,. \tag{13}$$

The second covariant derivative $\mathrm{D}\mathrm{D}\xi$ of a 1-form ξ on Σ can be expressed in terms of the tensor $\tau = \mathcal{B}\xi$, via the identity $(\mathrm{D}_w\mathrm{D}\xi)(u,v) = \xi R^{\mathrm{D}}(v,u)w + (\partial\tau)(v,u,w)$, valid for all vector fields u, v, w on Σ, where $2(\partial\tau)(v,u,w) = (\mathrm{D}_u\tau)(v,w) + (\mathrm{D}_w\tau)(v,u) - (\mathrm{D}_v\tau)(u,w)$. (In coordinates, this reads $\xi_{j,kl} = R_{jkl}{}^{s}\xi_s + (\tau_{jl,k} + \tau_{jk,l} - \tau_{kl,j})/2$, with $\tau_{jk} = \xi_{k,j} + \xi_{j,k}$, and easily follows, since the Ricci identity (7-i), in its coordinate form $\xi_{j,kl} - \xi_{j,lk} = R_{lkj}{}^{s}\xi_s$, gives $\tau_{jl,k} + \tau_{jk,l} - \tau_{kl,j} = 2\xi_{j,kl} + R_{kjl}{}^{s}\xi_s + R_{klj}{}^{s}\xi_s + R_{ljk}{}^{s}\xi_s$, while $R_{klj}{}^{s}\xi_s + R_{ljk}{}^{s}\xi_s = R_{kjl}{}^{s}\xi_s$ by the first Bianchi identity.) In particular,

$$\mathrm{D}\mathrm{D}\xi = -\,\xi R^{\mathrm{D}} \quad \text{whenever} \quad \mathcal{B}\xi = 0\,. \tag{14}$$

The equality $\mathrm{DD}\xi = -\xi R^{\mathrm{D}}$ means that $(\mathrm{D}_w\mathrm{D}\xi)(u, v) = \xi R^{\mathrm{D}}(v, u)w$ for all vector fields u, v, w on Σ. The 2-form $\mathrm{D}\xi$ is viewed here as a 0-form valued in 2-forms, which makes $\mathrm{DD}\xi$ a 1-form valued in 2-forms, while ξR^{D}, defined in the lines following (7), is now treated as a 1-form valued in 2-forms, in agreement with Remark 1.3.

In the following lemma we solve the equation $\mathcal{B}\xi = 0$, under additional assumptions on Σ and D, using the approach of (i) in Section 1.

LEMMA 4.1. *Given a projectively flat torsionfree connection* D *on a surface* Σ *such that the Ricci tensor* ρ^{D} *is symmetric, let us define a connection* ∇ *in the vector bundle* $\mathcal{X} = (T^*\Sigma)^{\wedge 2} \oplus T^*\Sigma$ *by* $\nabla(\Theta, \xi) = (\mathrm{D}\Theta - \xi \wedge \rho^{\mathrm{D}}, \mathrm{D}\xi - \Theta)$. *Then* ∇ *is flat, and the assignment* $\xi \mapsto (\Theta, \xi)$ *given by* $\Theta = \mathrm{D}\xi$ *is a linear isomorphism of the space* $\operatorname{Ker}\mathcal{B}$ *in* (13) *onto the space of all* ∇*-parallel sections* (Θ, ξ) *of* $\mathcal{X}$.

The pair (Θ, ξ) in Lemma 4.1 is a section of $\mathcal{X}$, with the components that are a real-valued 2-form Θ and a real-valued 1-form ξ on Σ. The meaning of the equality $\nabla(\Theta, \xi) = (\mathrm{D}\Theta - \xi \wedge \rho^{\mathrm{D}}, \mathrm{D}\xi - \Theta)$ is $\nabla_u(\Theta, \xi) = (\mathrm{D}_u\Theta - \xi \wedge \rho^{\mathrm{D}}(u, \cdot), \mathrm{D}_u\xi - \Theta(u, \cdot))$ for every vector field u tangent to Σ. We precede the proof of Lemma 4.1 with a remark.

REMARK 4.2. On a manifold Σ with a fixed torsionfree connection D, a 1-form ξ such that $\mathcal{B}\xi = 0$ is uniquely determined by its value ξ_y and covariant derivative $(\mathrm{D}\xi)_y$ at any given point $y \in \Sigma$. Indeed, by (14), the pair (Θ, ξ), with $\Theta = \mathrm{D}\xi$, then is a ∇-parallel section of the vector bundle $\mathcal{X} = (T^*\Sigma)^{\wedge 2} \oplus T^*\Sigma$, for the connection ∇ in $\mathcal{X}$ given by $\nabla(\Theta, \xi) = (\mathrm{D}\Theta + \xi R^{\mathrm{D}}, \mathrm{D}\xi - \Theta)$. (Notation as above; ξR^{D}, appearing in (7), is treated here as a 1-form valued in 2-forms, cf. Remark 1.3.)

PROOF OF LEMMA 4.1. From (1) with $\psi = (\Theta, \xi)$ and (4), $R^{\nabla}(u, v)\psi = (\Theta', \xi')$, where $\Theta' = \hat{R}(u, v)\Theta + \xi \wedge [(d^{\mathrm{D}}\rho^{\mathrm{D}})(u, v)] + (\Theta \wedge \rho^{\mathrm{D}})(u, v)$ and ξ' is defined by $\xi' = -(d^{\mathrm{D}}\mathrm{D}\xi + \xi \wedge \rho^{\mathrm{D}})(u, v)$, with $d^{\mathrm{D}}\mathrm{D}\xi$ as in (7) and $\hat{R}$ denoting the curvature tensor of the connection that D induces in $(T^*\Sigma)^{\wedge 2}$. Flatness of ∇ now follows, since $\hat{R} = 0$ (see (a)–(d) in Section 1), $d^{\mathrm{D}}\rho^{\mathrm{D}} = 0$ by Theorem 3.3, $\Theta \wedge \rho^{\mathrm{D}} = 0$ due to (6), and $\xi' = 0$ in view of (8-ii).

The assignment $\xi \mapsto (\Theta, \xi)$ is obviously injective. That it maps some subspace of $\operatorname{Ker}\mathcal{B}$ *onto* the space of ∇-parallel sections of $\mathcal{X}$ is also clear: if $\nabla(\Theta, \xi) = 0$, then $\mathrm{D}\xi = \Theta$, and so $\mathcal{B}\xi = \Theta + \Theta^* = 0$. Finally, $\nabla(\mathrm{D}\xi, \xi) = 0$ whenever $\mathcal{B}\xi = 0$, as a consequence of Remark 4.2 and (8-i). □

LEMMA 4.3. *Suppose that* D *is a projectively flat torsionfree connection on a simply connected surface* Σ, *the Ricci tensor* ρ^{D} *is symmetric, and* α *is a fixed* D*-parallel area form on* Σ, *cf.* (b) *in Section* 1. *Let* $\operatorname{Ker}\mathcal{B}$ *be the space given by* (13). *Then*

(i) $\dim \operatorname{Ker}\mathcal{B} = 3$,

(ii) *a mapping* $F : \Sigma \to \operatorname{Ker}\mathcal{B}$ *can be defined by letting* $F(y)$, *for* $y \in \Sigma$, *be the unique* $\xi \in \operatorname{Ker}\mathcal{B}$ *with* $\xi_y = 0$ *and* $(\mathrm{D}\xi)_y = \alpha_y$,

(iii) *the mapping* $F : \Sigma \to \operatorname{Ker}\mathcal{B}$ *defined in* (ii) *is an immersion, and*

(iv) *its differential* $dF_y : T_y\Sigma \to \operatorname{Ker}\mathcal{B}$ *at any* $y \in \Sigma$ *sends* $v \in T_y\Sigma$ *to the element* $\eta = dF_y v$ *of* $\operatorname{Ker}\mathcal{B}$ *characterized by* $\eta_y = -\alpha_y(v, \cdot)$ *and* $(\mathrm{D}\eta)_y = 0$.

PROOF. Assertions (i) and (ii) are immediate from Lemma 4.1. Now let $t \mapsto y = y(t)$ be a curve in Σ, and let $\xi = \xi(t) \in \operatorname{Ker}\mathcal{B}$ equal $F(y(t))$. With $v = v(t)$ and $\eta = \eta(t)$ standing for $\dot{y}$ and $d\xi/dt$, we thus have $\eta = dF_y v$ for each t. Differentiating the equalities $\xi_y = 0$ and $(\mathrm{D}\xi)_y = \alpha_y$ covariantly along the curve, and noting that $\mathrm{D}\alpha = 0$, we get $0 = \eta_y + \mathrm{D}_v\xi = \eta_y + \alpha_y(v, \cdot)$ and $(\mathrm{D}\eta)_y + (\mathrm{D}_v\mathrm{D}\xi)_y = 0$. However, $(\mathrm{D}_v\mathrm{D}\xi)_y = 0$ by (14), since $\xi_y = 0$. This proves (iv). Consequently, (iii) follows, as $\eta_y \neq 0$, by (iv), whenever $y \in \Sigma$ and $\eta = dF_y v \in \operatorname{Ker}\mathcal{B}$ for $v \in T_y\Sigma \smallsetminus \{0\}$. □

REMARK 4.4. For $\Sigma, \mathrm{D}, \alpha$ and $\mathcal{B}$ as in Lemma 4.3 and another such quadruple $\Sigma', \mathrm{D}', \alpha', \mathcal{B}'$, let $H : \Sigma \to \Sigma'$ be an affine diffeomorphism (a diffeomorphism sending D onto D′). Under the push-forward linear isomorphism $\operatorname{Ker}\mathcal{B} \to \operatorname{Ker}\mathcal{B}'$, induced by H, the immersion F of Lemma 4.3 obviously corresponds to the composite in which the analogous immersion $F' : \Sigma' \to \operatorname{Ker}\mathcal{B}'$ is followed by the isomorphism $\operatorname{Ker}\mathcal{B}' \to \operatorname{Ker}\mathcal{B}'$ of multiplication by the constant c such that $H^*\alpha' = c^{-1}\alpha$.

The following lemma will be needed in Section 6.

LEMMA 4.5. *Given a projectively flat torsionfree connection* D *on a simply connected surface* Σ *such that the Ricci tensor* ρ^{D} *is symmetric, let* $F : \Sigma \to \operatorname{Ker}\mathcal{B}$ *be as in Lemma* 4.3. *For any vector field* v *on* Σ, *the following two conditions are equivalent*:

(a) v *is* D*-parallel*;

(b) *the function* $y \mapsto dF_y v_y$, *valued in* $\operatorname{Ker}\mathcal{B}$, *is constant on* Σ.

PROOF. If v is D-parallel, so is the 1-form $\eta = -\alpha(v, \cdot)$, where α is a fixed D-parallel area form on Σ. The same η thus satisfies the conditions listed in Lemma 4.3(iv) at all points $y \in \Sigma$ simultaneously, so that $dF_y v_y = \eta$ for all $y \in \Sigma$, and (b) follows. Conversely, if the function $y \mapsto dF_y v_y$ is constant, and equal to $\eta \in \operatorname{Ker}\mathcal{B}$, then, by Lemma 4.3(iv), η is D-parallel and $\eta = -\alpha(v, \cdot)$. Hence v is D-parallel as well. □

For a centroaffine connection D on a surface Σ, defined as in Section 3 for some $\mathcal{V}$ and Φ with $r = 2$, we have an explicit description of the space $\operatorname{Ker}\mathcal{B}$ given by (13). Namely, any fixed volume form $\Omega \in (\mathcal{V}^*)^{\wedge 3} \smallsetminus \{0\}$ gives rise to an isomorphism $\mathcal{V} \to \operatorname{Ker}\mathcal{B}$ sending $\mathbf{w} \in \mathcal{V}$ to the 1-form ξ on Σ with $\xi_y(v) = \Omega(\mathbf{w}, \Phi(y), d\Phi_y v)$ for $y \in \Sigma$ and $v \in T_y\Sigma$. That $\xi \in \operatorname{Ker}\mathcal{B}$ is easily seen in the notation of (11). Specifically, ξ has the components $\xi_j = \xi(\mathbf{r}_j) = \Omega(\mathbf{w}, \mathbf{r}, \mathbf{r}_j)$, and so, by (11), $\xi_{j,k} = \partial_k\xi_j - \Gamma_{kj}^s\xi_s = \Omega(\mathbf{w}, \mathbf{r}_k, \mathbf{r}_j)$, which is skew-symmetric in j, k. The assignment $\mathbf{w} \mapsto \xi$ is injective: if $\mathbf{w} \in \mathcal{V}$ is nonzero, the transversality assumption about Φ guarantees that $\mathbf{w}$, $\Phi(y)$ and $d\Phi_y v$ are linearly independent for some $y \in \Sigma$ and $v \in T_y\Sigma$. Since $\dim \operatorname{Ker}\mathcal{B} \leq 3$ by Lemma 4.3, it follows that $\mathbf{w} \mapsto \xi$ is an isomorphism.

Furthermore, the immersion $\Phi : \Sigma \to \mathcal{V}$ is *equiaffine* relative to D, in the sense that, for some (or any) fixed $\Omega \in (\mathcal{V}^*)^{\wedge 3} \smallsetminus \{0\}$, the formula $\alpha_y(u, v) = \Omega(\Phi(y), d\Phi_y u, d\Phi_y v)$, for $y \in \Sigma$ and $v \in T_y\Sigma$, defines a D-parallel area form α on Σ. This is clear, since $\alpha_{jk} = \Omega(\mathbf{r}, \mathbf{r}_j, \mathbf{r}_k)$, and so (11) gives $\alpha_{jk,l} = \partial_l\alpha_{jk} - \Gamma_{lj}^s\alpha_{sk} - \Gamma_{lk}^s\alpha_{js} = 0$. (Note that Ω is constant, and $\Omega(\mathbf{r}_l, \mathbf{r}_j, \mathbf{r}_k) = 0$ as $\mathbf{r}_l, \mathbf{r}_j, \mathbf{r}_k$ are tangent to Σ.)

REMARK 4.6. Let us fix Ω, define α, and identify $\mathcal{V}$ with $\operatorname{Ker}\mathcal{B}$ as above. Then $F = \Phi$, for the immersion F of Lemma 4.3. This is clear from the relations $\alpha_{jk} = \Omega(\mathbf{r}, \mathbf{r}_j, \mathbf{r}_k) = \xi_{k,j}$, cf. Remark 1.1.

REMARK 4.7. If an immersion F of a closed surface Σ in a real 3-space $\mathcal{V}$ is transverse to all lines through 0, then Σ is diffeomorphic to S^2 and F is an embedding. Indeed, let $S = \{\eta \in \mathcal{V}; |\eta| = 1\}$ for a fixed Euclidean norm $|\ |$ in $\mathcal{V}$. The locally diffeomorphic mapping $\Sigma \ni y \mapsto F(y)/|F(y)| \in S$ must be a covering, and hence a diffeomorphism.

5. Local classification. Since every immersion is, locally, an embedding, the following theorem provides a complete local classification of projectively flat torsionfree connections on surfaces with symmetric Ricci tensors, which is the 2-dimensional case of a more general result of Kurita [14].

THEOREM 5.1. *There exists a natural bijective correspondence between the equivalence classes of*

(a) *pairs* (Σ, D) *formed by a simply connected surface* Σ *and a projectively flat torsionfree connection* D *on* Σ *such that the Ricci tensor of* D *is symmetric and the immersion* $F : \Sigma \to \operatorname{Ker}\mathcal{B}$ *defined in Lemma* 4.3 *is an embedding*,

and the equivalence classes of

(b) *simply connected surfaces* S *embedded in a fixed* 3*-dimensional real vector space* $\mathcal{V}$ *and transverse to all lines in* $\mathcal{V}$ *containing* 0.

The equivalence relation in question consists in being congruent under a specific class of transformations: *affine diffeomorphisms of surfaces with connections for* (a), *linear isomorphisms* $\mathcal{V} \to \mathcal{V}$ *for* (b).

Explicitly, the bijective correspodence assigns to the equivalence class of a pair (Σ, D) *the equivalence class of the surface* $S = H(F(\Sigma)) \subset \mathcal{V}$, *where* H *is any linear isomorphism* $\operatorname{Ker}\mathcal{B} \to \mathcal{V}$. *The inverse assignment sends a surface* $S \subset \mathcal{V}$ *to* (Σ, D), *where* $\Sigma = S$ *and* D *is the centroaffine connection of* S, *described in Section* 3.

