Combinatorics and topology of graphs on surfaces

IUPUI Colloquium
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Two parts: ① Combinatorics and ② Topology

replace X by H

① **Matroids** M

Def. 1 (Rank) $r: 2^M \to \mathbb{Z}$

(R1): $r(\emptyset)=0$. (R2): $r(X \cup y) = \begin{cases} r(X), \text{ or} \\ r(X)+1 \end{cases}$ $y \notin X$

(R3): $y, z \notin X$, $r(X\cup y) = r(X \cup z) = r(X) \Rightarrow r(X \cup \{y,z\}) = r(X)$.

skip

Def. 2 (Independent sets $\mathcal{I} \subset 2^M$)

(I1) $X \subset Y \in \mathcal{I} \Rightarrow X \in \mathcal{I}$

(I2) $X, Y \in \mathcal{I}$, $|X| = |Y|+1 \Rightarrow \exists x \in X \setminus Y : Y \cup x \in \mathcal{I}$

Def 3. Circuits $\mathcal{C} \subset 2^M$

(C1) No proper subset of a circuit is a circuit
$X \subsetneq Y \in \mathcal{C} \Rightarrow X \notin \mathcal{C}$

(C2) $C_1 \neq C_2 \in \mathcal{C}$, $c \in C_1 \cap C_2 \Rightarrow \exists C_3 \in \mathcal{C} : C_3 \subset (C_1 \cup C_2) \setminus c$

Def 4 Bases $\mathcal{B} \subset 2^M$

(B1) $X \subsetneq Y \in \mathcal{B} \Rightarrow X \notin \mathcal{B}$

(B2) $B_1, B_2 \in \mathcal{B}$, $b_1 \in B_1 \setminus B_2 \Rightarrow \exists b_2 \in B_2 \setminus B_1$:
$(B_1 \setminus b_1) \cup b_2 \in \mathcal{B}$

X is independent if $r(X) = |X|$, maximal independent set is a base

X is a circuit if $r(X) = |X|-1$

Matroids ← H. Whitney 1935

graphs → Matroids

set of vectors in a vector space → Matroids

if G is a graph

$M = E(G) = \mathcal{C}(G)$ cycle matroid of G

$r(H) := v(H) - c(H)$

v: # vertices, c: # connected components.

Bond matroid $\mathcal{B}(G)$ on $E(G)$; (circuits of $\mathcal{B}(G)$) = (bonds of G)

~~a base B of $\mathcal{B}$~~

$B \subset E(G)$ is a base of $\mathcal{B}(G)$ $\iff$ $E(G)\setminus B$ is a base of $\mathcal{C}(G)$

Rank reformulation: $r_{M^*}(X) = |X| + r_M(M\setminus X) - r_M(M)$

$\mathcal{B}(G) = (\mathcal{C}(G))^*$ $\qquad \Downarrow$

$r_M(M) + r_{M^*}(M^*) = |M|$

Whitney planarity criteria $\qquad$ example

G is planar $\iff$ $\mathcal{B}(G) = \mathcal{C}(G^*)$ $\qquad G =$  $\quad G^* =$ 

$r(\mathcal{C}(G)) = 0 \qquad r(\mathcal{B}(G)) = 2$

Matroid perspective $\qquad M \xrightarrow{1-1} M'$

(any circuit of M) $\longmapsto$ $\bigcup$(circuits of M')

$r_M(X) - r_M(Y) \geq r_{M'}(X) - r_{M'}(Y)$ for $Y \subseteq X$

M. Las Vergnas $\quad T_{M\to M'} := \sum_{H \subseteq M} (x-1)^{r_{M'}(M') - r_{M'}(H)} (y-1)^{n_M(H)} z^{(r_M(M) - r_M(H)) - (r_{M'}(M') - r_{M'}(H))}$

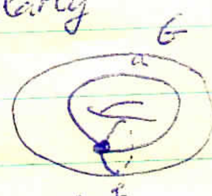
$n_M(H) := |H| - r_M(H)$

Properties Tutte polynomial

- $T_M(x,y) = T_{M\to M}(x,y,z)$
- $T_M(x,y) = T_{M\to M'}(x,y,x-1)$
- $T_{M'}(x,y) = (y-1)^{r_M(M) - r_{M'}(M')} \, T_{M\to M'}\left(x,y,\frac{1}{y-1}\right)$

Example $\qquad G \hookrightarrow \Sigma$ orientable surface

cellularly



$M := \mathcal{B}(G^*) \longrightarrow M' := \mathcal{C}(G)$

$$LV_{G\subset\Sigma}(x,y,z) = T_{\mathcal{B}(G^*)\to\mathcal{C}(G)}(x,y,z)$$

For example $LV = z^2+2z+1 = (z+1)^2$

② Krushkal polynomial

$$P_{G\subset\Sigma}(X,Y,A,B) := \sum_{\substack{H\subseteq G \\ \text{spanning} \\ \text{subgraphs}}} X^{c(H)-c(G)}\, Y^{k(H)}\, A^{\overbrace{s(H)/2}^{g(H)}}\, B^{\overbrace{s^{\perp}(H)/2}^{g(\bar{H})}}$$

$c(H) = \#$ connected components

$k(H) = \dim\left(\ker\left(H_1(H,\mathbb{R}) \longrightarrow H_1(\Sigma,\mathbb{R})\right)\right)$

$s(H) = 2\cdot g(H)$

$$V = \operatorname{im}\left(H_1(H,\mathbb{R}) \to H_1(\Sigma,\mathbb{R})\right) \subset H_1(\Sigma,\mathbb{R})$$

$g(H) = \frac{1}{2}\dim\left(V/V\cap V^{\perp}\right)$, $\quad g(\bar{H}) = \frac{1}{2}\dim\left(V^{\perp}/V\cap V^{\perp}\right)$

In example $\quad P = B+2+A$

③ Theorem $\quad LV_{G\subset\Sigma}(x,y,z) = z^{g}\cdot P_{G\subset\Sigma}(x-1,y-1,z^{-1};z)$

Lemma 1: $\quad k(H) = n_M(H)$

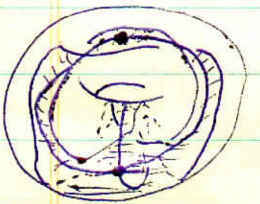
Lemma 2: $\quad 2g = r_M(G) - r_{M^*}(G)$

Lemma 3: $\quad g + g(H) - g(\bar{H}) = r_M(H) - r_{M^*}(H)$

4 Kruskal polynomial in knot theory

Kauffman bracket of a alternating link in a thickened surface.

$$[L](A,B,d) = A^{g+v(G)-e(G)} B^{-g+n(G)} d^{g+c(G)-1} * P_{G_L \hookrightarrow \Sigma}\left(\frac{Bd}{A}, \frac{Ad}{B}, \frac{A}{Bd}\frac{B}{Ad}\right)$$



checkerboard coloring $\longrightarrow$

$L \hookrightarrow \Sigma \times I$
maximal embedding
(each region is a disk)

G_L
Tait graph.

$$J_L(t) = (-1)^w t^{3w/4} [L]\left(t^{-1/4}, \cancel{t}^{1/4}, -d^{1/2} - d^{-1/2}\right)$$