Polyak-Viro formulas for coefficients of the Conway polynomial

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(joint work with Michael (Cap) Khoury, Alfred Rossi)

This work has been done by students Michael (Cap) Khoury and Alfred Rossi during the Summer 2006 VIGRE working group "Knots and Graphs": (`http://www.math.ohio-state.edu/~chmutov/wor-gr-su06/wor-gr.htm`) at the Ohio State University, funded by NSF grant DMS-0135308. The problem was to describe Polyak-Viro arrow diagram formulas for coefficients of the Conway polynomial. Later we realized that this description is equivalent to F. Jaeger's state model [Ja].

1. Jaeger's state model for the Conway polynomial

Let us first reformulate the Jaeger model on a language suitable for our purpose.

A subset S of the crossings of a knot diagram K is said to be *one-component* if the curve obtained from K by smoothing all crossings of S according to orientation has one component.

Assume that the diagram K has a base point and S is a one-component subset of crossing. Let us travel along K starting with the base point. Suppose we approach to the first crossing of the subset S along an overpass. Let us jump down to the underpass and continue to travel along K (more precisely along the oriented smoothing of K at the crossing). Repeat the procedure for the remaining crossings of S. If it is possible to trace the whole curve K like this, always jumping down at the first approaching to a crossing of S and jumping up at the second approaching to it, then the subset S is said to be *jump down*.

Define the *down* polynomial, in variable t, as

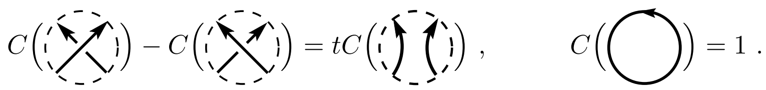
$$C_{\mathrm{down}}(K) \quad := \sum_{\substack{S \quad \text{jump down} \\ \text{one-component}}} \Big(\prod_{\mathsf{x}\in S} \mathrm{wr}(\mathsf{x})\Big)\, t^{|S|} \ ,$$

where $\mathrm{wr}(\mathsf{x})$ is the local writhe of the crossing x. If S is the empty set, then we set the product to be equal 1 by definition. Therefore the free term of $C_{\mathrm{down}}(K)$ always equals 1.

Corollary of the main Theorem. *The Conway polynomial $C(K)$ a knot K is equal to the down polynomial of its diagram,*

$$C(K) = C_{down}(K) \ .$$

Let us remind that the Conway polynomial is defined by the equations



Similarly to $C_{\mathrm{down}}(K)$, one can define the *up* polynomial $C_{\mathrm{up}}(K)$. It turns out that for all classical knots $C_{\mathrm{down}}(K) = C_{\mathrm{up}}(K)$. However this fails for virtual knots.

2. Gauss diagrams and Polyak-Viro formulas

Definition 2.1. *A Gauss diagram is a chord diagram with oriented chords and with numbers* $+1$ *or* -1 *assigned to each chord.*

With a knot diagram we associate a Gauss diagram whose outer circle is the parameterizing circle S^1 of our knot, a chord is drawn for each double point of the diagram, each chord is oriented from the overpass to the underpass and the local writhe number is assigned to each double point (chord). M. Polyak and O. Viro suggested [PV] the following approach to represent knot invariants in terms of Gauss diagrams.

Definition 2.2. *An arrow diagram is a based chord diagram with oriented chords.*

Definition 2.3. *Let* A *be an arrow diagram and let* G *be a Gauss diagram, both with base points. A homomorphism* φ *from* A *to* G*,* $\varphi \in \mathrm{Hom}(A, G)$*, is an injective map of chords of* A *to chords of* G *which respects the orientation of the chords and their positions to the base points.*

Definition 2.4. *The pairing between a based arrow diagram and a based Gauss diagram is defined by*

$$\langle A, G\rangle := \sum_{\varphi\in \mathrm{Hom}(A,G)} \ \prod_{c \text{ chord in } A} \mathrm{sign}(\varphi(c)) \ .$$