PROOF. That, for a pair (Σ, D) as in (a), $S = H(F(\Sigma))$ has the properties named in (b), is obvious from Lemma 4.3. Conversely, for S and $\mathcal{V}$ as in (b), the conditions listed in (a) are satisfied by the centroaffine connection D on $\Sigma = S$. (See Remarks 3.4 and 4.6, for the inclusion mapping $\Phi : \Sigma \to \mathcal{V}$.) Both assignments are well defined on equivalence classes (cf. Remark 4.4). It now remains to be shown that the two mappings between sets of equivalence classes are each other's inverses.

First, if $S, \mathcal{V}$ are as in (b), then S coincides, by Remark 4.6, with the image $F(\Sigma)$ of the immersion F in Lemma 4.3 for the centroaffine connection D on $\Sigma = S$.

Conversely, for (Σ, D) as in (a), the diffeomorphism $F : \Sigma \to F(\Sigma)$ sends D onto the centroaffine connection on the surface $F(\Sigma)$. Indeed, let us suppose that a vector field $t \mapsto w = w(t) \in T_{y(t)}\Sigma$ is tangent to Σ along a curve $t \mapsto y = y(t)$ in Σ, while the symbols $\dot{y} = \dot{y}(t)$ and $\eta = \eta(t)$ denote dy/dt and $dF_y\dot{y} = \dot{\xi}$ (i.e., $d\xi/dt$), and $\zeta = \zeta(t)$ stands for $dF_y w$. Differentiating the equality $\zeta_y = -\alpha_y(w, \cdot)$ covariantly along the curve, and noting that $(\mathrm{D}\zeta)_y = 0$ (see Lemma 4.3(iv)), we obtain $\dot{\zeta}_y = -\alpha_y(\mathrm{D}_{\dot{y}}w, \cdot)$. Thus, again by Lemma 4.3(iv), $\dot{\zeta} = d\zeta/dt$ differs from the dF_y-image of $\mathrm{D}_{\dot{y}}w$, at $y = y(t)$, by an element of $\mathrm{Ker}\,\mathcal{B}$ which is a 1-form vanishing at y, and hence is equal to a scalar times the normal vector $F(y(t))$. Thus, dF_y sends $\mathrm{D}_{\dot{y}}w$ to the tangent component of $\dot{\zeta}_y$. □

6. Special classes of connections. In this section we use Theorem 5.1 to classify projectively flat torsionfree connections D on surfaces Σ, with symmetric Ricci tensors ρ^{D}, which satisfy further restrictive conditions. The conditions in question are local symmetry ($\mathrm{D}R^{\mathrm{D}} = 0$, which is equivalent, by (5), to $\mathrm{D}\rho^{\mathrm{D}} = 0$) and (Ricci-)recurrence. Both results are well-known, cf. [17, 15].

THEOREM 6.1. *Let* D *be a projectively flat torsionfree connection on a simply connected surface* Σ, *for which the Ricci tensor* ρ^{D} *is symmetric and the immersion* F *defined in Lemma* 4.3 *is an embedding of* Σ *in the* 3*-dimensional vector space* $\mathrm{Ker}\,\mathcal{B}$.

(i) *Suppose, in addition, that* ρ^{D} *is* D*-parallel. Then there exists a symmetric bilinear form* $\langle\,,\rangle$ *in* $\mathrm{Ker}\,\mathcal{B}$ *such that* $\langle F(y), F(y)\rangle = 1$ *for every* $y \in \Sigma$ *and* $\langle\,,\rangle$ *has the algebraic type of the direct sum of a positive-definite form in dimension* 1 *and* ρ^{D}, *at any point of* Σ. *Thus, the image* $F(\Sigma)$ *is a relatively open subset of an algebraic surface* S *in* $\mathrm{Ker}\,\mathcal{B}$, *and* S *itself may be described as follows.*

(a) *If* $\rho^{\mathrm{D}} = 0$, *i.e.,* D *is flat,* S *is a plane not containing* 0.

(b) *If* ρ^{D} *is* D*-parallel, of rank* 1, *and positive semidefinite,* S *is an elliptic cylinder and its center axis contains* 0.

(c) *If* ρ^{D} *is* D*-parallel, of rank* 1, *and negative semidefinite,* S *is a hyperbolic cylinder whose center axis contains* 0.

(d) *If* ρ^{D} *is* D*-parallel, nondegenerate, and positive definite or negative definite or, respectively, indefinite,* S *is an ellipsoid or a two-sheeted hyperboloid or, respectively, a one-sheeted hyperboloid, centered at* 0.

(ii) *Conversely, if* $F(\Sigma)$ *is contained in an algebraic surface* S *in* $\mathrm{Ker}\,\mathcal{B}$ *with the properties listed in* (a), (b), (c) *or* (d), *then* ρ^{D} *is* D*-parallel and has the algebraic type named in* (a) *through* (d).

PROOF. Using a D-parallel area form α on Σ (cf. (a)–(d) in Section 1), we define $\langle\,,\rangle$ to be the symmetric bilinear mapping assigning to $\xi, \eta \in \mathrm{Ker}\,\mathcal{B}$ the function $\langle\xi, \eta\rangle$ on Σ given by $\langle\xi, \eta\rangle = \phi\psi + \rho^{\mathrm{D}}(v, w)$, with the functions ϕ, ψ and vector fields v, w such that $\mathrm{D}\xi = \phi\alpha$, $\mathrm{D}\eta = \psi\alpha$, $\xi = \alpha(v, \cdot)$ and $\eta = \alpha(w, \cdot)$.

If $\mathrm{D}\rho^{\mathrm{D}} = 0$, then, for any pair $\xi, \eta \in \mathrm{Ker}\,\mathcal{B}$, the function $\langle\xi, \eta\rangle$ is constant. To see this, let us fix an arbitrary vector field u on Σ, and assume that neither ξ nor η is identically zero. For v, w, ϕ, ψ determined by ξ and η as above, $\alpha(\mathrm{D}_u v, \cdot) = \mathrm{D}_u\xi = \phi\alpha(u, \cdot)$, so that $\mathrm{D}_u v = \phi u$, and, similarly, $\mathrm{D}_u w = \psi u$. Also, by (14) and (8-i), $(d_u\phi)\alpha = \mathrm{D}_u\mathrm{D}\xi = \xi \wedge \rho^{\mathrm{D}}(u, \cdot)$, where d_u is the directional derivative. Hence $(d_u\phi)\xi = (d_u\phi)\alpha(v, \cdot) = -\rho^{\mathrm{D}}(u, v)\xi$, as $\xi(v) = \alpha(v, v) = 0$. Noting that, by Remark 4.2, $\xi \neq 0$ on a dense open subset of Σ, we now obtain $d_u\phi = -\rho^{\mathrm{D}}(u, v)$, and, similarly, $d_u\psi = -\rho^{\mathrm{D}}(u, w)$. Combining our formulae for $\mathrm{D}_u v$, $\mathrm{D}_u w$, $d_u\phi$ and $d_u\psi$, we see that $d_u\langle\xi, \eta\rangle = 0$, as required.

Assertions (i) and (a)–(d) are now immediate. Conversely, under the assumption of (ii), let $\langle\,,\rangle$ be a symmetric bilinear form on $\operatorname{Ker}\mathcal{B}$ such that $\langle\xi, \xi\rangle = 1$ for every $\xi \in F(\Sigma)$. In the notation of (11) we thus have $\langle\mathbf{r}, \mathbf{r}\rangle = 1$, and partial differentiation gives $\langle\mathbf{r}, \mathbf{r}_j\rangle = 0$. Applying $\langle\mathbf{r}, \cdot\rangle$ to (11), we now get $b_{jk} = \langle\mathbf{r}, \mathbf{r}_{jk}\rangle$, that is, $b_{jk} = -\langle\mathbf{r}_j, \mathbf{r}_k\rangle$. Hence, again by (11), $b_{jk,l} = \partial_l b_{jk} - \Gamma_{lj}^s b_{sk} - \Gamma_{lk}^s b_{js}$ vanishes identically. As $b = -\rho^{\mathrm{D}}$ (see Remark 3.4(c)), this completes the proof. □

A tensor field B on a manifold endowed with a fixed connection D is called *recurrent* if, for every tangent vector field v, the tensors B and $\mathrm{D}_v B$ are linearly dependent at every point. Equivalently, in the open set U on which $B \neq 0$, one has $\mathrm{D}B = \zeta \otimes B$ for some 1-form ζ. The connection D itself is said to be *recurrent*, or *Ricci-recurrent*, if its curvature tensor (or, respectively, Ricci tensor) is recurrent. When this is the case for the Levi-Civita connection of a pseudo-Riemannian manifold (M, g), one refers to (M, g) as a *recurrent* or, respectively, *Ricci-recurrent* manifold. Finally, (M, g) is called *conformally recurrent* if its Weyl conformal curvature tensor is recurrent.

For a torsionfree connection with a symmetric Ricci tensor on a surface, being recurrent means, by (5), the same as being Ricci-recurrent. Such connections are called here 'Ricci-recurrent' rather than 'recurrent' (which will be convenient later, in Part III).

THEOREM 6.2. *Suppose that* D *is a projectively flat torsionfree connection on a simply connected surface* Σ *and the Ricci tensor* ρ^{D} *is symmetric, while* $\rho^{\mathrm{D}} \neq 0$ *and* $\mathrm{D}\rho^{\mathrm{D}} \neq 0$ *everywhere in* Σ. *In addition, let the immersion* $F : \Sigma \to \operatorname{Ker}\mathcal{B}$ *defined in Lemma* 4.3 *be an embedding. Then the following four conditions are equivalent*:

(a) D *is Ricci-recurrent*,
(b) *there exists a nonzero* D*-parallel vector field on* Σ,
(c) Σ *admits a nonzero* D*-parallel* 1*-form*,
(d) *the image surface* $S = F(\Sigma)$ *is a cylinder, that is, a union of mutually parallel line segments in* $\operatorname{Ker}\mathcal{B}$.

PROOF. If D is Ricci-recurrent, and so $\mathrm{D}\rho^{\mathrm{D}} = \zeta \otimes \rho^{\mathrm{D}}$ for some nowhere-vanishing 1-form ζ, the Codazzi equation $d^{\mathrm{D}}\rho^{\mathrm{D}} = 0$ gives $\zeta \wedge \rho^{\mathrm{D}} = 0$. Thus, ζ spans the image $\mathcal{I}$ of the vector-bundle morphism $T\Sigma \to T^*\Sigma$ sending each vector field u to $\rho^{\mathrm{D}}(u, \cdot)$. (In fact, $\zeta \wedge \rho^{\mathrm{D}}(u, \cdot) = 0$, and so $\rho^{\mathrm{D}}(u, \cdot)$ equals a function times ζ.) Hence $\mathcal{I}$ is a line subbundle of $T^*\Sigma$ and, as ρ^{D} is recurrent, $\mathcal{I}$ must be D-parallel (invariant under D-parallel transports), which in turn implies that the 1-form ζ spanning $\mathcal{I}$ is recurrent, i.e., $\mathrm{D}\zeta = \eta \otimes \zeta$ for some 1-form η. The Ricci identity (8-ii) gives $d^{\mathrm{D}}\mathrm{D}\zeta = 0$, so that $0 = d^{\mathrm{D}}\mathrm{D}\zeta = d^{\mathrm{D}}(\eta\otimes\zeta) = (d\eta)\otimes\zeta$. Consequently, $d\eta = 0$ and, as Σ is simply connected, $\eta = df$ for some function $f : \Sigma \to \boldsymbol{R}$. Since $\mathrm{D}\zeta = df \otimes \zeta$, the 1-form $\xi = e^{-f}\zeta$ is D-parallel. Therefore, (a) implies (c).

Equivalence of (b) and (c) is clear, since a fixed D-parallel area form α on Σ leads to a correspondence $v \mapsto \xi = \alpha(v, \cdot)$ between vector fields v and 1-forms ξ. On the other hand,

(b) and (d) are equivalent as a consequence of Lemma 4.5. (Note that, if $F(\Sigma)$ is a cylinder, some nonzero vector in $\operatorname{Ker}\mathcal{B}$ is tangent to $F(\Sigma)$ at every point.)

Finally, (c) implies (a). Indeed, if $\mathrm{D}\xi = 0$ for a 1-form $\xi \neq 0$, the Ricci identity (8-ii) yields $\xi \wedge \rho^{\mathrm{D}} = 0$, and so the line subbundle $\mathcal{I}$ of $T^*\Sigma$ spanned by ξ contains the image of the vector-bundle morphism $u \mapsto \rho^{\mathrm{D}}(u, \cdot)$. (See the beginning of this proof.) As $\mathcal{I}$ is D-parallel, ρ^{D} must be recurrent. □

REMARK 6.3. If a projectively flat torsionfree connection D on a simply connected surface Σ is Ricci-recurrent, ρ^{D} is symmetric, and $F(\Sigma)$ contains no nontrivial line segment, then either ρ^{D} or $-\rho^{\mathrm{D}}$ is a positive-definite metric on Σ with nonzero constant Gaussian curvature. In fact, ρ^{D} must be parallel and definite, for if it were parallel but not definite, or non-parallel, $F(U)$ would be a union of line segments, for some nonempty open set $U \subset \Sigma$. (See Theorem 6.1(i) and (d) in Theorem 6.2.) Thus, $\pm\rho^{\mathrm{D}}$ is a Riemannian metric with the Levi-Civita connection D, and $\mathrm{D}\rho^{\mathrm{D}} = 0$.

7. Equiaffine connections. Let D be a torsionfree connection on a surface Σ. By a D-parallel *area element* we mean any nonzero D-parallel differential 2-form $\pm\alpha$ on Σ defined, at each point, only up to a sign. (The sign $\pm$ indicates its double-valuedness.) We will call D an *equiaffine* connection if a D-parallel area element exists on Σ.

By (a) through (d) in Section 1, D is *locally* equiaffine if and only if its Ricci tensor ρ^{D} is symmetric. In the case where Σ is simply connected, symmetry of ρ^{D} implies (global) equiaffinity of D on Σ, and a single-valued D-parallel area form on Σ exists as well.

REMARK 7.1. For an equiaffine torsionfree connection D on a *closed* surface Σ, every diffeomorphism $\Sigma \to \Sigma$ which is affine (that is, sends D onto itself) is also *unimodular* in the sense of preserving some (or any) D-parallel area element $\pm\alpha$ on Σ. In fact, an affine diffeomorphism obviously sends $\pm\alpha$ onto a constant multiple $\pm c\alpha$. However, since $\pm\alpha$ constitutes a smooth positive measure density, invariance of the area $\int_\Sigma \alpha$ under diffeomorphisms implies that $c = \pm 1$.

Given a surface Σ with a projectively flat torsionfree connection D such that ρ^{D} is symmetric, let $\hat{\mathrm{D}}$ be the pullback of D to the universal covering surface $\hat{\Sigma}$ of Σ. While $\hat{\mathrm{D}}$ is always equiaffine, it is clear that D is equiaffine on Σ if and only if the affine diffeomorphisms $\hat{\Sigma} \to \hat{\Sigma}$ forming the deck transformation group $\pi_1\Sigma$ are all unimodular. On the other hand, every deck transformation in $\pi_1\Sigma$ gives rise to the corresponding push-forward linear isomorphism $\operatorname{Ker}\hat{\mathcal{B}} \to \operatorname{Ker}\hat{\mathcal{B}}$ of the 3-dimensional space $\operatorname{Ker}\hat{\mathcal{B}}$ defined as in (13) for $\hat{\mathrm{D}}$ and $\hat{\Sigma}$ rather than D and Σ. (Cf. Lemma 4.3(i).) In other words, $\pi_1\Sigma$ has a natural linear representation in $\operatorname{Ker}\hat{\mathcal{B}}$, and the immersion $\hat{F} : \hat{\Sigma} \to \hat{\mathcal{B}}$ defined as in Lemma 4.3 is obviously equivariant relative to it.

REMARK 7.2. For $\Sigma, \mathrm{D}, \hat{\Sigma}$ and $\hat{\mathrm{D}}$ as in the last paragraph, D is equiaffine on Σ if and only if the representation of $\pi_1\Sigma$ in $\operatorname{Ker}\hat{\mathcal{B}}$ just described consists of operators with determinant ± 1. (In fact, according to Remark 4.6 and the paragraph preceding it, choosing a $\hat{\mathrm{D}}$-parallel area form in $\hat{\Sigma}$ amounts to fixing a volume form in $\operatorname{Ker}\hat{\mathcal{B}}$.)

