Partial duality of graphs on surfaces

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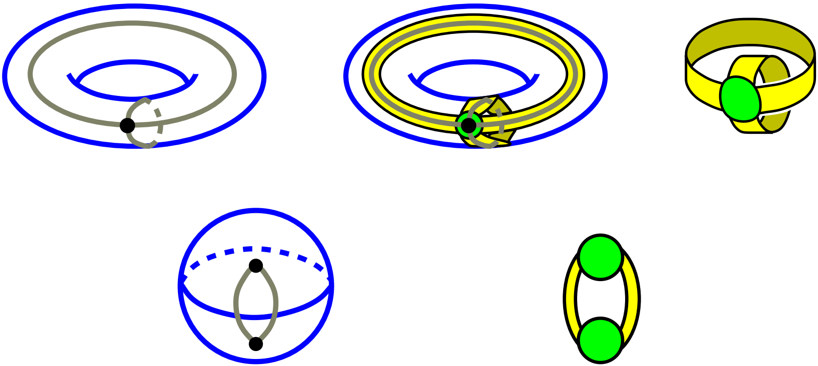
Special Session on Graphs Embedded in Surfaces, and Their Symmetries

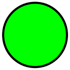
Saturday, October 2, 2010
5:00 — 5:20 p.m.

Graphs on surfaces

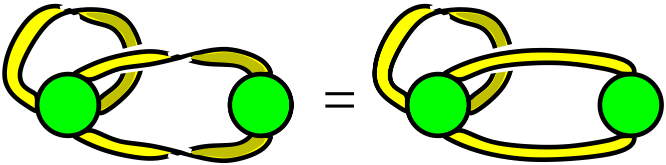


Ribbon graphs

A ribbon graph R is a surface represented as a union of

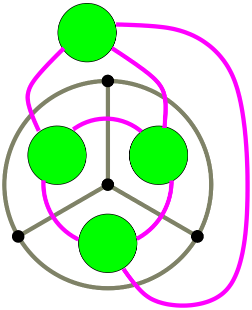
vertices-discs  and edges-ribbons 

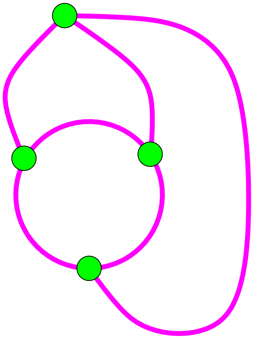
- discs and ribbons intersect by disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.

