Algebra and Geometry around Knots and Braids

Euler Institute, St. Petersburg, Russia

September 10 – 14, 2007

Thistlethwaite's theorem for virtual links

Sergei Chmutov

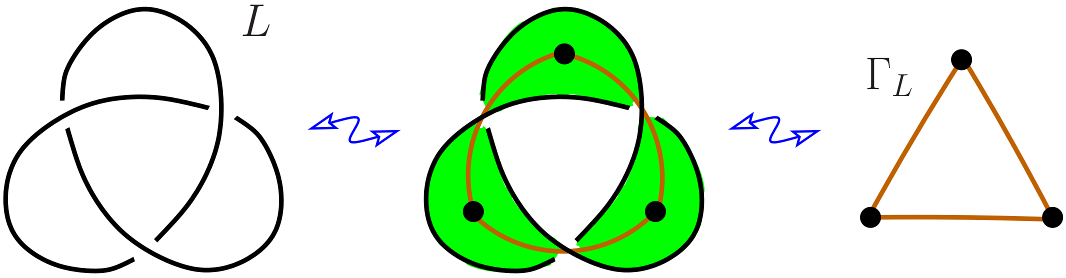
The Ohio State University, Mansfield

Joint work with *Jeremy Voltz*

M. B. Thistlethwaite,

L. Kauffman, K.Murasugi, F.Jaeger

Up to a sign and a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_{\Gamma_L}(-t, -t^{-1})$.



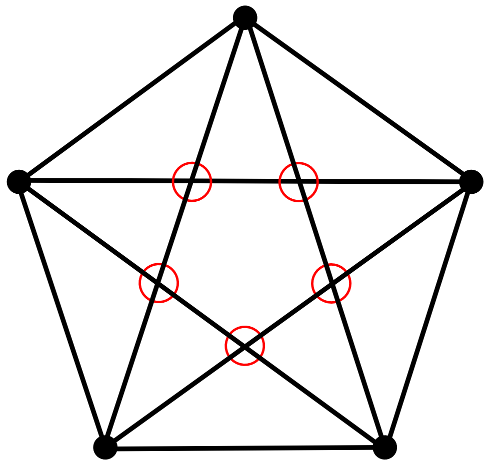
$$V_L(t) = t + t^3 - t^4 = -t^2(-t^{-1} - t + t^2)$$

$$T_{\Gamma_L}(x, y) = y + x + x^2$$

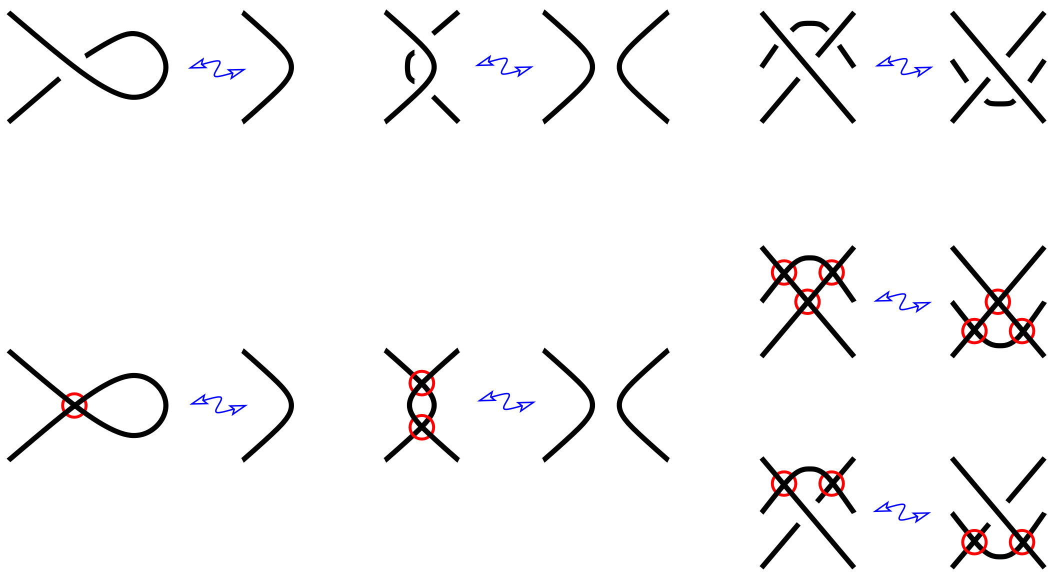
$$T_{\Gamma_L}(-t, -t^{-1}) = -t^{-1} - t + t^2$$