Let Σ be a fixed closed surface. Any projectively flat torsionfree connection D on Σ such that ρ^{D} is symmetric can be constructed as follows, in terms of the universal covering surface $\hat{\Sigma}$ of Σ and the action on $\hat{\Sigma}$ of the deck transformation group $\pi_1\Sigma$. We begin by choosing a linear representation of $\pi_1\Sigma$ in a real 3-space $\mathcal{V}$. In view of Theorem 5.1, we now only need to prescribe an equivariant immersion $\hat{\Phi} : \hat{\Sigma} \to \mathcal{V}$, transverse to all lines through 0. To this end, we choose a fundamental domain for the action of $\pi_1\Sigma$ on $\hat{\Sigma}$, in the form of a curvilinear polygon $Q \subset \hat{\Sigma}$ on which the action of $\pi_1\Sigma$ realizes standard identifications between some pairs of edges. (See [25, pp. 148–149].) Our $\hat{\Phi}$, chosen arbitrarily on a small neighborhood of one vertex of Q, is then propagated to neighborhoods of other vertices, with the aid of linear transformations assigned to standard generators of $\pi_1\Sigma$. Next, $\hat{\Phi}$ is extended from (smaller) neighborhoods of the vertices to narrow tubular neighborhoods of a half of the total number of edges, chosen so as to be either pairwise disjoint (Σ nonorientable), or so that each selected edge shares a vertex with exactly one other selected edge (Σ orientable). Generators of $\pi_1\Sigma$ are used, again, to propagate the immersion $\hat{\Phi}$ to neighborhoods of the remaining edges. Finally, we extend $\hat{\Phi}$ to the interior of Q, leaving it unchanged near the boundary, and from there to all of $\hat{\Sigma}$, in a manner uniquely determined by the requirement of equivariance.

Not all steps of this process are always possible, as the transversality requirement may preclude extensibility of the immersion. For instance, if Σ is a closed surface and we try to immerse in $\mathcal{V}$ not just $\hat{\Sigma}$, but also Σ itself (which amounts to choosing the trivial representation), the immersion $\hat{\Phi}$ will not exist except when Σ is diffeomorphic to the 2-sphere (Remark 4.7).

REMARK 7.3. The divergence formula $\int_\Sigma (\mathrm{div}^{\mathrm{D}} w)\alpha = 0$ for compactly supported C^1 vector fields w on a manifold Σ remains valid also when D is just a torsionfree connection on Σ with a D-parallel volume element $\pm\alpha$ (defined analogously as an area element in the case of surfaces).

In fact, using a finite partition of unity, we may reduce the question to the case where the compact support of w is contained in the domain of a coordinate system y^j. Since $\mathrm{D}\alpha = 0$, we have $\Gamma_{jk}^k = \partial_j \log|\alpha_{1\ldots r}|$, where $r = \dim\Sigma$, and so $\int_\Sigma (\mathrm{div}^{\mathrm{D}} w)\alpha$ equals the Lebesgue integral of the Euclidean divergence $\partial_j(w^j|\alpha_{1\ldots r}|)$, which vanishes, since so do the integrals of the individual terms, for each fixed j.

8. The 2-sphere and projective plane. For the 2-sphere S^2 we have the following classification result, cf. [20, p. 91].

COROLLARY 8.1. *The assignment described at the end of Theorem* 5.1 *establishes a bijective correspondence between the equivalence classes of*

(i) *projectively flat torsionfree connections* D *with symmetric Ricci tensors on the* 2-*dimensional sphere* Σ,

and those of

(ii) 2-*spheres* S *embedded in a fixed* 3-*dimensional real vector space* $\mathcal{V}$ *and transverse to all lines in* $\mathcal{V}$ *passing through* 0.

The equivalence relations are provided by: *affine diffeomorphisms between connections on* Σ *in* (i), *linear isomorphisms* $\mathcal{V} \to \mathcal{V}$ *in* (ii).

PROOF. This is immediate from Theorem 5.1 and Remark 4.7. $\square$

Connections $\mathrm{D}, \tilde{\mathrm{D}}$ on manifolds $\Sigma, \tilde{\Sigma}$ are called *projectively equivalent* if some diffeomorphism $\Sigma \to \tilde{\Sigma}$ sends the geodesics of D onto (re-parametrized) geodesics of $\tilde{\mathrm{D}}$.

On the 2-sphere Σ there exists just one projective equivalence class of projectively flat torsionfree connections D with symmetric Ricci tensors (cf. [20, Sect. 1.7]). Indeed, the diffeomorphism $\Sigma \to S$ described in Remark 4.7 for $F = \Phi$ and $\mathcal{V} = \operatorname{Ker} \mathcal{B}$ (see Lemma 4.3) sends the geodesics of D (cf. Remark 3.4(a)) onto the great circles in S.

REMARK 8.2. The assertion of Corollary 8.1 remains true also when one replaces the phrase 2-*sphere* in (i) with *real projective plane*, and adds to the end of (ii) the clause *as well as invariant under multiplication by* -1.

Indeed, let $\Psi : S^2 \to S^2$ be the fixed-point free involution corresponding to the two-fold covering $S^2 \to \boldsymbol{R}\mathrm{P}^2$, and let D be the pullback to S^2 of a given projectively flat torsionfree connection with a symmetric Ricci tensor on $\boldsymbol{R}\mathrm{P}^2$. Thus, $\Psi^*\alpha = -\alpha$ for any fixed D-parallel area form α on S^2, since Ψ is an orientation-reversing affine diffeomorphism (Remark 7.1). From Remark 4.4 we now get $F(\Psi(y)) = -F(y)$ for every $y \in S^2$ and the immersion F of Lemma 4.3, while, by Remark 4.7, F is an embedding.

9. Projectively flat 2-tori and Klein bottles. For any fixed real number $a \neq 1$, the set $S = \{(z, |z|^a); z \in \boldsymbol{C} \smallsetminus \{0\}\}$ is a surface in the real vector space $\mathcal{V} = \boldsymbol{C} \times \boldsymbol{R}$, transverse to all lines containing 0. Let D denote the centroaffine connection on S (see Section 3). The assignment $(z, |z|^a) \mapsto z$ is a diffeomorphism between S and the multiplicative group $\boldsymbol{C} \smallsetminus \{0\}$, which turns S into an Abelian Lie group. The group translations are affine diffeomorphisms $S \to S$, and, for $a = -2$ only, they are all unimodular (Section 7): under our identification $\boldsymbol{C} \smallsetminus \{0\} \approx S$, the translation by $q \in \boldsymbol{C} \smallsetminus \{0\}$ corresponds to the restriction to S of the linear automorphism $(z, t) \mapsto (qz, |q|^a t)$ of $\mathcal{V} = \boldsymbol{C} \times \boldsymbol{R}$, which has the determinant $|q|^{a+2}$ (cf. Remarks 4.4 and 4.6; note that automorphisms leaving S invariant give rise to affine diffeomorphisms $S \to S$). Similarly, for any $q \in (0, \infty)$, the restriction to S of the linear transformation $(z, t) \mapsto (q^{1/2}\bar{z}, q^{a/2}t)$ is an affine diffeomorphism, unimodular if $a = -2$.

REMARK 9.1. Unless $a = 0$, the above surface S contains no nontrivial line segment. Indeed, for a segment in $\boldsymbol{C} \smallsetminus \{0\}$ parametrized by $t \mapsto z = ct + b$, where b, c are nonzero complex numbers, $|z|^a$ cannot be a linear function of the parameter t, since by raising $|z|^a$ to the power $2/a$ one obtains the polynomial $|z|^2 = |ct + b|^2$ of degree 2 in t.

The connections on the 2-torus, discussed below, are among those described by Opozda [18, Example 2.10]. Our description is different, for reasons dictated by our applications.

EXAMPLE 9.2. Any $q \in \boldsymbol{C}$ such that $0 \neq |q| \neq 1$ obviously generates a multiplicative subgroup G of $\boldsymbol{C} \smallsetminus \{0\}$, isomorphic to $\boldsymbol{Z}$, for which the quotient Lie group $(\boldsymbol{C} \smallsetminus \{0\})/G$ is a torus. For $a \neq 1$ and S as above, this fact, combined with the isomorphic identification $\boldsymbol{C} \smallsetminus \{0\} \approx S$, gives rise to a torus group $\Sigma = S/\boldsymbol{Z}$, where the action of $\boldsymbol{Z}$ on S corresponds to G acting on $\boldsymbol{C} \smallsetminus \{0\}$. The centroaffine connection D on S, being translation-invariant, descends to a translation-invariant projectively flat torsionfree connection on Σ, also denoted by D. Again, D is equiaffine on the torus Σ if $a = -2$. If, in addition, $q \in (1, \infty)$, the affine diffeomorphism $(z, t) \mapsto (q^{1/2}\bar{z}, q^{a/2}t)$ mentioned above, whose square is the generator of the $\boldsymbol{Z}$ action, commutes with the $\boldsymbol{Z}$ action on S, and so it descends to a fixed-point-free *affine involution* $\Psi : \Sigma \to \Sigma$, while D then gives rise to a projectively flat torsionfree connection, again denoted by D, on the Klein bottle $\Sigma/\boldsymbol{Z}_2$, with the action of $\boldsymbol{Z}_2$ on Σ generated by Ψ. Finally, D is not Ricci-recurrent unless $a = 0$. Otherwise, Remarks 9.1 and 6.3 would contradict the Gauss-Bonnet theorem.

10. A differential equation. Let D be a projectively flat torsionfree connection on a surface Σ such that the Ricci tensor ρ^{D} is symmetric. We denote by $\mathcal{L}$ the second-order partial differential operator sending each twice-covariant symmetric tensor field τ on Σ to a differential 2-form $\mathcal{L}\tau$ on Σ valued in 2-forms, defined, in the notation of Remark 1.4 and Section 2, by

$$\mathcal{L}\tau = dd^{\mathrm{D}}\tau + \tau \wedge \rho^{\mathrm{D}} . \tag{15}$$

The main result of this section, Theorem 10.2(i), provides a simple topological condition sufficient for solvability of the linear equation $\mathcal{L}\tau = A$, where the unknown τ is a twice-covariant symmetric tensor on a surface Σ with a connection D satisfying specific assumptions. Theorem 10.2(ii) establishes the extent to which a solution τ is nonunique, by describing the solutions of the associated linear homogeneous equation. In local coordinates, the condition $\mathcal{L}\tau = A$ means that A_{jklm} is equal to

$$\tau_{mk,lj} - \tau_{lk,mj} - \tau_{mj,lk} + \tau_{lj,mk} + \tau_{mk}R_{lj} - \tau_{lk}R_{mj} - \tau_{mj}R_{lk} + \tau_{lj}R_{mk} . \tag{16}$$

Here A is a given four-times covariant tensor with $A_{kjlm} = -A_{jklm} = A_{jkml}$. According to Theorem 10.2(ii), when Σ is simply connected, (16) vanishes if and only if $\tau_{jk} = \xi_{k,j} + \xi_{j,k}$ for some 1-form ξ on Σ.

We begin with a lemma.

LEMMA 10.1. *Suppose that* D *is a projectively flat torsionfree connection on a surface* Σ, *the Ricci tensor* ρ^{D} *of* D *is symmetric*, $\mathcal{B}$ *and* $\mathcal{L}$ *are the operators given by* (12) *and* (15), *while* ∇ *denotes the flat connection in* $\mathcal{X} = (T^*\Sigma)^{\wedge 2} \oplus T^*\Sigma$, *defined in Lemma* 4.1.

(a) *For a fixed differential* 2*-form* A *on* Σ *valued in* 2*-forms, the existence of a twice-covariant symmetric tensor field* τ *on* Σ *with* $\mathcal{L}\tau = A$ *is equivalent to the* d^{∇}*-exactness of the* $\mathcal{X}$*-valued* 2*-form* $(A, 0)$ *on* Σ.

(b) *Any twice-covariant symmetric tensor field* τ *on* Σ *gives rise to the* $\mathcal{X}$*-valued* 1*-form* $(d^{\mathrm{D}}\tau, \tau)$, *and then*

i) *$\mathcal{L}\tau = 0$ if and only if $(d^{\mathrm{D}}\tau, \tau)$ is d^{∇}-closed,*
ii) *$(d^{\mathrm{D}}\tau, \tau)$ is d^{∇}-exact if and only if $\tau = \mathcal{B}\xi$ for some real-valued 1-form ξ.*

PROOF. By (4), $d^{\nabla}(\zeta, \lambda) = (d\zeta + \lambda \wedge \rho^{\mathrm{D}}, d^{\mathrm{D}}\lambda - \zeta)$ whenever ζ is a 1-form on Σ valued in 2-forms and λ is a twice-covariant tensor on Σ viewed as a $T^*\Sigma$-valued 1-form, cf. Remark 1.1, so that the pair (ζ, λ) is an $\mathcal{X}$-valued 1-form. (The term $-\zeta$ in $d^{\mathrm{D}}\lambda - \zeta$, treated here, in accordance with Remark 1.3, as a 2-form valued in 1-forms, arises, since, by summing ζ cyclically over its arguments, we get a real-valued differential 3-form on the surface Σ, that is, 0.) Therefore, d^{∇}-exactness of $(A, 0)$ in (a) means that $A = \mathcal{L}\lambda$ for some twice-covariant tensor λ, while, by (6), $\mathcal{L}\lambda$ remains unchanged when λ is replaced by its symmetric part, which proves (a). Next, for τ as in (b), the above formula for $d^{\nabla}(\zeta, \lambda)$, applied to $\zeta = d^{\mathrm{D}}\tau$ and $\lambda = \tau$, yields (i).

On the other hand, d^{∇}-exactness of $(d^{\mathrm{D}}\tau, \tau)$ means that $(d^{\mathrm{D}}\tau, \tau) = \nabla(\Theta, 2\xi)$ or, equivalently, $d^{\mathrm{D}}\tau = \mathrm{D}\Theta - 2\xi \wedge \rho^{\mathrm{D}}$ and $\tau = 2\mathrm{D}\xi - \Theta$, for some (real-valued) 2-form Θ and 1-form ξ. Taking the transpose of the last equality (Remark 1.1) we get $\tau^* = 2(\mathrm{D}\xi)^* + \Theta$, and so $\tau = (\tau + \tau^*)/2 = \mathrm{D}\xi + (\mathrm{D}\xi)^* = \mathcal{B}\xi$.

Conversely, if $\tau = \mathrm{D}\xi + (\mathrm{D}\xi)^*$ for a 1-form ξ, we get the d^{∇}-exactness relations $d^{\mathrm{D}}\tau = \mathrm{D}\Theta - 2\xi \wedge \rho^{\mathrm{D}}$ and $\tau = 2\mathrm{D}\xi - \Theta$ with the 2-form $\Theta = \mathrm{D}\xi - (\mathrm{D}\xi)^*$ (that is, $\Theta = d\xi$). In fact, the second relation is obvious, and the first follows from (7) and (8). □

THEOREM 10.2. *Given a projectively flat torsionfree connection* D *with a symmetric Ricci tensor on a simply connected surface Σ, let $\mathcal{B}$, $\mathcal{L}$ be as in* (12) *and* (15).

(i) *If Σ is noncompact and A is any differential 2-form on Σ valued in 2-forms, then $\mathcal{L}\tau = A$ for some twice-covariant symmetric tensor τ on Σ.*

(ii) *The twice-covariant symmetric tensors τ with $\mathcal{L}\tau = 0$ are precisely the tensors $\mathcal{B}\xi$ for all real-valued 1-forms ξ on Σ.*

(iii) *A 1-form ξ on Σ is determined by the tensor $\mathcal{B}\xi$ uniquely up to adding an element of the 3-dimensional vector space* $\operatorname{Ker}\mathcal{B}$.

PROOF. Assertion (i) is immediate from Lemma 10.1(a): $\dim\Sigma = 2$, so that the $(T^*\Sigma)^{\wedge 2}$-valued 2-form $(A, 0)$ is d^{∇}-closed, and hence d^{∇}-exact, as $H^2(\Sigma, \boldsymbol{R}) = \{0\}$ due to noncompactness of Σ. (Cf. the lines preceding Remark 1.1.) Similarly, (ii) follows from Lemma 10.1(b), since Σ is simply connected, and so d^{∇}-closedness of $(d^{\mathrm{D}}\tau, \tau)$ is equivalent to its d^{∇}-exactness, while Lemma 4.3(i) gives (iii). □

REMARK 10.3. The differential operator $\mathcal{L}$ given by (15) (and (16)), sending twice-covariant symmetric tensor fields to differential 2-forms valued in 2-forms, is well-defined on manifolds Σ of any dimension $r \geq 2$, even though our rationale for writing d instead of d^{D} applies only when $r = 2$. The principal symbol of $-\mathcal{L}/2$ equals that of the quasi-linear operator sending a metric to its four-times covariant curvature tensor (cf. (3)). In some cases, the equality extends beyond the principal symbols; see (c) in Section 16.