In general, if you take an arbitrary arrow diagram A, the value $\langle A, G(K)\rangle$ is not uniquely defined by the knot K. Nevertheless, for we can extend the pairing to a linear combination of arrow diagrams

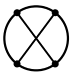
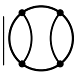
$$\langle \sum_i \lambda_i A_i, G\rangle := \sum_i \lambda_i \langle A_i, G\rangle$$

by linearity. Then some linear combinations of arrow diagrams may yield knot invariants by this construction. Moreover, with a slight generalization of arrow diagrams pairing, there is a general theorem due to M. Goussarov [G, GPV] stating that any Vassiliev invariant can be obtained from a suitable linear combination of arrow diagrams (possibly with signed chords).

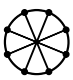
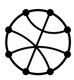
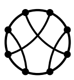
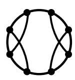
3. Main Theorem

Definition 3.1. *A chord diagram* D *is said to be one-component if after parallel doubling of each chord the resulting curve will have one component,* $|D| = 1$*.*

Example 3.2. There is only one one-component chord diagram with two chords:

$\Big|$  $\Big| = 1 \Longleftarrow$  , $\Big|$  $\Big| = 3 \Longleftarrow$  .

With four chords, there are four one-component diagrams:

$d^4_1 =$  , $d^4_5 =$  , $d^4_6 =$  , and $d^4_7 =$  .

Definition 3.3. Choosing a base point we can turn a one-component chord diagram into an arrow diagram according to the following rule. *Starting from the base point we travel along the diagram with doubled chords. In this journey we pass both copies of each chord in opposite directions. Choose an arrow on a chord which correspond to the direction of the first passage of the copies of the chord.* Here is an example.

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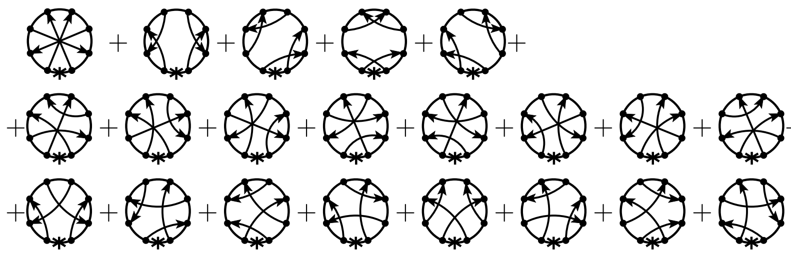
Definition 3.4. Let us define the *Conway combination* $\mathfrak{C}_{2n}$ of arrow diagrams as a sum of all based arrow diagrams with $2n$ arrows obtained from one-component chord diagrams by the rule above. For example,

$\mathfrak{C}_2 \;:=$  ,

$\mathfrak{C}_4 \;:=$  .

Note that for a given one-component chord diagram we have to consider all possible choices for the base point. However, some choices may lead to the same arrow diagram. In $\mathfrak{C}_{2n}$ we list them without repetitions.

Main Theorem. *For $n \geqslant 1$, the coefficient c_{2n} of t^{2n} in the Conway polynomial of a knot K with the Gauss diangram G is equal to*

$$c_{2n} = \langle \mathfrak{C}_{2n}, G \rangle \ .$$

References

[G] M. Goussarov, *Finite type invariants are presented by Gauss diagram formulas*, Sankt-Petersburg Department of Steklov Mathematical Institute preprint (translated from Russian by O. Viro), December 1998.
`http://www.math.toronto.edu/~drorbn/Goussarov/`, see also
`http://www.pdmi.ras.ru/~duzhin/VasBib/`.

[GPV] M. Goussarov, M. Polyak and O. Viro, *Finite type invariants of classical and virtual knots*, *Topology* **39** (2000) 1045–1068.

[Ja] F. Jaeger, *A combinatorial model for the Homfly polynomial*, *European J.Combinatorics* **11** (1990) 549–558.

[PV] M. Polyak and O. Viro, *Gauss diagram formulas for Vassiliev invariants*, Int. Math. Res. Notes **11** (1994) 445–454.