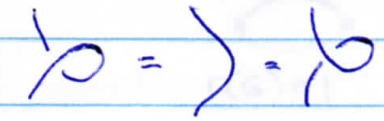
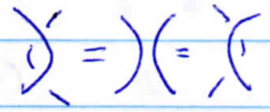
Invitation to Research. OSU Feb. 15, 2012
4:30–5:30
CH 240

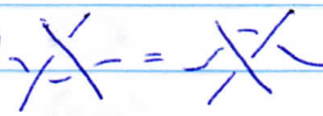
Combinatorics of knots

Planar isotopy

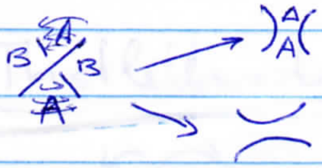
Reidemeister moves

I 

II 

III 

The Jones polynomial



A-splitting

B splitting

A ~~states~~ state is choise of a splitting at every crossing

$\alpha(S) = \#$ A splittings

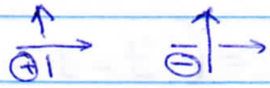
$\beta(S) = \#$ B splitting

$\delta(S) = \#$ circles ~~of~~ the splitted curve

Kauffman bracket

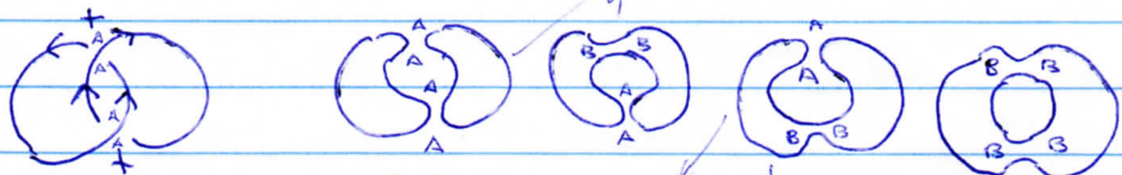
$$\underset{\text{link}}{L} \rightsquigarrow [L](A,B,d) := \sum_S A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1}$$

Writhe of L $w(L) := \sum_{\text{crossings}} \pm 1$,



$$J_L(t) := (-1)^{w(L)}\, t^{3w(L)/4}\, [L]\left(t^{-1/4}, t^{1/4}, -t^{1/2}-t^{-1/2}\right)$$

Example



$w(L) = 2$ $\quad A^2 d \quad AB \quad AB \quad B^2 d$

$$J_L(t) = t^{3/2}\left(t^{-1/2}(-t^{1/2}-t^{-1/2}) + 2 + t^{1/2}(-t^{1/2}-t^{-1/2})\right)$$

$$= t^{3/2}\left(\cancel{-1} - t^{-1} \cancel{+2} - t^{1} \cancel{-1}\right) = \boxed{-t^{1/2} - t^{5/2}}$$

The Tutte polynomial ~~of~~

G a graph.

$$T_G(x,y) := \sum_{\substack{F\subseteq E(G)\\ \text{spanning subgraph}}} (x-1)^{r(G)-r(F)}\,(y-1)^{n(F)}$$

where $r(F) = \#\text{vertices} - k$, $k = \#\text{con. comp.}$

$n(F) = \#\text{edges} - r(F)$

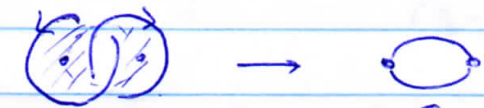
$= \beta_1(F)$

Example

$r(G)=1$ $\quad (x-1) \quad 1 \quad 1 \quad (y-1)$

$$T_G(x,y) = x+y$$

Thistlethwaite Theorem



L $\quad$ G

$$J_L(t) = (-1)^w\, t^{\frac{3w - r(G) + n(G)}{4}} \left(-t^{\cancel{3}\,1/2} - t^{-1/2}\right)^{k(G)-1} T_G(-t,-t^{-1})$$

For example: $w(\cancel{L})=2$

$r(G)=1$

$n(G)=1$

$$J_L(t) = t^{\frac{6-1+1}{4}} \cdot (-t - t^{-1}) = t^{3/2}(-t-t^{-1})$$

$$[L](A,B,d) = A^{r(G)}\,B^{n(G)}\,d^{k(G)-1}\,T_G\left(\frac{Bd}{A}+1, \frac{Ad}{B}+1\right)$$

Khovanov Homology

Lecture 2
Feb. 22, 2012
4:30 – 5:30
CH 240

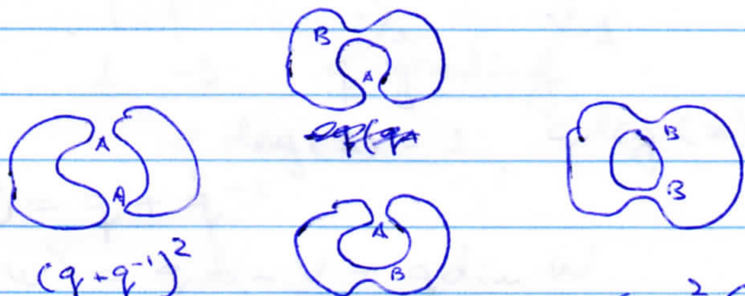
$$A := (-q)^{-1/2}\ ; \quad B := (-q)^{1/2}\ ; \quad d = q + q^{-1}$$

$$q = -A^{-2} \quad = \quad q = -B^2 \qquad\qquad q = -t^{1/2}$$

angulat bracket $\langle L\rangle(q) := (-q)^{n/2}(q+q^{-1})[L](q)$

$$:= \sum_s (-q)^{\beta(s)}(q+q^{-1})^{\delta(s)}$$

Example



$(q+q^{-1})^2$ $\quad$ $2(-q)(q+q^{-1})$ $\quad$ $(-q)^2(q+q^{-1})^2$

$$\langle L\rangle = q^2 \cancel{+2} + q^{-2} \cancel{-2q^2} \cancel{-2} + q^4 \cancel{+2q^2} + 1 = (q+q^{-1})(\cancel{q} + q^{-1} \cancel{-2q} + q^2(q+q^{-1}))$$