11. The case of closed surfaces. The construction in Section 21 requires solvability of the equation

$$\mathcal{L}\tau = \varepsilon\alpha \otimes \alpha \quad \text{with} \quad \varepsilon \in \{1, -1\}, \tag{17}$$

where the unknown is a twice-covariant symmetric tensor field τ on a surface Σ carrying an equiaffine projectively flat torsionfree connection D, while $\pm\alpha$ is a D-parallel area element (Section 7), and $\mathcal{L}$ is given by (15). The value of ε is of no consequence for solvability of (17), since $\mathcal{L}$ is linear; in other words, solving (17) means, up to a factor, finding τ such that $\mathcal{L}\tau$ is parallel and nonzero. We have the following result.

THEOREM 11.1. *On every closed surface Σ, (17) holds for some non-Ricci-recurrent, equiaffine, projectively flat torsionfree connection* D, *some twice-covariant symmetric tensor field τ, and a* D*-parallel area element $\pm\alpha$.*

We precede the proof of Theorem 11.1, given in Section 12, with three lemmas.

A fixed D-parallel area element $\pm\alpha$ on a surface Σ with an equiaffine torsionfree connection D, cf. Section 7, can be used to identify twice-covariant symmetric tensors τ on Σ (or, 2-forms A on Σ valued in 2-forms) with twice-contravariant symmetric tensors T on Σ (or, respectively, functions $\psi : \Sigma \to \boldsymbol{R}$). Specifically, such τ and T are sections of the vector bundles $[T^*\Sigma]^{\odot 2}$ and $[T\Sigma]^{\odot 2}$, while $\pm\alpha$ gives rise to an isomorphism $T\Sigma \to T^*\Sigma$ acting on vector fields v by $v \mapsto \alpha(v, \cdot)$, and hence defined at each point only up to a sign; however, the isomorphism $[T\Sigma]^{\odot 2} \to [T^*\Sigma]^{\odot 2}$ induced by it, and sending T to τ, is single-valued. In coordinates, $\tau_{jk} = \alpha_{jl}\alpha_{km}T^{lm}$. Similarly, our A are sections of a line bundle in which $\alpha \otimes \alpha$ is a global single-valued trivializing section, and so we can identify A with ψ such that $A = \psi\alpha \otimes \alpha$.

LEMMA 11.2. *Let $\pm\alpha$ be a* D*-parallel area element on a surface Σ with an equiaffine projectively flat torsionfree connection* D *having the Ricci tensor ρ^{D}. Under the above identifications, the operator $\mathcal{L}$ in* (15) *corresponds to an operator $\mathcal{F}$ sending twice-contravariant symmetric tensor fields T to functions $\Sigma \to \boldsymbol{R}$. Explicitly, the operator $\mathcal{F}$ is given by $\mathcal{F}T = \mathrm{div}^{\mathrm{D}}(\mathrm{div}^{\mathrm{D}}T) + \langle\rho^{\mathrm{D}}, T\rangle$, where $\langle\,,\rangle$ stands for the natural pairing between covariant and contravariant* 2*-tensors. In coordinates, $\mathcal{F}T = T^{jk}{}_{,jk} + T^{jk}R_{jk}$.*

PROOF. Let us fix a D-parallel area element $\pm\alpha$ on Σ, and let α^{-1} be the reciprocal bivector of α, with the components α^{jk} such that $\alpha^{jl}\alpha_{lk} = \delta^j_k$. Thus, $\alpha^{jk}\alpha_{jk} = -2$, and, whenever A_{jklm} are the components of a 2-form A valued in 2-forms, $4\psi = \alpha^{jk}\alpha^{lm}A_{jklm}$ for the function ψ such that $A = \psi\alpha \otimes \alpha$. Now, if $\mathcal{L}\tau = A$, the components A_{jklm} are given by (16). Hence, for reasons of skew-symmetry, $4\psi = \alpha^{jk}\alpha^{lm}(\tau_{mk,lj} + \tau_{mk}R_{lj})$, which equals $T^{jk}{}_{,jk} + T^{jk}R_{jk}$ as α^{-1} is D-parallel and $T^{jk} = \alpha^{jl}\alpha^{km}\tau_{ml}$. $\square$

REMARK 11.3. In the notation of Lemma 11.2, we have $\mathcal{F}T = 1$ for the non-Ricci-recurrent equiaffine translation-invariant projectively flat torsionfree connection D on the 2-torus Σ described in Example 9.2 (with $a = -2$) and a suitable translation-invariant twice-contravariant symmetric tensor field T on Σ. Indeed, it suffices to choose T so that the constant $\langle \rho^{\mathrm{D}}, T \rangle$ equals 1, as the function $\mathrm{div}^{\mathrm{D}}(\mathrm{div}^{\mathrm{D}} T)$, being translation-invariant (that is, constant), must vanish by the divergence formula (Remark 7.3).

LEMMA 11.4. *Given a projectively flat torsionfree connection* D *on a surface Σ, a* D*-parallel area element $\pm\alpha$, and a function $f : \Sigma \to (0, \infty)$, let $\tilde{\mathrm{D}}$ be the projectively flat torsionfree connection on Σ with $\tilde{\mathrm{D}} = \mathrm{D} + 2\xi \odot \mathrm{Id}$ for $\xi = -d \log f$, cf. Lemma* 3.1. *The formula $\tilde{\alpha} = f^{-3}\alpha$ then defines a $\tilde{\mathrm{D}}$-parallel area element $\pm\tilde{\alpha}$ on Σ. Furthermore, if $\mathcal{L}$ and $\mathcal{F}$ are the operators described in* (15) *and Lemma* 11.2, *while $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{F}}$ stand for their analogues corresponding to $\tilde{\mathrm{D}}$ and $\tilde{\alpha}$, then*

(a) $\tilde{\mathcal{L}}(f^{-2}\tau) = f^{-2}\mathcal{L}\tau$ *for any symmetric covariant* 2*-tensor τ on Σ,*

(b) $\tilde{\mathcal{F}}(f^4 T) = f^4 \mathcal{F} T$ *for any symmetric contravariant* 2*-tensor T on Σ.*

PROOF. The $\tilde{\mathrm{D}}$-divergence of any vector field w (or, twice-contravariant symmetric tensor field T) clearly equals $\mathrm{div}^{\mathrm{D}} w + (r + 1)\xi(w)$ (or, $\mathrm{div}^{\mathrm{D}} T + (r + 3)T\xi$) whenever D and $\tilde{\mathrm{D}}$ are torsionfree connections on a manifold Σ of dimension r and $\tilde{\mathrm{D}} = \mathrm{D} + 2\xi \odot \mathrm{Id}$. Using Lemma 3.1(i), we now easily obtain $\tilde{\mathcal{F}} T = \mathcal{F} T + (r^2 + 5r + 2)\langle T, \xi \otimes \xi \rangle + 4\langle T, \mathrm{D}\xi \rangle + (2r + 4)\xi(\mathrm{div}^{\mathrm{D}} T)$. When $r = 2$ and $\xi = -d \log f$, this yields (b). As we then obviously have $\tilde{\mathrm{D}}\tilde{\alpha} = 0$ for $\tilde{\alpha} = f^{-3}\alpha$, assertion (a) follows. □

LEMMA 11.5. *Let g a Riemannian metric of constant Gaussian curvature K on a closed surface Σ, and let $\mathcal{F}$ be the operator defined in Lemma* 11.2 *with* D *replaced by the Levi-Civita connection ∇ of g. If $K < 0$, or Σ is nonorientable and $K > 0$, then $\mathcal{F}$ is a surjective operator from the space of C^∞ twice-contravariant symmetric tensor fields on Σ onto the space of C^∞ functions $\Sigma \to \boldsymbol{R}$.*

PROOF. Let g^{-1} be the reciprocal metric, with the components g^{jk}. The operator given by $f \mapsto \mathcal{F}(fg^{-1})$ clearly equals $\Delta + 2K$, that is, sends any C^∞ function f to $\Delta f + 2Kf$, where Δ is the Laplacian of g, acting by $\Delta f = g^{jk} f_{,jk}$. Since $\Delta + 2K$ is self-adjoint and elliptic, its surjectivity will be immediate once we establish its injectivity.

Injectivity of $\Delta + 2K$ is clear when $K < 0$, as $\Delta + 2K$ then is a negative operator. Suppose now that Σ is nonorientable and $K > 0$. The two-fold covering of Σ is the round sphere S^2 of curvature K, on which $2K$ is the lowest positive eigenvalue of $-\Delta$, and the corresponding eigenspace $\mathrm{Ker}(\Delta + 2K)$ consists of restrictions of linear functionals on $\boldsymbol{R}^3$ to S^2 treated as a sphere in $\boldsymbol{R}^3$ centered at 0. Eigenfunctions thus cannot descend to the projective plane Σ, as they change sign under the antipodal involution, and injectivity of $\Delta + 2K$ on Σ follows. This completes the proof. □

Note that $\mathcal{F}$ is not surjective in the remaining cases: if $K = 0$, or Σ is orientable and K is a positive constant, the image of $\mathcal{F}$ is the L^2-orthogonal complement of the kernel of $\Delta + 2K$. Indeed, $\mathrm{Ker}(\Delta + 2K) = \mathrm{Ker}\,\mathcal{F}^*$ for the adjoint $\mathcal{F}^*$ of $\mathcal{F}$, given by $\mathcal{F}^*\psi = \nabla d\psi + K\psi g$ for any function ψ.

12. Proof of Theorem 11.1. To prove Theorem 11.1, it suffices, by Lemma 11.2, to exhibit a connection D on Σ with the stated properties and a twice-contravariant symmetric tensor field T on Σ such that $\mathcal{F}T = 1$, i.e., $\operatorname{div}^{\mathrm{D}}(\operatorname{div}^{\mathrm{D}}T) + \langle \rho^{\mathrm{D}}, T\rangle = 1$.

When Σ is diffeomorphic to the 2-torus, D and T exist according to Remark 11.3.

If our surface is diffeomorphic to the Klein bottle, we may denote it by $\Sigma/\mathbf{Z}_2$ (rather than Σ), and let Σ, D, T be as in Remark 11.3, the $\mathbf{Z}_2$ action on the torus Σ being generated by the affine involution Ψ of Example 9.2. As $\mathcal{F}T = \mathcal{F}T' = 1$, where T' is the push-forward of T under Ψ, the Ψ-invariant tensor field $(T + T')/2$ on Σ descends to the required tensor field on the Klein bottle $\Sigma/\mathbf{Z}_2$.

If Σ is not diffeomorphic to the 2-torus, or the Klein bottle, or the 2-sphere, we may choose on Σ a Riemannian metric g of constant Gaussian curvature and denote by ∇ its Levi-Civita connection. The required connection D on Σ can now be obtained from ∇ by a projective modification. Namely, we set $\mathrm{D} = \nabla + 2\xi \odot \mathrm{Id}$ (notation of Lemma 3.1), with $\xi = -\,d\log f$ for a suitable function $f : \Sigma \to (0,\infty)$. In view of Lemma 11.4(a) with $\tilde{\mathrm{D}}$, D replaced by D, ∇, the operator $\mathcal{F}$ corresponding to D is surjective since so is, by Lemma 11.5, the analogue of $\mathcal{F}$ corresponding to ∇.

We still need to verify here that, for a suitable choice of f, the connection D will not be Ricci-recurrent. Although this might be justified by very general reasons (namely, Ricci-recurrence of D would amount to imposing on f a system of partial differential equations), a direct geometric argument is also possible. Specifically, given a nonempty contractible open set $U \subset \Sigma$ for which the immersion $F : U \to \operatorname{Ker}\mathcal{B}$ of Lemma 4.3 is an embedding, a compactly supported small deformation of the image $F(\Sigma)$ yields a surface $\hat{S}$ in $\operatorname{Ker}\mathcal{B}$, again transverse to lines through 0, and such that the radial projection $\pi : F(U) \to \hat{S}$ is a diffeomorphism; π sends F-images of geodesics in U onto (re-parametrized) geodesics of the centroaffine connection $\tilde{\mathrm{D}}$ on $\hat{S}$. (See Remark 3.4(a).) Let the connection D on Σ now be the result of replacing ∇, just on U, with the pullback of $\tilde{\mathrm{D}}$ under the composite $\pi \circ F$. By Lemma 3.1, $\mathrm{D} = \nabla + 2\xi \odot \mathrm{Id}$ with $\xi = -\,d\log f$ for some function $f : \Sigma \to (0,\infty)$ equal to 1 outside a compact subset of U. Choosing $\hat{S}$ so that some nonempty open subset of $\hat{S}$ is not contained in a quadric surface, and at the same time contains no nontrivial line segment, we now ensure that D is not Ricci-recurrent. (Cf. Theorems 6.1 and 6.2.)

Finally, the existence of the required D and T on the 2-sphere is immediate from their existence on the projective plane.

PART II. NULL PARALLEL DISTRIBUTIONS

A general local-coordinate form of pseudo-Riemannian metrics with null parallel distributions was found by Walker [27]; for a coordinate-free version, see [7]. In this part we discuss a certain class of null parallel distributions without using Walker's theorem directly. However, our ultimate application (in Part III) of the results obtained here does lead to a special type of Walker coordinates; see formulae (29) in Section 22.

13. Curvature conditions. Let $\mathcal{P}$ be a null parallel distribution of dimension r on an n-dimensional pseudo-Riemannian manifold (M, g). Thus, the g-orthogonal complement $\mathcal{P}^\perp$ is a parallel distribution of dimension $n - r$. If the sign pattern of g has i_- minuses and i_+ pluses, then

$$\text{(18)} \qquad \text{(i)} \quad r \le \min(i_-, i_+)\,, \quad \text{(ii)} \quad \mathcal{P} \subset \mathcal{P}^\perp\,, \quad \text{(iii)} \quad r \le n/2\,.$$

In fact, $\mathcal{P}$ is null, which gives (18-ii) and hence $r \le n-r$, that is, (18-iii). Now (18-i) follows: in a pseudo-Euclidean space with the sign pattern as above, $i_\pm$ is the maximum dimension of a subspace on which the inner product is positive/negative semidefinite.

Every null parallel distribution $\mathcal{P}$ satisfies the curvature conditions

$$\text{(19)} \quad \text{(a)} \quad R(\mathcal{P}, \mathcal{P}^\perp, \cdot, \cdot) = 0\,, \quad \text{(b)} \quad R(\mathcal{P}, \mathcal{P}, \cdot, \cdot) = 0\,, \quad \text{(c)} \quad R(\mathcal{P}^\perp, \mathcal{P}^\perp, \mathcal{P}, \cdot) = 0\,,$$

where (19-a) states that $R(v, u, w, w') = 0$ for all vector fields v, u, w, w' such that v is a section of $\mathcal{P}$ and u is a section of $\mathcal{P}^\perp$, and similarly for (19-b), (19-c). Indeed, given such v, u, w, w', (1) implies that $R(w, w')v$ is a section of $\mathcal{P}$, and so it is orthogonal to u. This gives (19-a), while (19-a) and (18-ii) yield (19-b). Finally, (19-a) and the first Bianchi identity imply (19-c).

We will focus our discussion on the case where, in addition to (19), the conditions

$$\text{(20)} \qquad \text{(i)} \quad R(\mathcal{P}, \cdot, \mathcal{P}^\perp, \cdot) = 0\,, \quad \text{(ii)} \quad R(\mathcal{P}^\perp, \mathcal{P}^\perp, \cdot, \cdot) = 0$$

hold for a null parallel distribution $\mathcal{P}$ on a pseudo-Riemannian manifold.

Our interest in (20) arises from the fact that a 2-dimensional null parallel distribution $\mathcal{P}$ with (20) exists on every conformally symmetric pseudo-Riemannian manifold (M, g) with rank $W = 1$. (See Lemma 17.3(ii).)

REMARK 13.1. A vector field w on the total space M of a bundle is π-projectable onto the base manifold Σ, where $\pi : M \to \Sigma$ is the bundle projection, if and only if, for every vertical vector field u on M, the Lie bracket $[w, u]$ is also vertical. This is easily verified in local coordinates for M that make π appear as a Euclidean projection.