Virtual links

Virtual crossings

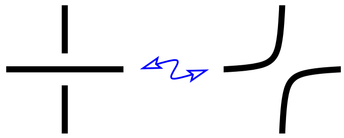


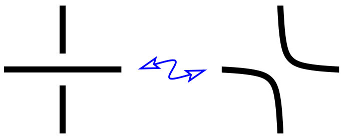
Reidemeister moves



The Kauffman bracket

Let L be a virtual link diagram.

A-splitting: 

B-splitting: 

A *state* S is a choice of either A- or B-splitting at every classical crossing.

$\alpha(S) = \#(\text{of } A\text{-splittings in } S)$

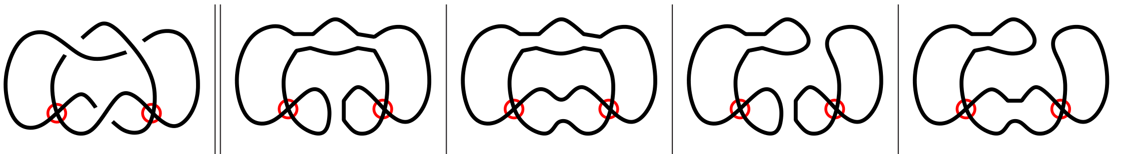
$\beta(S) = \#(\text{of } B\text{-splittings in } S)$

$\delta(S) = \#(\text{of circles in } S)$

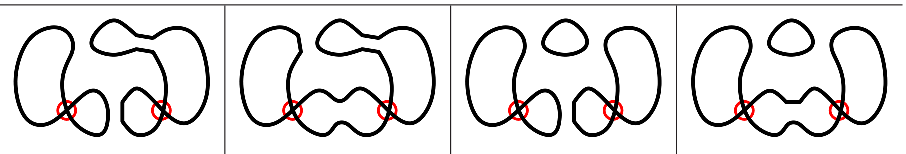
$$[L](A, B, d) \ := \sum_S A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1}$$

$$J_L(t) \ := (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

Example



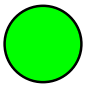
(α, β, δ)	$(3, 0, 1)$	$(2, 1, 2)$	$(2, 1, 2)$	$(1, 2, 1)$



	$(2, 1, 2)$	$(1, 2, 1)$	$(1, 2, 3)$	$(0, 3, 2)$

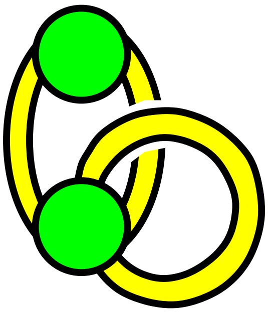
$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d;\qquad J_L(t) = 1$