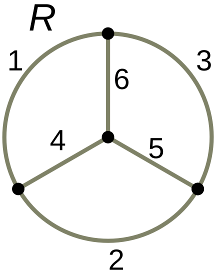


Duality

$R^* = R^{\{1,2,3,4,5,6\}}$

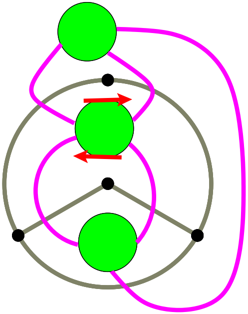


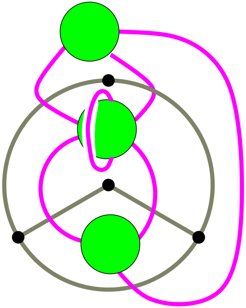




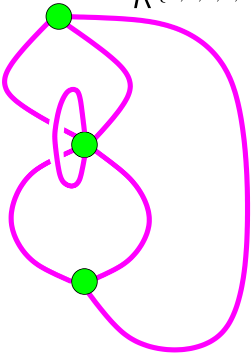
$$R^{\{1,2,3,4,5\}} = ???$$

Partial duality

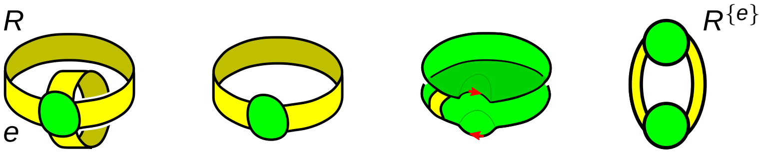


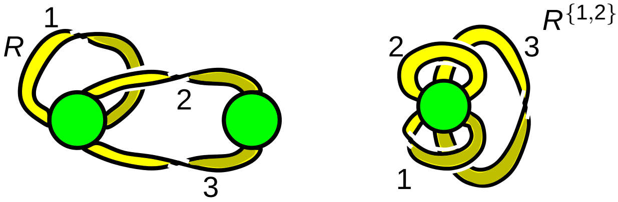


$R^{\{1,2,3,4,5\}}$



Examples





Properties

Let $A \subseteq E(R)$ for a ribbon graph R.

(a) $R^{\emptyset} = R$.

(b) $R^{E(R)} = R^*$.

(c) $\left(R^A\right)^A = R$.

(d) For an edge $e \notin A$, $R^{A \cup \{e\}} = \left(R^A\right)^{\{e\}} = \left(R^{\{e\}}\right)^A$.

(e) $\left(R^A\right)^{A'} = R^{(A \cup A') \setminus (A \cap A')}$.

(f) Partial duality preserves orientability.

Bollobás-Riordan polynomial

Let F be a ribbon graph;

- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;
- $\mathrm{bc}(F)$ be the number of boundary components of F;

$$B_R(X,Y,Z) := \sum_{F\subseteq R} \Big(\prod_{e\in F} x_e\Big)\Big(\prod_{e\in R\setminus F} y_e\Big) X^{r(R)-r(F)} Y^{n(F)} Z^{k(F)-bc(F)+n(F)}$$

Relation to the Tutte polynomial

$x_e = y_e = 1$

$$B_R(x-1, y-1, 1) = T_R(x, y)$$

If R is planar (genus zero): $B_R(x-1, y-1, z) = T_R(x, y)$

Signed graphs:

$+$-edge: $x_e := 1, \quad y_e := 1.$

$-$-edge: $x_e := \sqrt{X/Y}, \qquad y_e := \sqrt{Y/X}.$

Duality theorem

The restriction of the polynomial $(YZ)^{v(R)}B_R(X,Y,Z)$ *to the surface* $XYZ^2=1$ *is invariant under the partial duality:*

$$(YZ)^{v(R)}B_R(X,Y,Z)\Big|_{XYZ^2=1} = (YZ)^{v(R')}B_{R'}(X,Y,Z)\Big|_{XYZ^2=1}$$

where $R' := R^A$ *with the weights correspondence*

$$x'_e = \begin{cases} x_e & \text{if } e \notin A \\ y_e XZ & \text{if } e \in A, \end{cases} \qquad y'_e = \begin{cases} y_e & \text{if } e \notin A \\ x_e YZ & \text{if } e \in A. \end{cases}$$

Idea of the proof

$$(YZ)^{v(R)}B_R(X,Y,Z) \;=\; \sum_F M_R(F)$$

The weight correspondence gives

a one-to-one correspondence:
$$\begin{array}{ccc} F & \subseteq & E(R) \\ \updownarrow & & \| \\ F' & \subseteq & E(R') \end{array}$$

$$F' = (F \cup A) \setminus (F \cap A)\,.$$

$$M_R(F)\Big|_{XYZ^2=1} \;=\; M_{R'}(F')\Big|_{XYZ^2=1}\,.$$

Corollaries

Corollary 1. *Let* $g := k(R) - \chi(\widetilde{R})/2$, *where* $\widetilde{R}$ *is a closed surface obtained from* R *by capping all boundary components. Then*

$$X^g B_R(\{x_e, y_e\}, X, Y, Z)\Big|_{XYZ^2=1} = Y^g B_{R^*}(\{y_e, x_e\}, Y, X, Z)\Big|_{XYZ^2=1}$$

Corollary 2. *Let* R *be a connected plane graph. Then*

$$T_R(x, y) = T_{R^*}(y, x)$$