14. Projectability of the Levi-Civita connection. We will use the following assumption. The clause about the leaves of $\mathcal{P}^\perp$ follows from the other hypotheses if one replaces M by a suitable neighborhood of any given point.

(21) $\mathcal{P}$ is an r-dimensional null parallel distribution on a pseudo-Riemannian manifold (M, g) with $\dim M = n$, and $\mathcal{P}^\perp$ has contractible leaves which are the fibres of a bundle projection $\pi : M \to \Sigma$ for some r-dimensional manifold Σ.

Given a null parallel distribution $\mathcal{P}$ on a pseudo-Riemannian manifold (M, g), we say that the Levi-Civita connection ∇ of g is *$\mathcal{P}^\perp$-projectable*, cf. [12], if, for π, Σ chosen, locally, as in (21), and for any vector fields w, v in M such that w is π-projectable and v is a section of $\mathcal{P}$ parallel along $\mathcal{P}^\perp$, the section $\nabla_w v$ of $\mathcal{P}$ is parallel along $\mathcal{P}^\perp$ as well.

LEMMA 14.1. *A null parallel distribution $\mathcal{P}$ on a pseudo-Riemannian manifold (M, g) satisfies condition* (20-i) *if and only if the Levi-Civita connection ∇ of (M, g) is $\mathcal{P}^\perp$-projectable.*

PROOF. For w, v as above and any section u of $\mathcal{P}^\perp$, (1) gives $\nabla_u \nabla_w v = R(w, u)v$, as the other two terms in (1) vanish: v is parallel along $\mathcal{P}^\perp$, and so $\nabla_u v = \nabla_{[w,u]} v = 0$, where $[w, u]$ is a section of $\mathcal{P}^\perp = \operatorname{Ker} d\pi$ due to π-projectability of w and Remark 13.1. □

Let us now assume (21), and let the Levi-Civita connection ∇ be $\mathcal{P}^\perp$-projectable.

(i) For any π-projectable vector field w, if u is a section of $\mathcal{P}^\perp$, then so is $\nabla_u w$.

(ii) Sections v of $\mathcal{P}$ parallel along $\mathcal{P}^\perp$ are in a natural bijective correspondence Λ with sections ξ of $T^*\Sigma$. It assigns to v the 1-form $\xi = \Lambda(v)$ on Σ such that $\xi((d\pi)w) = g(v, w)$ for any π-projectable vector field w on M. (Here and below $(d\pi)w$ denotes the vector field on Σ onto which w projects.)

(iii) For any $\phi : \Sigma \to \boldsymbol{R}$ treated as a function on M constant along $\mathcal{P}^\perp$, the g-gradient $v = \nabla\phi$ is a section of $\mathcal{P}$ parallel along $\mathcal{P}^\perp$ and $\Lambda(v) = \xi$ for the section $\xi = d\phi$ of $T^*\Sigma$.

(iv) ∇ gives rise to a torsionfree connection D in the tangent bundle $T\Sigma$, which we call the $\mathcal{P}^\perp$-*projected connection* for g and $\mathcal{P}$, and which is characterized by $\mathrm{D}_{(d\pi)w}\xi = \Lambda(\nabla_w v)$ whenever $\xi = \Lambda(v)$, for π-projectable vector fields w on M and sections v of $\mathcal{P}$ parallel along $\mathcal{P}^\perp$.

(v) $\Lambda(R(w, w')v) = -\xi R^{\mathrm{D}}((d\pi)w, (d\pi)w')$ if $\xi = \Lambda(v)$, with ξR^{D} as in (7), for sections v of $\mathcal{P}$ parallel along $\mathcal{P}^\perp$, the curvature tensors R of g and R^{D} of the $\mathcal{P}^\perp$-projected connection D on Σ, and π-projectable vector fields w, w' on M.

(vi) $\mathcal{P}$ is $\tilde{g}$-parallel for the conformally related metric $\tilde{g} = f^{-2}g$ on M, whenever $f : \Sigma \to (0, \infty)$, so that f may be treated as a function on M constant along $\mathcal{P}^\perp$. The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ then is $\mathcal{P}^\perp$-projectable, and its $\mathcal{P}^\perp$-projected connection is $\tilde{\mathrm{D}} = \mathrm{D} + 2d\phi \odot \mathrm{Id}$ with $\phi = -\log f$ (notation of Lemma 3.1).

Indeed, (i) follows as $\nabla_u w = [u, w] + \nabla_w u$, while $[u, w]$ (or $\nabla_w u$) is a section of $\mathcal{P}^\perp$ by Remark 13.1 (or, respectively, since $\mathcal{P}^\perp$ is parallel). Next, ξ in (ii) is well defined: two choices of w having the same $(d\pi)w$ differ by a section of $\mathcal{P}^\perp$ (orthogonal to v). Also, the function $g(v, w) : M \to \boldsymbol{R}$ descends to Σ. Namely, $d_u[g(v, w)] = 0$ for any section u of $\mathcal{P}^\perp$, as $\nabla_u v = 0$ and, by (i), $g(v, \nabla_u w) = 0$. Injectivity of Λ is obvious; its surjectivity easily follows, since, by (19-c) and (1), the connections induced by ∇ in the restrictions of $\mathcal{P}$ to the leaves of $\mathcal{P}^\perp$ are all flat. This proves (ii). Next, (iii) is obvious from (ii), as $v = \nabla\phi$, being orthogonal to $\mathcal{P}^\perp$, is a section of $\mathcal{P}$, and hence so is $\nabla_w v$ for any vector field w, which gives $g(\nabla_u v, w) = g(\nabla_w v, u) = 0$ for sections u of $\mathcal{P}^\perp$. In (iv), D is clearly well-defined, and it is torsionfree: the second covariant derivative $\mathrm{D}d\phi$ of any function $\phi : \Sigma \to \boldsymbol{R}$ is symmetric, as (iii) yields $(\mathrm{D}d\phi)((d\pi)w, (d\pi)w') = g(\nabla_w v, w') = (\nabla d\phi)(w, w')$ for $v = \nabla\phi$ and π-projectable vector fields w, w' on M. Finally, (v) follows from (iv), (1), (4) and (7-i), while (a) in Section 2, (iii) and (i) imply (vi).

15. Pullbacks of covariant tensors. We will say that a k-times covariant tensor field λ on a manifold M *annihilates* a distribution $\mathcal{P}$ on M if $k \geq 1$ and $\lambda(v_1, \ldots, v_k) = 0$ whenever $v_1, \ldots, v_k$ are vector fields on M and v_j is a section of $\mathcal{P}$ for some $j \in \{1, \ldots, k\}$.

LEMMA 15.1. *Suppose that we have* (21) *and* (20-i), D *is the* $\mathcal{P}^\perp$*-projected connection on* Σ, *cf. Section* 14, *and* τ *is a* k*-times covariant tensor field on* Σ.

(a) *If* $k \geq 1$, *the pullback* $\pi^*\tau$ *of* τ *to* M *annihilates* $\mathcal{P}^\perp$.

(b) *For* $k \geq 0$, *we have* $\nabla(\pi^*\tau) = \pi^*(\mathrm{D}\tau)$, *with both covariant derivatives treated as* $(k+1)$*-times covariant tensor fields.*

Also, if λ *is a* k*-times covariant tensor field on* M *and* $k \geq 1$, *while* λ *and* $\nabla\lambda$ *both annihilate* $\mathcal{P}^\perp$, *then* $\lambda = \pi^*\tau$ *for a unique* k*-times covariant tensor field* τ *on* Σ.

PROOF. Since $\mathcal{P}^\perp = \operatorname{Ker} d\pi$, (a) follows. To obtain (b), we need only to consider the cases $k = 0$ and $k = 1$, as the class of covariant tensor fields τ on Σ with $\nabla(\pi^*\tau) = \pi^*(\mathrm{D}\tau)$ is closed under both addition (with any fixed k) and tensor multiplication. For $k = 0$, (b) is obvious; on functions, $\nabla = \mathrm{D} = d$. If $k = 1$, so that τ is a section of $T^*\Sigma$, (ii) in Section 14 gives $\pi^*\tau = g(v, \cdot)$ for a section v of $\mathcal{P}$ parallel along $\mathcal{P}^\perp$, and then $\nabla(\pi^*\tau) = \pi^*(\mathrm{D}\tau)$ by (iv) in Section 14. This proves (b). In the final clause, we set $\tau((d\pi)w_1, \ldots, (d\pi)w_k) = \lambda(w_1, \ldots, w_k)$ for π-projectable vector fields w_j on M. That $d_u[\lambda(w_1, \ldots, w_k)] = 0$ for any section u of $\mathcal{P}^\perp$ now follows, since $\nabla\lambda$ annihilates $\mathcal{P}^\perp$ (and so $\nabla_u\lambda = 0$), while each $\nabla_u w_j$ is a section of $\mathcal{P}^\perp$ by (i) in Section 14. □

16. Riemann extensions. Let $M = T^*\Sigma$ be the total space of the cotangent bundle of a manifold Σ. Our convention is that, as a set, $T^*\Sigma = \{(y, \eta);\ y \in \Sigma \text{ and } \eta \in T_y\Sigma\}$. Any fixed connection D on Σ gives rise to the pseudo-Riemannian metric h^{D} on $T^*\Sigma$ characterized by the condition

$$h^{\mathrm{D}}_x(w, w) = 2w^{\mathrm{vrt}}(d\pi_x w^{\mathrm{hrz}}) \quad \text{for } x = (y, \eta) \in M = T^*\Sigma \text{ and } w \in T_xM\,, \tag{22}$$

w^{vrt}, w^{hrz} being the vertical and D-horizontal components of w, with $w^{\mathrm{vrt}} \in T^*_y\Sigma$ due to the identification between the vertical space at x and the fibre $T^*_y\Sigma$. In other words, all vertical and all D-horizontal vectors are h^{D}-null, while $h^{\mathrm{D}}_x(\zeta, w) = \zeta(d\pi_x w)$ for $x = (y, \eta) \in T^*\Sigma = M$, a vertical vector $\zeta \in \operatorname{Ker} d\pi_x = T^*_y\Sigma$, and any vector $w \in T_xM$. Here $\pi : T^*\Sigma \to \Sigma$ is the bundle projection.

Our h^{D} is one of Patterson and Walker's *Riemann extension metrics* [19]. In the coordinates y^j, p_j for $T^*\Sigma$ obtained from an arbitrary local coordinate system y^j in Σ,

$$h^{\mathrm{D}} = 2dp_j dy^j - 2p_j \Gamma^j_{kl} dy^k dy^l\,, \tag{23}$$

where the products of differentials stand for symmetric products, and Γ^k_{jl} are the components of D in the coordinates y^j. Namely, due to the coordinate expression for the covariant derivative, the horizontal lift of a vector field w tangent to Σ has the components $\dot{y}^j = w^j$ and $\dot{p}_l = \Gamma^j_{kl} w^k p_j$ in the coordinates y^j, p_j. Thus, (23) implies (22).

We will not use the easily-verified facts that the vertical distribution $\mathcal{P}$ on $T^*\Sigma$ is h^{D}-parallel, its Levi-Civita connection is $\mathcal{P}^\perp$-projectable in the sense of Section 14, and the $\mathcal{P}^\perp$-projected connection on Σ coincides with the original D.

Another interesting example [27] arises when Σ is a manifold of dimension $r \ge 2$ with a global coordinate system y^j and with a fixed twice-covariant symmetric tensor field λ, while $\tilde{\mathrm{D}}$ is the flat torsionfree connection on Σ with component functions in the coordinates y^j all equal to zero. On $N = T^*\Sigma$ we have the pseudo-Riemannian metric

$$\tilde{h} = 2dq_j dy^j - 2\lambda_{kl} dy^k dy^l \quad \text{with } \lambda_{kl} \text{ not depending on the coordinates } q_j\,, \tag{24}$$

in the standard coordinate system for $N = T^*\Sigma$, this time denoted by y^j, q_j. Then

(a) the vertical subbundle $\mathcal{P}$ of TN is a null parallel distribution on the pseudo-Riemannian manifold $(N, \tilde{h})$, and it is spanned by $\tilde{h}$-parallel vector fields,

(b) the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{h}$ is $\mathcal{P}^\perp$-projectable (cf. Section 14), and the corresponding $\mathcal{P}^\perp$-projected connection is $\tilde{\mathrm{D}}$, and

(c) $\tilde{h}$ is Ricci-flat, while its four-times covariant curvature and Weyl tensors are given by $\tilde{R} = \tilde{W} = \pi^*(\tilde{\mathcal{L}}\lambda)$, with $\tilde{\mathcal{L}}$ as in Remark 10.3 (for $\tilde{\mathrm{D}}$ rather than D).

(See also Lemma 20.3.) As usual, $\pi : N \to \Sigma$ is the bundle projection.

Indeed, with the index ranges $j, k, l, m \in \{1, \dots, r\}$ and $\mu, \nu \in \{r+1, \dots, 2r\}$, let $c_{\mu j}$ be a nonsingular $r \times r$ matrix of constants. In the new coordinates y^j, y^μ for N defined by $q_j = c_{\mu j} y^\mu$, the components of $\tilde{h}$ are $\tilde{h}_{jk} = -2\lambda_{jk}$, $\tilde{h}_{\mu\nu} = 0$ and $\tilde{h}_{\mu j} = \tilde{h}_{j\mu} = c_{\mu j}$, so that all three-times "covariant" Christoffel symbols of $\tilde{h}$ vanish except $\tilde{\Gamma}_{jkl} = \lambda_{jk,l} - \lambda_{kl,j} - \lambda_{jl,k}$, the comma denoting partial differentiation. This gives (a) and (b); the y^μ coordinate vector fields are parallel and span $\operatorname{Ker} d\pi$. Also (cf. (3)), $\tilde{R}_{jklm} = \tilde{\Gamma}_{jlm,k} - \tilde{\Gamma}_{klm,j} = \lambda_{mk,lj} - \lambda_{lk,mj} - \lambda_{mj,lk} + \lambda_{lj,mk}$ are the only (possibly) nonzero curvature components, which implies (c), Ricci-flatness of $\tilde{h}$ now being clear, since $\tilde{h}^{jk} = 0$.

PART III. CONFORMALLY SYMMETRIC MANIFOLDS

This part deals with the central topic of the present paper. In Sections 17 through 20 we provide successive steps leading to a proof, in Section 22, of our main classification result.

17. Basic properties. Given a pseudo-Riemannian manifold (M, g) of dimension $n \ge 4$ which is conformally symmetric (that is, $\nabla W = 0$), let $\operatorname{rank} W$ be the rank of its Weyl tensor acting on exterior 2-forms at each point, and let $\mathcal{P}$ be the parallel distribution on M spanned by all vector fields of the form $W(u, v)v'$ for arbitrary vector fields u, v, v' on M.

LEMMA 17.1. *For any conformally symmetric pseudo-Riemannian manifold (M, g), the following three conditions are equivalent*:

(i) $\operatorname{rank} W = 1$,

(ii) *the parallel distribution $\mathcal{P}$ introduced above is two-dimensional and null*,

(iii) $W = \varepsilon\, \omega \otimes \omega$ *for some $\varepsilon = \pm 1$ and some parallel differential 2-form $\omega \ne 0$ on M, defined, at each point, only up to a sign, and having rank 2 at every point.*

Each of conditions (i) *through* (iii) *holds if* (M, g) *is essentially conformally symmetric*, *but not Ricci-recurrent*, *as defined in the Introduction.*

PROOF. Since W acting on 2-forms is self-adjoint, (i) implies (iii): $\operatorname{rank}\omega = 2$ as $\omega \wedge \omega = 0$ by the first Bianchi identity for $W = \varepsilon\,\omega \otimes \omega$. If (iii) holds, $\mathcal{P}$ is the image of the vector-bundle morphism acting by $u \mapsto \omega u$ (see (10) for $\tau = \omega$), and so $\dim \mathcal{P} = 2$. As contractions of W vanish, all vectors in $\mathcal{P}$ are null. Thus, (iii) yields (ii). Finally, if we assume (ii) and choose, locally, a differential 2-form ω' such that the image of the morphism $u \mapsto \omega' u$ is $\mathcal{P}$, then ω' will span the image of W acting on 2-forms, proving (i). The final clause is a known result [6, Theorem 9(ii)]. □

LEMMA 17.2. *In any conformally symmetric pseudo-Riemannian manifold* (M, g) *of dimension* $n \geq 4$ *such that* $\operatorname{rank} W = 1$,

(a) *the scalar curvature* s *is identically zero*,

(b) *the Ricci tensor* ρ *satisfies the Codazzi equation* $d^{\nabla}\rho = 0$, *cf. Remark* 1.2,

(c) $R = W + (n-2)^{-1} g \wedge \rho$, *with notation as at the end of Section* 2,

(d) *for every vector field* u, *the image* ρu, *defined by* (10), *is a section of the null parallel distribution* $\mathcal{P}$ *appearing in Lemma* 17.1(ii).

