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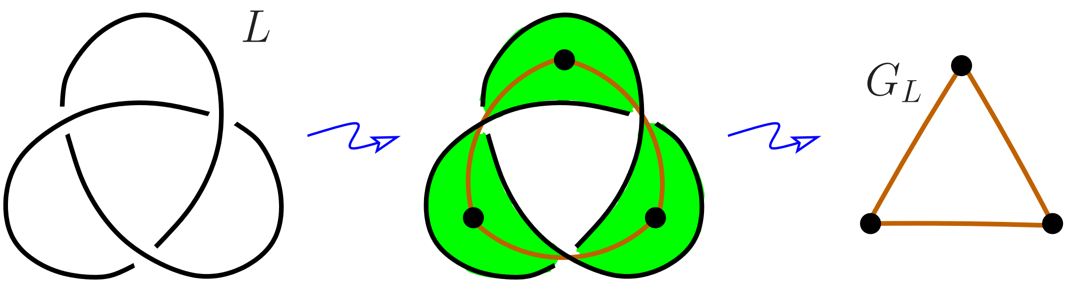
Up to a sign and a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_{G_L}(-t, -t^{-1})$.

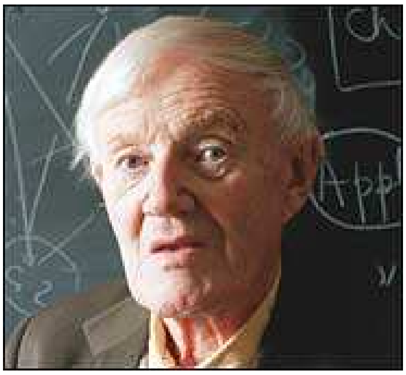


$$V_L(t) = t + t^3 - t^4 = -t^2(-t^{-1} - t + t^2)$$

$$T_{G_L}(x, y) = y + x + x^2$$

$$T_{G_L}(-t, -t^{-1}) = -t^{-1} - t + t^2$$

The Tutte polynomial



Let • F be a graph;

- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;

$$T_G(x,y) \;:=\; \sum_{F \subseteq E(G)} (x-1)^{r(G)-r(F)}(y-1)^{n(F)}$$

Properties.

$T_G = T_{G-e} + T_{G/e}$	if e is neither a bridge nor a loop ;
$T_G = xT_{G/e}$	if e is a bridge ;
$T_G = yT_{G-e}$	if e is a loop ;
$T_{G_1 \sqcup G_2} = T_{G_1 \cdot G_2} = T_{G_1} \cdot T_{G_2}$	for a disjoint union, $G_1 \sqcup G_2$ and a one-point join, $G_1 \cdot G_2$;

$T_\bullet = 1$.

$T_G(1,1)$ is the number of spanning trees of G ;

$T_G(2,1)$ is the number of spanning forests of G ;

$T_G(1,2)$ is the number of spanning connected subgraphs of G ;

$T_G(2,2) = 2^{|E(G)|}$ is the number of spanning subgraphs of G .

The Bollobás-Riordan polynomial

Let • F be a ribbon graph;

- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;
- $\mathrm{bc}(F)$ be the number of boundary components of F;
- $s(F) := \dfrac{e_-(F) - e_-(\overline{F})}{2}$.

$$R_G(x,y,z) \ := \sum_F x^{r(G)-r(F)+s(F)} y^{n(F)-s(F)} z^{k(F)-\mathrm{bc}(F)+n(F)}$$

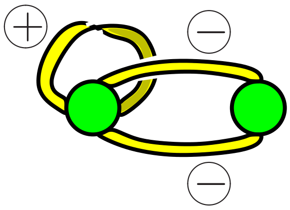
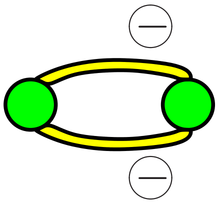
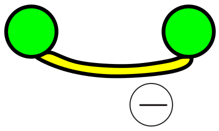
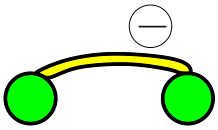
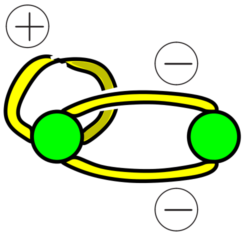
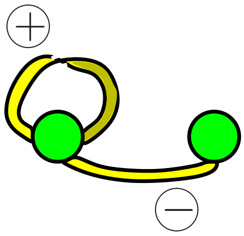
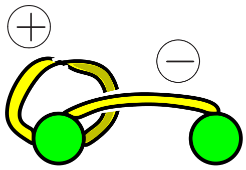
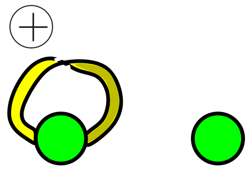
Relations to the Tutte polynomial.

$$R_G(x-1, y-1, 1) = T_G(x,y)$$

If G is planar (genus zero):

$$R_G(x-1, y-1, z) = T_G(x,y)$$

Example.

				
$(k, r, n, \mathrm{bc}, s)$	$(1, 1, 1, 2, 1)$	$(1, 1, 0, 1, 0)$	$(1, 1, 0, 1, 0)$	$(2, 0, 0, 2, -1)$
				
	$(1, 1, 2, 1, 1)$	$(1, 1, 1, 1, 0)$	$(1, 1, 1, 1, 0)$	$(2, 0, 1, 2, -1)$

- $r(F) := v(F) - k(F)$;
- $n(F) := e(F) - r(F)$;
- $\mathrm{bc}(F)$ is the number of boundary components;
- $s(F) := \dfrac{e_-(F) - e_-(\overline{F})}{2}$.

$$R_G(x, y, z) = x + 2 + y + xyz^2 + 2yz + y^2 z \ .$$

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