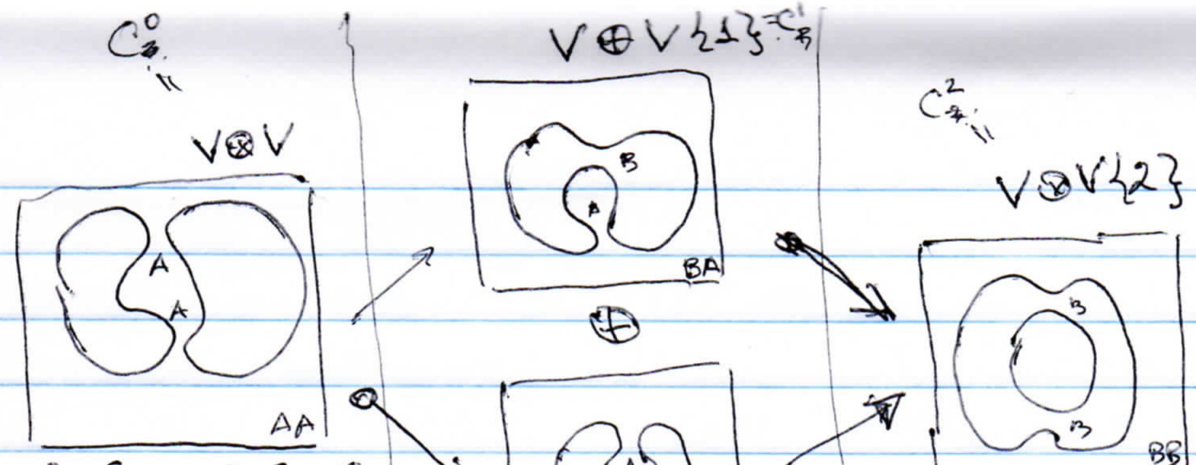
$$= q^4 + q^2 + 1 + q^{-2} = (1+q^{-2})(q^4+1) \qquad n_- = 0,\ n_+ = 2$$

$$\boxed{J_L(q) = (-1)^{n_-} q^{n_+ - 2n_-}(q+q^{-1})^{-1} \cdot \langle L\rangle} = q^2(q+q^{-1})^{-1}(q^4+q^2+1+q^{-2})$$

$$= q^{8}\cancel{(q+q^{-2})^2}(1+q^4) = -t^{1/2}\cancel{(1+t^{-1})^2}(1+t^2)$$

$$\cancel{= -t^{1/2}(1+2t^{-1}+t^{-2})(1+t^2)}$$

$$\cancel{= -t^{3/2}(1+2t^{-1}+t^{-2}+t^2+2t+1)}$$



$C^0 = C^0_2 \oplus C^0_0 \oplus C^0_{-2}$

circle $\langle \text{OO} \rangle \; \langle \text{OO}, \text{OO} \rangle \; \{\text{OO}\}$

$\bigcirc : V := \mathbb{R}[x]/(x^2) = \langle 1, x \rangle = \langle 1 \rangle \oplus \langle x \rangle$

$\langle 1 \rangle = V_1, \quad \langle x \rangle = V_{-1}$

$1 \quad -1$ q-grading

$q\deg(1) = 1, \quad q\deg(x) = -1$

$$q\text{-}\dim(V) = q + q^{-1}$$

$$q\dim(V \oplus W) = q\dim V + q\dim W$$

$$q\dim(V \otimes W) = q\dim V \cdot q\dim W$$

$$W_m = (W\{\ell\})_{m+\ell}$$

$$q\dim C^0_* = (q + q^{-1})^2$$

$$q\dim C^1 = 2q(q + q^{-1})$$

$$q\dim C^2 = q^2(q + q^{-1})^2$$

$$\chi(C^\bullet_*) = \langle L \rangle = q^4 + q^2 + 1 + q^{-2}$$

d. iterated

Multiplication m in V: $\quad m : V \otimes V \longrightarrow V$

$$1 \otimes 1 \longmapsto 1$$
$$1 \otimes x \longmapsto x$$
$$x \otimes 1 \longmapsto x$$
$$x \otimes x \longmapsto 0$$

Comultiplication Δ in V: $\quad \Delta : V \longrightarrow V \otimes V$

$$1 \longmapsto 1 \otimes x + x \otimes 1$$
$$x \longmapsto x \otimes x$$

$C^i_j \to C^{i+1}_j$

$H^{i,j}(L)$

$j \backslash i$	0	1	2
4			$\mathbb{R}$
2			$\mathbb{R}$
0	$\mathbb{R}$		
−2	$\mathbb{R}$		

$$\chi(H^{\bullet,\bullet}) = q^4 + q^2 + 1 + q^{-2}$$