PROOF. For ω as in Lemma 17.1(iii), $R_{jkm}{}^{s}\omega_{sl} = R_{jkl}{}^{s}\omega_{sm}$ by (9) with $\lambda = \omega$. Thus, $R_k^s\omega_{sl} = R_{jkls}\omega^{sj}$. Summing cyclically over j, k, l, we get $R_{jkls}\omega^{sj} = 0$ (as $\mathcal{P}$ is the image of ω, and so $R_{klsj}\omega^{sj} = 0$ by (19-b)). Hence $R_k^s\omega_{sl} = 0$. Also, $W = \varepsilon\,\omega \otimes \omega$, so that $W_{jkl}{}^{s}\omega_{sm} = \omega_{jk} W_{l}{}^{s}{}_{sm} = 0$. As $W = R - (n-2)^{-1} g \wedge \sigma$ (see the end of Section 2), we thus have $0 = (n-1)(n-2) W_{klsj}\omega^{sj} = 2\mathrm{s}\omega_{kl}$, and (a) follows. Next, since $W_{jkl}{}^{s}\omega_{sm}$, $R_k^s\omega_{sl}$ and s all vanish, symmetry of $R_{jkl}{}^{s}\omega_{sm}$ in l, m, established above, implies analogous symmetry of $R_{jl}\omega_{km} - R_{kl}\omega_{jm}$. Choosing a basis of a given tangent space in which $\omega_{jl} = 0$ unless $\{j, l\} = \{1, 2\}$, then setting $k = 1$, $m = 2$, and using the latter symmetry for $j, l > 2$, or $j = 2$ and $l > 2$, we obtain $R_{jl} = 0$ unless $j, l \in \{1, 2\}$. As $\mathcal{P}$ is the image of ω, this yields (d). Finally, $\operatorname{div}^{\nabla} W = 0$ (i.e., $W_{jkl}{}^{s}{}_{,s} = 0$), since $\nabla W = 0$. Hence, by the second Bianchi identity, the Codazzi equation holds for the Schouten tensor σ (see the end of Section 2). As (a) gives $\sigma = \rho$, (b) and (c) follow. □

For the case $\nabla R \neq 0$ in Lemma 17.2, see also [5, Theorem 7] and [6, Theorem 7].

LEMMA 17.3. *For any conformally symmetric pseudo-Riemannian manifold* (M, g) *with* $\operatorname{rank} W = 1$, *and for* $\omega, \mathcal{P}$ *as in Lemma* 17.1, *let* ρ *denote the Ricci tensor. Then*

(i) $\omega, \rho, \nabla\rho$ *and* W *all annihilate* $\mathcal{P}^{\perp}$, *in the sense of Section* 15,

(ii) $\mathcal{P}$ *satisfies the curvature conditions* (20).

PROOF. To obtain (i), we fix a vector field v. Since $\nabla\rho$ is totally symmetric by Lemma 17.2(b), we only need to verify that the images of the vector-bundle morphisms $TM \to TM$, obtained from ω, ρ and $\nabla_v\rho$ by index raising (cf. (10)), are all contained in $\mathcal{P}$. (The case of W in (i) is immediate from that of ω, since $W = \varepsilon\,\omega \otimes \omega$.) For ω, the last claim is obvious: $\mathcal{P}$ is the image of ω. For ρ, it follows from Lemma 17.2(b). For $\nabla_v\rho$, it is immediate from the assertion about ρ, since $\mathcal{P}$ is parallel. This proves (i). Finally, (i) applied to W implies (ii), by Lemma 17.2(c), since $g(\mathcal{P}, \mathcal{P}^{\perp}) = 0$, while ρ annihilates $\mathcal{P}^{\perp}$ and $\mathcal{P}$ by (i) and (18-ii). □

THEOREM 17.4. *Let (M, g) be a conformally symmetric pseudo-Riemannian manifold of dimension $n \geq 4$ such that* rank $W = 1$, *and let $\mathcal{P}$ be the* 2*-dimensional null parallel distribution on M, described above. We may assume, locally, that the leaves of $\mathcal{P}^{\perp}$ are the fibres of a bundle projection $\pi : M \to \Sigma$ for some surface Σ, and so, by Lemma* 17.3(ii), *Σ carries the $\mathcal{P}^{\perp}$-projected torsionfree connection* D *defined in Section* 14. *Then*

(i) *the Ricci tensor ρ^{D} of* D *is symmetric*,

(ii) *g has the Ricci tensor $\rho = (n-2)\pi^*\rho^{\mathrm{D}}$*,

(iii) *$d^{\mathrm{D}}\rho^{\mathrm{D}} = 0$, that is,* D *is projectively flat, cf. Theorem* 3.3,

(iv) *$\mathrm{D}\rho^{\mathrm{D}} = 0$ at $\pi(x) \in \Sigma$ if and only if $\nabla R = 0$ at $x \in M$*,

(v) *ρ^{D} is recurrent, in the sense of Section* 6, *if and only if so is ρ*,

(vi) *the* 2*-form ω of Lemma* 17.1(iii) *equals $\pi^*\alpha$ for a* D*-parallel area form α defined, locally in Σ, up to a sign.*

PROOF. By Lemma 17.3(i) with $\nabla\omega = 0$, the final clause of Lemma 15.1 applies to both $\lambda = \omega$ and $\lambda = \rho$. We thus get (vi) (where $\mathrm{D}\alpha = 0$ in view of Lemma 15.1(b) for $\tau = \alpha$, as $\nabla\omega = 0$), and $\rho = (n-2)\pi^*\tau$ for some symmetric covariant 2-tensor τ on Σ.

Lemma 17.2(c) gives $(n-2)R(w, w')v = g(w, v)\rho w' - g(w', v)\rho w$ for any section v of $\mathcal{P}^{\perp}$ and vector fields w, w' on M. (Notation of (10); the terms involving W and ρv vanish as W and ρ annihilate $\mathcal{P}^{\perp}$, by Lemma 17.3(i).) If w and w' are π-projectable, while v is a section of $\mathcal{P}$ parallel along $\mathcal{P}^{\perp}$, this equality, (v) in Section 14 and the relation $\rho = (n-2)\pi^*\tau$ give $R^{\mathrm{D}} = \tau \wedge \mathrm{Id}$ on Σ (cf. (5)); by contraction, we get $\tau = \rho^{\mathrm{D}}$, which proves (i) and (ii). Now Lemmas 15.1(b) and 17.2(b) yield (iii) through (v). □

18. Conformal changes of the metric. Given a conformally symmetric pseudo-Riemannian manifold (M, g) with rank $W = 1$, let us consider the following conditions imposed on a function $f : M \to \boldsymbol{R}$:

(25)
a) $(2-n)\nabla df = f\rho$, where $n = \dim M$ and ρ is the Ricci tensor,
b) the gradient ∇f is a section of the distribution $\mathcal{P}$ (see Section 17).

REMARK 18.1. Condition (25-b) for a null parallel distribution $\mathcal{P}$ on any pseudo-Riemannian manifold (M, g) and a function $f : M \to \boldsymbol{R}$ implies that $\Delta f = g(v, v) = 0$ for the section $v = \nabla f$ of $\mathcal{P}$. Indeed, the image of $\nabla v : TM \to TM$ is contained in $\mathcal{P}$, and, since ∇v is self-adjoint, its kernel contains $\mathcal{P}^{\perp}$ (and $\mathcal{P}$); we thus get $\Delta f = 0$ evaluating $\mathrm{tr}\nabla v$ in a basis of any tangent space T_xM, a part of which spans $\mathcal{P}_x$.

LEMMA 18.2. *Let (M, g) be a conformally symmetric pseudo-Riemannian manifold of dimension $n \geq 4$ such that* rank $W = 1$.

(i) *Some neighborhood of any point of M admits a function $f > 0$ with* (25), *and, for any such f, the conformally related metric $\tilde{g} = f^{-2}g$ is Ricci-flat.*

(ii) *If M is simply connected, then the vector space of functions $f : \Sigma \to \boldsymbol{R}$ with* (25) *is* 3*-dimensional, and such a function f is uniquely determined by its value and gradient at any given point x, which are arbitrary elements of $\boldsymbol{R}$ and $\mathcal{P}_x$.*

PROOF. The formula $\overline{\nabla}(\xi, f) = (\nabla\xi + (n-2)^{-1} f\rho, df - \xi)$, with ρ standing for the Ricci tensor of g, defines a connection $\overline{\nabla}$ in the vector bundle $T^*M \oplus (M \times \boldsymbol{R})$. (Notation as in the lines preceding Remark 4.2; sections of $T^*M \oplus (M \times \boldsymbol{R})$ are pairs (ξ, f) consisting of a 1-form ξ and a function f.) The curvature tensor $\bar{R}$ of $\overline{\nabla}$, evaluated from (1) with $\psi = (\xi, f)$ and (4), is given by $\bar{R}(u, v)\psi = (\xi'(u, v), 0)$, where $\xi' = -d^\nabla\nabla\xi - (n-2)^{-1}(\xi \wedge \rho + f d^\nabla\rho)$ with $d^\nabla\nabla\xi$ as in (7) for $\Sigma = M$ and $\mathrm{D} = \nabla$. By (7), $d^\nabla\nabla\xi = \xi R$. As $d^\nabla\rho = 0$ (see Lemma 17.2(b)), $\xi' = -\xi R - (n-2)^{-1}\xi \wedge \rho$.

Let the subbundle $\mathcal{Y}$ of $T^*M \oplus (M \times \boldsymbol{R})$ be the direct sum $\mathcal{Z} \oplus (M \times \boldsymbol{R})$, where $\mathcal{Z}$ is the subbundle of T^*M whose sections are the 1-forms ξ on M that annihilate $\mathcal{P}^\perp$ in the sense of Section 15. The subbundle $\mathcal{Y}$ is ∇-parallel, that is, invariant under ∇-parallel transports, since the distribution $\mathcal{P}^\perp$ is parallel. Consequently, $\mathcal{Y}$ is also $\overline{\nabla}$-parallel, due to the definition of $\xi \wedge \rho$ in Section 2 and the fact that ρ annihilates $\mathcal{P}^\perp$ (see Lemma 17.3(i)). Thus, $\overline{\nabla}$ has a restriction to a connection in $\mathcal{Y}$. Next, given a section (ξ, f) of $\mathcal{Y}$, Lemma 17.2(c) allows us to replace R in ξR by $(n-2)^{-1} g \wedge \rho$. (Indeed, $\xi = g(v, \cdot)$, where v is a section of $\mathcal{P}$, so that $W(v, \cdot, \cdot, \cdot) = 0$ and $\rho v = 0$ by Lemma 17.3(i) and (18-ii).) However, $\xi(g \wedge \rho) = -\xi \wedge \rho$, as $\rho v = 0$. Thus, $\xi' = 0$ (see the last paragraph), and so the restriction of $\overline{\nabla}$ to $\mathcal{Y}$ is flat.

Finally, if M is simply connected, the assignment $f \mapsto (df, f)$ is a linear isomorphism of the space $\mathcal{S}$ of all functions $f : \Sigma \to \boldsymbol{R}$ with (25) onto the space $\mathcal{S}'$ of $\overline{\nabla}$-parallel sections of $\mathcal{Y}$. Namely, injectivity of $f \mapsto (df, f)$ is obvious; that it maps $\mathcal{S}$ into $\mathcal{S}'$, and is surjective, follows from the definition of $\overline{\nabla}$. Our assertion is now immediate, as Remark 18.1 and (c) in Section 2 imply Ricci-flatness of $\tilde{g}$. □

PROPOSITION 18.3. *If (M, g) is a conformally symmetric pseudo-Riemannian manifold of dimension $n \geq 4$ such that* $\operatorname{rank} W = 1$, *and $f : M \to \boldsymbol{R}$ is a positive function with* (25), *then the conformally related metric $\tilde{g} = f^{-2} g$ on M is, locally, a Riemannian product of a Ricci-flat metric $\tilde{h}$ with the signature $--++$ on a four-manifold N and a flat metric γ on a manifold V of dimension $n-4$. In addition,*

(i) *the* 2*-dimensional null parallel distribution $\mathcal{P}$ on M defined in Section* 17 *is tangent to the N factor,*

(ii) *both $\mathcal{P}$ and $\mathcal{P}^\perp$ are spanned, locally, by $\tilde{g}$-parallel vector fields,*

(iii) *f is constant in the direction of the V factor.*

PROOF. From (a) in Section 2 for any vector field u and any section v of $\mathcal{P}^\perp$, with $\phi = -\log f$ and $w = \nabla\phi$, we get $\tilde{\nabla}_u v = \nabla_u v + g(u, w)v - g(u, v)w$, where $g(v, w)u = 0$, since $w = \nabla\phi = -f^{-1}\nabla f$ is, by (25-b), a section of $\mathcal{P}$. The subbundles $\mathcal{P}^\perp$ and $\mathcal{P}$ of TM, with $\mathcal{P} \subset \mathcal{P}^\perp$, are thus $\tilde{\nabla}$-parallel (invariant under $\tilde{\nabla}$-parallel transports). Furthermore, the restriction of $\tilde{\nabla}$ to the subbundle $\mathcal{P}^\perp$ is flat: as $\tilde{g} = f^{-2} g$ is Ricci-flat (Lemma 18.2(i)), its four-times covariant curvature tensor $\tilde{R}$ equals its Weyl tensor $\tilde{W} = f^{-2} W$, and so it annihilates $\mathcal{P}^\perp$ by Lemma 17.3(i). This proves (ii).

Let U be a simply connected neighborhood of any given point of M. The $\tilde{g}$-parallel (that is, $\tilde{\nabla}$-parallel) sections of $\mathcal{P}^\perp$ defined on U form a vector space $\mathcal{E}$ of dimension $n-2$, on which $\tilde{g}$ is a degenerate symmetric bilinear form. The 2-dimensional $\tilde{g}$-nullspace $\mathcal{E}'$ of $\mathcal{E}$ (i.e., the $\tilde{g}$-orthogonal complement of $\mathcal{E}$) consists of all $\tilde{g}$-parallel sections of $\mathcal{P}$. Since $\tilde{g}$ descends to a (nondegenerate) pseudo-Euclidean inner product on $\mathcal{E}/\mathcal{E}'$, we may choose a $(n-4)$-dimensional vector subspace of $\mathcal{E}/\mathcal{E}'$, on which $\tilde{g}$ is nondegenerate, and realize it as the projection image of some subspace $\mathcal{X}$ of $\mathcal{E}$ with $\dim \mathcal{X} = n-4$ such that $\tilde{g}$ is nondegenerate on $\mathcal{X}$. The distribution of dimension $n-4$ on U spanned by the vector fields forming $\mathcal{X}$ is clearly $\tilde{g}$-parallel and nondegenerate. Hence, by a classical result of Thomas [26], it is a factor distribution, tangent to some factor manifold (V, γ), in a local Riemannian-product decomposition of (M, g). Obviously, γ is flat, as elements of $\mathcal{X}$ constitute γ-parallel vector fields spanning the tangent bundle of V. Denoting by (N, h) the other factor manifold, we get (i) (since $\mathcal{P}$ is orthogonal to all sections of $\mathcal{P}^\perp$, including elements of $\mathcal{X}$), as well as (iii) (which follows from (i) and (25-b)). □

REMARK 18.4. Locally, condition (25) can be conveniently rephrased in terms of the quotient surface Σ with the $\mathcal{P}^\perp$-projected connection D (cf. Theorem 17.4). Namely, (25-b) means that f may be treated as a function $\Sigma \to \boldsymbol{R}$, and then (25) is equivalent to the condition $\mathrm{D}df = -f\rho^{\mathrm{D}}$. This is immediate from Theorem 17.4(ii) and Lemma 15.1(b) applied to both $\lambda = f$ and $\lambda = df$.

Lemma 18.2(ii) could therefore be derived from an analogous statement about the equation $\mathrm{D}df = -f\rho^{\mathrm{D}}$ on Σ, which can be proved by the same argument, and constitutes the 2-dimensional case of a more general result of Gardner, Kriele and Simon [11].

19. Reduction of the dimension. Our next step in classifying conformally symmetric manifolds with rank $W=1$, in dimensions $n \geq 4$, consists in reducing the problem to the case $n=4$.