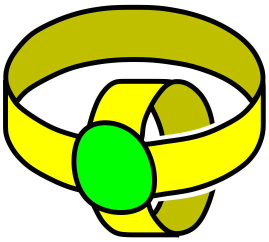
Ribbon graphs

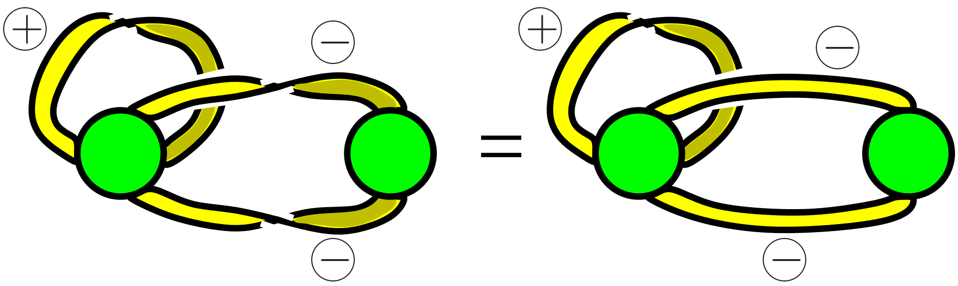
A ribbon graph G is a surface represented as a union of vertices-discs  and edges-ribbons 

- discs and ribbons intersect by disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.

Examples







The Bollobás-Riordan polynomial

Let
- F be a ribbon graph;
- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;
- $\mathrm{bc}(F)$ be the number of boundary components of F;
- $s(F) := \dfrac{e_-(F) - e_-(\overline{F})}{2}$.

$$R_G(x, y, z) \;:=\; \sum_F x^{r(G)-r(F)+s(F)} y^{n(F)-s(F)} z^{k(F)-\mathrm{bc}(F)+n(F)}$$

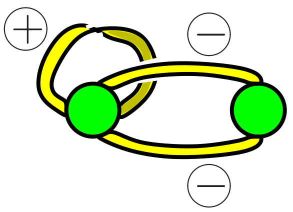
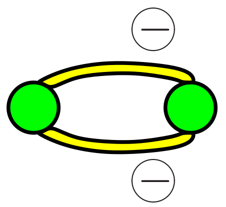
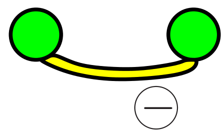
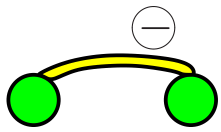
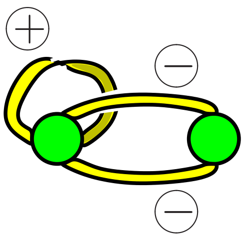
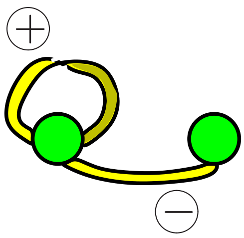
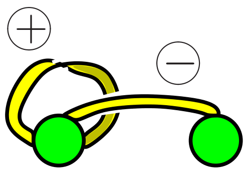
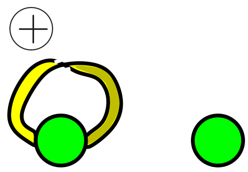
Relations to the Tutte polynomial.

$$R_G(x-1, y-1, 1) = T_G(x, y)$$

If G is planar (genus zero):

$$R_G(x-1, y-1, z) = T_G(x, y)$$

Example

				
$(k, r, n, \mathrm{bc}, s)$	$(1,1,1,2,1)$	$(1,1,0,1,0)$	$(1,1,0,1,0)$	$(2,0,0,2,-1)$
				
	$(1,1,2,1,1)$	$(1,1,1,1,0)$	$(1,1,1,1,0)$	$(2,0,1,2,-1)$

- $r(F) := v(F) - k(F)$;
- $n(F) := e(G) - r(F)$;
- $\mathrm{bc}(F)$ is the number of boundary components;
- $s(F) := \dfrac{e_-(F) - e_-(\overline{F})}{2}$.

$$R_G(x, y, z) = x + 2 + y + xyz^2 + 2yz + y^2 z \ .$$

Theorem

Let L be a virtual link diagram, G_L be the corresponding signed ribbon graph, and $n := n(G_L)$, $r := r(G_L)$, $k := k(G_L)$. Then

$$[L] = A^n B^r d^{k-1} R_{G_L}\left(\frac{Ad}{B}, \frac{Bd}{A}, \frac{1}{d}\right) .$$