For pseudo-Riemannian manifolds (N, h), (V, γ) and a C^∞ function $f : N \to (0, \infty)$, the *warped product* with the *base* (N, h), *fibre* (V, γ), and *warping function* f is the pseudo-Riemannian manifold (M, g) given by $M = N \times V$ and $g = h + f^2\gamma$. (Here g, f, h also stand for their own pullbacks to $N \times V$.) See [1, p. 237] for more details. Note that $g = h + f^2\gamma$ is conformally related to the product metric $f^{-2}g = f^{-2}h + \gamma$.

In the next lemma, all Weyl tensors are four-times covariant tensors.

LEMMA 19.1. *Given pseudo-Riemannian manifolds* $(N, \tilde{h})$ *and* (V, γ) *such that* $\tilde{h}$ *is Ricci-flat and* γ *is flat, while* $\dim N \geq 4$, *let* f *be a positive function on N, and let* W^h *denote the Weyl tensor of the conformally related metric* $h = f^2\tilde{h}$ *on N. The Weyl tensor W of the warped product metric* $g = h + f^2\gamma$ *on* $N \times V$ *then equals the pullback of* W^h *under the projection* $N \times V \to V$.

This is clear from the conformal transformation rule for W. Namely, W is related to the four-times covariant Weyl and curvature tensors $\tilde{W}$, $\tilde{R}$ of the product metric $\tilde{g} = f^{-2}g = \tilde{h} + \gamma$ by $W = f^2\tilde{W} = f^2\tilde{R}$, where $\tilde{W} = \tilde{R}$, since $\tilde{g}$ is Ricci-flat. Next, $\tilde{R}$ is the pullback of the curvature tensor R' of $\tilde{h}$ under $N \times V \to V$, while, as $\tilde{h}$ is Ricci-flat, R' equals the Weyl tensor W' of $\tilde{h} = f^{-2}h$, that is, $f^{-2}W^h$.

THEOREM 19.2. *In any conformally symmetric pseudo-Riemannian manifold* (M, g) *of dimension* $n \geq 4$ *such that* $\operatorname{rank} W = 1$, *every point has a connected neighborhood isometric to a warped product* $(N \times V, h + f^2\gamma)$, *for any solution* $f > 0$ *to* (25), *and*

(a) (N, h) *is a conformally symmetric manifold with* $\dim N = 4$ *and* $\operatorname{rank} W^h = 1$,

(b) (V, γ) *is flat*, $\dim V = n - 4 \geq 0$, *and* f *is constant in the* V*-factor direction*,

(c) f *treated as a function on* (N, h) *satisfies* (25) *with* n *replaced by* 4.

Conversely, *every warped product* $(M, g) = (N \times V, h + f^2\gamma)$ *with* (a)–(c) *is conformally symmetric*, *has* $\operatorname{rank} W = 1$, *and* f *satisfies* (25) *on* (M, g).

PROOF. Let (M, g) satisfy the conditions $\nabla W = 0$ and $\operatorname{rank} W = 1$. By Proposition 18.3, every point of M has a connected neighborhood U such that $(U, f^{-2}g)$ is isometric to a Riemannian product manifold $(N \times V, \tilde{h} + \gamma)$, in which the factor $(N, \tilde{h})$ is 4-dimensional and Ricci-flat, while (V, γ) is flat; since f, which exists by Lemma 18.2(i), may be treated as a function on N (Proposition 18.3(iii)), (U, g) is isometric to the warped product $(N \times V, h + f^2\gamma)$, where $h = f^2\tilde{h}$.

As the N-factor submanifolds are totally geodesic in (M, g) and h represents their submanifold metrics, restricting to any of them the g-parallel 2-form ω appearing in Lemma 17.1(iii), we obtain an h-parallel 2-form ω^h on N. Next, the relation $W = \varepsilon\,\omega \otimes \omega \neq 0$ combined with Lemma 19.1 gives $W^h = \varepsilon\,\omega^h \otimes \omega^h \neq 0$, so that $\operatorname{rank} \omega^h = 2$ by the first Bianchi identity for W^h. Hence (N, h) is conformally symmetric and, by Lemma 17.1, $\operatorname{rank} W^h = 1$. Also, $\mathcal{P}$ (tangent to the N factor by Proposition 18.3(i)) is the distribution defined as in Section 17 for (N, h) rather than (M, g). By Remark 18.1, f treated as a function on N has vanishing h-Laplacian and its h-gradient is null, so that Ricci-flatness of $\tilde{h} = f^{-2}h$ and (c) in Section 2 give (25-a) with $n = 4$.

Conversely, let $(M, g) = (N \times V, h + f^2\gamma)$, with (a)–(c). As $W^h = \varepsilon\,\omega^h \otimes \omega^h \neq 0$ in (N, h), where ω^h is an h-parallel 2-form (Lemma 17.1(iii)), we get $W = \varepsilon\,\omega \otimes \omega \neq 0$, by Lemma 19.1, for the 2-form ω on M obtained as the pullback of ω^h under the factor projection $M \to N$. For the Levi-Civita connection $\tilde{\nabla}$ of the product metric $\tilde{g} = f^{-2}g = \tilde{h} + \gamma$ and any vector field u, we have $\tilde{\nabla}_u\omega = -2(d_u\phi)\omega + \omega(u, \cdot) \wedge d\phi$, where $\phi = -\log f$, since the same is true for the Levi-Civita connection of $\tilde{h}$ and ω^h (rather than ω) in view of (b) in Section 2 and the fact that ω^h is h-parallel. Using (b) in Section 2 again, we now see that ω is g-parallel. Thus, by Lemma 17.1, (M, g) is conformally symmetric and $\operatorname{rank} W = 1$. □

COROLLARY 19.3. *Theorem* 19.2 *remains true also when the dimensions* 4 *and* $n - 4$ *in* (a)–(c) *are replaced by* k *and* $n - k$, *for any given* $k \in \{4, \dots, n\}$.

PROOF. Writing γ in Theorem 19.2 as the product $\gamma'' + \gamma'$ of two flat metrics, we obtain a new warped-product decomposition of $g = h + f^2\gamma$, namely, $g = h' + f^2\gamma'$, with $h' = h + f^2\gamma''$ having the required properties as a consequence of Theorem 19.2. □

In Theorem 19.2 and Corollary 19.3, the existence of a local warped-product decomposition is a consequence of the conditions $\nabla W = 0$ and $\operatorname{rank} W = 1$ imposed on the given pseudo-Riemannian manifold. Hotloś [13] studied the case where an essentially conformally symmetric manifold is *assumed* to be a warped product. The final clause of our Theorem 19.2, and the generalized version of that final clause provided by Corollary 19.3, are closely related to one of his results [13, Theorem 4.2].

20. Dimension four. Four-dimensional conformally symmetric manifolds (N, h) with $\operatorname{rank} W^h = 1$, where W^h is the Weyl tensor of h, can be constructed as follows. Let there be given an equiaffine projectively flat torsionfree connection D on a surface Σ with a D-parallel area element $\pm\alpha$ (see Section 7), a factor $\varepsilon \in \{1, -1\}$, and a twice-covariant symmetric tensor field τ on Σ with $\mathcal{L}\tau = \varepsilon\alpha\otimes\alpha$. Locally, such τ always exists (Theorem 10.2(i)). We now set $(N, h) = (T^*\Sigma, h^{\mathrm{D}} - 2\tau)$, where h^{D} is the metric on $T^*\Sigma$ given by (22), and the symbol τ also stands for the pullback of τ to $T^*\Sigma$. It should be pointed out that the tensor τ is not really a parameter for the above construction; locally, up to an isometry, different choices of τ lead to the same metric $h^{\mathrm{D}} - 2\tau$. See Section 21.

THEOREM 20.1. *Every pseudo-Riemannian four-manifold* $(N, h) = (T^*\Sigma, h^{\mathrm{D}} - 2\tau)$ *obtained as above is conformally symmetric and* $\operatorname{rank} W^h = 1$, *for its Weyl tensor* W^h, *while* $\mathcal{P}$ *defined as in Section* 17 *coincides with the vertical distribution on* $T^*\Sigma$, *and* D *used in the construction is the same as the* $\mathcal{P}^\perp$*-projected connection in Theorem* 17.4.

Conversely, every point of any conformally symmetric pseudo-Riemannian four-manifold (N, h) *with* $\operatorname{rank} W^h = 1$ *has a connected neighborhood isometric to an open subset of a manifold* $(T^*\Sigma, h^{\mathrm{D}} - 2\tau)$ *constructed as above.*

We precede the proof of Theorem 20.1 with two lemmas. Conformal flatness of $\tilde{\mathrm{D}}$ in Lemma 20.2 is a result of Walker [28, p. 69]. Lemma 20.3 is due to Ruse [23] for $r = 2$ (which is the only case that we need), and to Walker [27] for arbitrary r.

LEMMA 20.2. *For a projectively flat torsionfree connection* D *on a manifold* Σ *whose Ricci tensor is symmetric, the metric* h^{D} *on* $T^*\Sigma$, *given by* (22), *is conformally flat. Furthermore, using Lemma* 3.1 *we can find, locally in* Σ, *a function* $f > 0$ *such that, for* $\xi = -d\log f$, *the connection* $\tilde{\mathrm{D}} = \mathrm{D} + 2\xi \odot \mathrm{Id}$ *is flat. Choosing any local coordinates* y^j *for* Σ *in which the components of* $\tilde{\mathrm{D}}$ *all vanish, and letting* y^j, p_j *be the corresponding coordinates in* $T^*\Sigma$, *we then have the symmetric-product relations*

$$(26)\qquad \begin{array}{ll} \text{a)}\ \ \Gamma_{kl}^j dy^k dy^l = 2f^{-1} df\, dy^j\,, & \text{b)}\ \ f^{-2}h^{\mathrm{D}} = 2d(f^{-2}p_j)dy^j\,, \\ \text{c)}\ \ f^{-1}R_{kl}dy^k dy^l = d[\partial_j(f^{-1})]dy^j\,, & \end{array}$$

where $\partial_j = \partial/\partial y^j$, *while* Γ_{kl}^j *and* R_{kl} *are the components of* D *and its Ricci tensor* ρ^{D}.

PROOF. Since $f\Gamma_{kl}^j = f_{,k}\delta_l^j + f_{,l}\delta_k^j$ (cf. Lemma 3.1), (a) follows. Now (23) gives (b). Conformal flatness of h^{D} is in turn obvious from (b): the metric $f^{-2}h^{\mathrm{D}}$, having constant component functions in the new coordinates $y^j, f^{-2}p_j$, is flat. Finally, by Remark 3.2,

$\mathrm{D}df = -f\rho^{\mathrm{D}}$, that is, $fR_{kl} = -f_{,kl}$, while $f_{,kl} = \partial_l\partial_k f - \Gamma_{kl}^j\partial_j f$. Thus, (a) implies (c), completing the proof. □

LEMMA 20.3. *Every pseudo-Riemannian metric $\tilde{h}$ on a manifold N of dimension $2r$ with an r-dimensional $\tilde{h}$-null distribution $\mathcal{P}$ spanned by $\tilde{h}$-parallel vector fields has, locally, the form* (24) *in some coordinates y^j, q_j such that the q_j coordinate vector fields span $\mathcal{P}$. In addition, the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{h}$ is $\mathcal{P}$-projectable, cf. Section* 14, *its $\mathcal{P}$-projected connection $\tilde{\mathrm{D}}$ on a local leaf space Σ for $\mathcal{P}$ is flat, and the components of $\tilde{\mathrm{D}}$ are all zero in the local coordinates for Σ provided by the functions y^j.*

PROOF. Letting U stand for increasingly small neighborhoods of any given point x of N, we may choose r functions y^j on U whose $\tilde{h}$-gradients $v_j = \tilde{\nabla}y^j$ are linearly independent $\tilde{h}$-parallel sections of $\mathcal{P}$. The Ricci identity (7-i) for $\mathrm{D} = \tilde{\nabla}$, its curvature tensor $\tilde{R}$, and $\xi = \tilde{h}(v_j, \cdot)$, shows that $\tilde{R}$ annihilates $\mathcal{P}$ (cf. Section 15), and so $\mathcal{P}$-projectability of $\tilde{\nabla}$ follows from Lemma 14.1. Being constant along $\mathcal{P}^\perp = \mathcal{P}$, our y^j descend to Σ, where they form a local coordinate system with $\tilde{\mathrm{D}}$-parallel differentials dy^j (by (iii)–(iv) in Section 14); thus, the component functions of the $\mathcal{P}$-projected connection $\tilde{\mathrm{D}}$ all vanish. Any fixed submanifold Σ' of U such that $x \in \Sigma'$ and the projection $\pi : U \to \Sigma$ sends Σ' diffeomorphically onto Σ gives rise to r functions q_j on U with $q_j = 0$ on $U \cap \Sigma'$ and $(dq_j)(v_k) = \delta_{jk}$. (As the vector fields v_j commute, they form coordinate vector fields on each leaf Y of $\mathcal{P}$ for the coordinates q_j on Y.) Thus, $\tau(v_k, \cdot) = 0$ for $\tau = \tilde{h} - 2dq_j dy^j$ and each k, and so $\tau = -2\lambda_{kl}dy^k dy^l$ for some λ_{kl}. In the coordinates y^j, y^μ for N obtained from y^j, q_j as at the end of Section 16, the relations $\tilde{\nabla}v_j = 0$ now give $\partial_\mu\lambda_{jk} = \tilde{\Gamma}_{jk\mu} = 0$, as $\tilde{h}_{j\mu} = c_{\mu j}$ are constants and $\tilde{h}_{jk} = -2\lambda_{jk}$. □

By (23), $h^{\mathrm{D}} - 2\tau = 2dp_j dy^j - 2(p_j\Gamma_{kl}^j + \tau_{kl})dy^k dy^l$ in the coordinates y^j, p_j for $T^*\Sigma$ corresponding to local coordinates y^j in Σ. This expression can be further simplified if y^j, rather than being arbitrary, are chosen, along with a function $f > 0$, as in Lemma 20.2. In the new coordinates y^j, $f^{-2}p_j$ for $T^*\Sigma$, we then get, from (26-b),

$$\tilde{h} = 2d(f^{-2}p_j)dy^j - 2f^{-2}\tau_{jk}dy^j dy^k\,, \tag{27}$$

where $\tilde{h} = f^{-2}(h^{\mathrm{D}} - 2\tau)$.

PROOF OF THEOREM 20.1. Our $\tilde{h}$ in (27) is a special case of (24). Hence, by (c) in Section 16, $\tilde{h}$ has the Weyl tensor $\tilde{W} = \pi^*(\tilde{\mathcal{L}}(f^{-2}\tau))$, for $\tilde{\mathcal{L}}$ associated with the connection $\tilde{\mathrm{D}}$ of Lemma 20.2, so that $\tilde{\mathrm{D}} = \mathrm{D} + 2\xi \odot \mathrm{Id}$ for $\xi = -d\log f$. Lemma 11.4(a) now gives $\tilde{\mathcal{L}}(f^{-2}\tau) = f^{-2}\mathcal{L}\tau$, and so the Weyl tensor of $h = h^{\mathrm{D}} - 2\tau = f^2\tilde{h}$ is $W^h = f^2\tilde{W} = \pi^*(\mathcal{L}\tau) = \varepsilon\,\omega^h \otimes \omega^h$, where $\omega^h = \pi^*\alpha$. Also, the vertical distribution $\mathcal{P}$ on $T^*\Sigma$ is $\tilde{h}$-null and $\tilde{h}$-parallel, while the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{h}$ is $\mathcal{P}^\perp$-projectable and $\tilde{\mathrm{D}}$ is its $\mathcal{P}^\perp$-projected connection. (See (a), (b) in Section 16.) Hence, by (vi) in Section 14, the same is true for $\mathcal{P}$ if one replaces $\tilde{h}$, $\tilde{\mathrm{D}}$ by h, D. As α is D-parallel, Lemmas 14.1 and 15.1(b) for $\tau = \alpha$ now imply that ω^h is h-parallel. The first part of Theorem 20.1 thus follows from Lemma 17.1. (The image of ω^h is $\mathcal{P}$, as ω^h annihilates $\mathcal{P} = \mathcal{P}^\perp$.)

For the second part of Theorem 20.1, let (N, h) be conformally symmetric, with $\dim N = 4$ and $\operatorname{rank} W^h = 1$. By Lemma 18.2(i), every $x \in N$ has a neighborhood U such that both (25) and (21) hold, for $r = 2$, some function $f : U \to (0, \infty)$, a suitable quotient surface Σ, and the distribution $\mathcal{P}$ defined in Section 17, with U instead of M in (21). According to Remark 18.4, f descends to a function $\Sigma \to \boldsymbol{R}$, and $\mathrm{D}df = -f\rho^{\mathrm{D}}$ for $f : \Sigma \to \boldsymbol{R}$ and the $\mathcal{P}^{\perp}$-projected connection D on Σ, which in turn means that the connection $\tilde{\mathrm{D}} = \mathrm{D} + 2\xi \odot \mathrm{Id}$ on Σ, with $\xi = -d \log f$, is flat (Remark 3.2).

Proposition 18.3(ii) (with $n = 4, \mathcal{P}^{\perp} = \mathcal{P}, g = h, \tilde{g} = \tilde{h}$) and Lemma 20.3 imply that, locally, $\tilde{h}$ has the form (24), for some twice-covariant symmetric tensor field λ on Σ, and local coordinates y^j in Σ, in which the components of $\tilde{\mathrm{D}}$ all vanish. By (c) in Section 16, $\tilde{W} = \pi^*(\tilde{\mathcal{L}}\lambda)$. Therefore, $\varepsilon\omega^h \otimes \omega^h = W^h = f^2\tilde{W} = \pi^*(\mathcal{L}\tau)$ for $\tau = f^2\lambda$ and $\mathcal{L}$ corresponding to D as in (15), with the successive equalities due to Lemma 17.1(iii), the relation $h = f^2\tilde{h}$, and Lemma 11.4(a). However, by Theorem 17.4(vi), $\omega^h = \pi^*\alpha$, where $\pm\alpha$ is a D-parallel area element on Σ. Hence $\mathcal{L}\tau = \varepsilon\alpha \otimes \alpha$. By (27), h and $h^{\mathrm{D}} - 2\tau$ are isometric, as they have the same form in suitable coordinates. □

All four-dimensional conformally symmetric manifolds with $\operatorname{rank} W = 1$ belong to a class of conformally recurrent four-manifolds for which Olszak [16] provided a local classification. Olszak's result states that the latter manifolds have, locally, the form $(T^*\Sigma, h^{\mathrm{D}} - 2\tau)$ for some torsionfree connection D on a surface Σ and some twice-covariant symmetric tensor field τ on Σ. To derive Theorem 20.1 from it, one needs to determine what requiring $h^{\mathrm{D}} - 2\tau$ to be conformally symmetric means for D and τ. As our discussion shows, the answer is: $\mathcal{L}\tau = \varepsilon\alpha \otimes \alpha$ for some D-parallel area element $\pm\alpha$.

21. The local structure theorem. Although a local classification of conformally symmetric manifolds with $\operatorname{rank} W = 1$ could now be easily derived from Theorems 19.2 and 20.1, we state it differently, for reasons explained in Remark 21.2 below. Namely, let the following objects be given:

(i) an integer $n \geq 4$,
(ii) a surface Σ with a projectively flat torsionfree connection D,
(iii) a D-parallel area element $\pm\alpha$ on Σ (see Section 7),
(iv) a sign factor $\varepsilon = \pm 1$,
(v) a real vector space V of dimension $n - 4$,
(vi) a pseudo-Euclidean inner product $\langle\,,\rangle$ on V.

We are also assuming the existence of a twice-covariant symmetric tensor τ on Σ with $\mathcal{L}\tau = \varepsilon\alpha \otimes \alpha$. Locally, such τ always exists, by Theorem 10.2(i).

The data (i)–(vi) give rise to the n-dimensional pseudo-Riemannian manifold

$$(M, g) = (T^*\Sigma \times V, h^{\mathrm{D}} - 2\tau + \gamma - \theta\rho^{\mathrm{D}})\,, \tag{28}$$

where h^{D} is the metric (22) on $T^*\Sigma$ and γ is the constant pseudo-Riemannian metric on V corresponding to the inner product $\langle\,,\rangle$, while $\theta : V \to \boldsymbol{R}$ is given by $\theta(v) = \langle v, v\rangle$. As before, the symbols for functions or covariant tensor fields on Σ, or on the factor manifolds $T^*\Sigma$ and V, are also used to represent their pullbacks to $T^*\Sigma \times V$.

For coordinate descriptions of the metric g in (28), see formulae (29) in Section 22.

We treat n, Σ, D, α, ε, V and $\langle\,,\rangle$ in (i)–(vi) as parameters for our construction, while τ is merely an object assumed to exist, even though the metric g in (28) clearly depends on τ. The reason is that, if the data (i)–(vi) are fixed, the metrics corresponding to two choices of τ are, locally, isometric to each other (Remark 22.1 below).

We can now state our local classification result; for a proof, see Section 22.

THEOREM 21.1. *The pseudo-Riemannian manifold* (28) *obtained as above from data* (i) *through* (vi) *with the stated properties is conformally symmetric and has* rank $W = 1$. *Also*,

(a) *the manifold* (28) *is locally symmetric*, *or Ricci-recurrent*, *if and only if so is* D,

(b) *the distribution* $\mathcal{P}$ *defined in Section* 17 *is tangent to the* $T^*\Sigma$ *factor*, *and coincides with the vertical distribution of* $T^*\Sigma$,

(c) *the* $\mathcal{P}^\perp$*-projected connection for* $\mathcal{P}$, *cf. Theorem* 17.4, *is the original* D.

Conversely, *in any conformally symmetric pseudo-Riemannian manifold with* rank $W =$ 1, *every point has a connected neighborhood isometric to an open subset of a manifold* (28) *constructed above from some data* (i) *through* (vi).

REMARK 21.2. A general local form of a conformally symmetric metric with rank $W = 1$ is the warped product $h + f^2\gamma$ with $h = h^{\mathrm{D}} - 2\tau$ and $\mathrm{D}df = -f\rho^{\mathrm{D}}$ on Σ (for details, see Theorems 19.2, 20.1 and Remark 18.4). As shown in Section 22, this is equivalent to (28). We chose to state the classification theorem in terms of (28) to avoid using, in addition to τ, yet another object (the function f), which is not a genuine parameter of the construction, as the local isometry type of the resulting metric does not depend on it.

22. Proof of Theorem 21.1. The metric g defined by (28) has the local-coordinate expressions

$$\text{(29)}\qquad \begin{aligned} &\text{i)}\quad g = 2dp_j dy^j - (2p_j\Gamma_{kl}^j + 2\tau_{kl} + \gamma_{ab}v^a v^b R_{kl})dy^k dy^l + \gamma_{ab}dv^a dv^b\,,\\ &\text{ii)}\quad f^{-2}g = 2dq_j dy^j - 2f^{-2}\tau_{jk}dy^j dy^k + \gamma_{ab}du^a du^b\,. \end{aligned}$$

In (29-i), y^j, p_j, v^a are product coordinates for $T^*\Sigma \times V$ formed by the coordinates y^j, p_j in $T^*\Sigma$ obtained as usual from any given local coordinates y^j in Σ, and linear coordinates v^a in V corresponding to a basis e_a, with $\gamma_{ab} = \langle e_a, e_b\rangle$, while Γ_{kl}^j and R_{kl} stand for the components of D and its Ricci tensor ρ^{D}. Thus, (29-i) is obvious from (23). As for (29-ii), we start from the coordinates y^j, p_j for $T^*\Sigma$ based on coordinates y^j in Σ that, instead of being arbitrary, are chosen (along with the function f) as in Lemma 20.2. We then replace the resulting product coordinates y^j, p_j, v^a in $T^*\Sigma \times V$ (with v^a as before) by the new coordinates y^j, q_j, u^a such that $2q_j = 2f^{-2}p_j + \gamma_{ab}v^a v^b f^{-3}\partial_j f$ and $u^a = f^{-1}v^a$. By (27), $f^{-2}(h^{\mathrm{D}} - 2\tau) = 2d(f^{-2}p_j)dy^j - 2f^{-2}\tau_{jk}dy^j dy^k$, while, at the same time, $f^{-2}(\gamma - \theta\rho^{\mathrm{D}}) = f^{-2}(\gamma_{ab}dv^a dv^b - \gamma_{ab}v^a v^b R_{kl}dy^k dy^l)$, and so (26-c) gives $f^{-2}(\gamma - \theta\rho^{\mathrm{D}}) = f^{-2}\gamma_{ab}dv^a dv^b - \gamma_{ab}v^a v^b f^{-1}d[\partial_j(f^{-1})]dy^j$, proving (29-ii).

In view of (29-ii), g is, locally, a warped product $h + f^2\gamma$ with the factor metrics $h = 2f^2 dq_j dy^j - 2\tau_{jk} dy^j dy^k$ (in dimension 4, coordinates: y^j, q_j) and $\gamma = \gamma_{ab} du^a du^b$ (dimension $n-4$, coordinates: u^a). Since $\gamma_{ab} = \langle e_a, e_b\rangle$ are constants, γ is flat. On the other hand, h is isometric to the metric $h^{\mathrm{D}} - 2\tau$ defined as in Section 20 using the data (i)–(iv) of Section 21. Namely, by (27), $h^{\mathrm{D}} - 2\tau$ equals $2f^2 d\hat{q}_j dy^j - 2\tau_{jk} dy^j dy^k$ in coordinates $y^j, \hat{q}_j$ formed by our y^j and some $\hat{q}_j$, while f and τ_{kl} depend only on y^j. By Theorem 20.1, (N, h) is conformally symmetric and $\operatorname{rank} W^h = 1$, so that Theorem 19.2 yields the same for (28), while Theorems 20.1 and 17.4(iv), (v) imply (a)–(c).

Conversely, let (M, g) be conformally symmetric, with $\operatorname{rank} W = 1$. Thus, locally, $g = h + f^2\gamma$ with h, γ, f as in Theorem 19.2, and, by Theorem 20.1, $h = h^{\mathrm{D}} - 2\tau$, for suitable D and τ. Now (27) gives $g = 2f^2 d(f^{-2}p_j)dy^j - 2\tau_{jk} dy^j dy^k + f^2\gamma_{ab} du^a du^b$ in some coordinates $y^j, f^{-2}p_j, u^a$, with constants γ_{ab} such that $\gamma = \gamma_{ab} du^a du^b$. (Note that $\mathrm{D}df = -f\rho^{\mathrm{D}}$, by Remark 18.4; hence, according to Remark 3.2, f is chosen as in Lemma 20.2.) This gives (29-ii) for $q_j = f^{-2}p_j$, and Theorem 21.1 follows.

REMARK 22.1. For (M, g) in (28) and $(M, g') = (T^*\Sigma \times V, h^{\mathrm{D}} - 2\tau' + \gamma - \theta\rho^{\mathrm{D}})$ constructed as in Section 21 from the same data (i)–(vi), but with (possibly) different 2-ten-break sors τ and τ', an isometry J of (M, g') onto (M, g) can be defined by the formula $J(y, \eta, v) = (y, \eta + \xi(y) - (d\langle L, Hv\rangle)_y, Hv + L(y))$ (notation of Section 16), for any function $L : \Sigma \to V$ with $\mathrm{D}dL = -L\rho^{\mathrm{D}}$, any $\langle\,,\rangle$-preserving linear isomorphism $H : V \to V$, and any 1-form ξ on Σ such that $2(\tau - \tau') = \mathcal{B}(\xi + d\langle L, L\rangle/4)$ (cf. (12)); $\langle L, Hv\rangle$ and $\langle L, L\rangle$ are functions $\Sigma \to \boldsymbol{R}$. In fact, $J^*g = g'$, since J^*g is obtained by replacing p_j and v^a in (29-i) with $p_j + \xi_j - \gamma_{ab} H^b_c v^c \partial_j L^b$ and $H^a_b x^b + L^a$. Cf. [19, §8].

Let Σ now be simply connected. First, (M, g') is isometric to (M, g), as one sees, choosing J with $L = 0$, $H = \mathrm{Id}$ and ξ such that $2(\tau - \tau') = \mathcal{B}\xi$ (which exists by Theorem 10.2(ii)). Secondly, for every $y \in \Sigma$, the set $\{y\} \times T_y\Sigma \times V$ is contained in an orbit of the isometry group of (M, g). In fact, L with $\mathrm{D}dL = -L\rho^{\mathrm{D}}$ and ξ' with $\mathcal{B}\xi' = 0$ realize all values $L(y) \in V$ and $\xi'_y \in T^*_y\Sigma$, cf. Remarks 18.4, 4.2 and Lemma 4.3.

23. Compactness the quotient surface. We do not know whether there exist any *compact* essentially conformally symmetric manifolds.[1] However, some such manifolds do have a compactness-type property, namely, the leaf space of $\mathcal{P}$ is a globally well-defined (Hausdorff) closed surface.

EXAMPLE 23.1. Given a closed surface Σ, an integer $n \geq 4$, and a metric signature $--\cdots++$ with n signs, containing at least two minuses and at least two pluses, there exists an essentially conformally symmetric manifold (M, g) of dimension n such that

(i) g is not Ricci-recurrent, and has the prescribed signature $--\cdots++$,

(ii) the leaves of the distribution $\mathcal{P}$ defined in Section 17 are the fibres of a bundle with the total space M and base Σ.

[1] (Added in proof.) They do exist: see our preprint math.DG/0702491.

Indeed, we may choose (M, g) to be the manifold (28) obtained from the construction in Section 21, using $\langle\,,\rangle$ of the appropriate signature, and D, along with $\pm\alpha$ and τ on our surface Σ, that satisfy the required assumptions; they exist by Theorem 11.1, while (i) follows from Theorem 21.1(a), since D is not Ricci-recurrent.

24. Further comments. Let us consider triples (M, g, x) formed by a manifold M, a point $x \in M$, and a conformally symmetric metric g on M having a fixed signature $--\cdots++$ with two or more minuses and two or more pluses, and satisfying the condition $\operatorname{rank} W = 1$. (Note that the signature determines $\dim M$.) We call two such triples (M, g, x) and (M', g', x') *equivalent* if some isometry of a neighborhood of x in M onto a neighborhood of x' in M' sends x to x', and we refer to the set of equivalence classes of this relation as the *local moduli space* of conformally symmetric metrics of the given signature with $\operatorname{rank} W = 1$.

We similarly define the local moduli space of equiaffine, projectively flat torsionfree connections on surfaces, to be the set of equivalence classes of quadruples $(\Sigma, \mathrm{D}, \pm\alpha, y)$ formed by a surface Σ with such a connection D (see Section 7), a D-parallel area element $\pm\alpha$ on Σ, and a point $y \in \Sigma$. The equivalence relation is defined similarly, except that isometries are replaced by (local) unimodular affine diffeomorphisms.

By Theorem 21.1, given $--\cdots++$, the former moduli space is in a natural one-to-one correspondence with the latter: the correspondence sends the equivalence class of (M, g, x) to that of $(\Sigma, \mathrm{D}, \pm\alpha, y)$ obtained in Theorem 17.4, with $y = \pi(x)$.

Our next comment concerns Ricci-recurrence. In Theorem 21.1 one cannot simply replace the condition $\operatorname{rank} W = 1$ by 'not being Ricci-recurrent' (both for (M, g) and for D in (iii) of Section 21), and still obtain a classification result with an analogous final clause. Namely, there is no principle of unique continuation, either for conformally symmetric metrics g with $\operatorname{rank} W = 1$, or for projectively flat torsionfree connections D on surfaces. (Thus, neither of the two can in general be made real-analytic by a suitable choice of local coordinates.) In fact, both g, D can be Ricci-recurrent on some nonempty open set, without being so everywhere. Examples are immediate from Theorem 6.2: it suffices to deform a cylinder surface so as to make it non-cylindrical just in a small subset. The construction of Section 21, applied to the corresponding centroaffine connection D, then yields a metric g with the stated property.

Finally, any conformally symmetric pseudo-Riemannian manifold (M, g) of dimension $n \geq 4$ with $\operatorname{rank} W = 1$, and any pseudo-Euclidean inner product $\langle\,,\rangle'$ on a k-dimensional real vector space V', treated as a translation-invariant metric γ' on V', give rise to the conformally symmetric metric $g' = g - (n-2)^{-1}\theta'\rho + \gamma'$ with $\operatorname{rank} W = 1$ on $M \times V$, where ρ is the Ricci tensor of g and $\theta' : V \to \boldsymbol{R}$ is defined by $\theta'(v) = \langle v, v\rangle'$. (Notation of (28).) In fact, g' can also be constructed as in Section 21, since, by Theorem 17.4(ii), we may replace ρ^{D} in (28) with $(n-2)^{-1}\rho$.

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