Quantum mathematics

$\sqrt{\pi}$

Wed. Oct. 10, 5:00

2000 — E. Witten (1990 + Drinfeld, Jones)

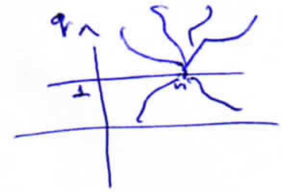
1900 — H. Poincaré

1800 — Gauss

1687 I. Newton

math

$$[n] := \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \dots + q^{n-1}$$



Quantum oscillator

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \qquad q = e^{\hbar}$$

$$\hbar \sim 10^{-34}$$

$[1] = 1$

$[2] = 1 + q$

$[3] = 1 + q + q^2 \quad \dots$

$$[n]! := [1]\cdot[2]\dots[n]$$

q-Pascal triangle $ab = ba$

$$(a+b)_q^n = \sum_{j=0}^{n} q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix} a^{n-j} b^j$$

If $yx = qxy$

$$(x+y)^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} x^j y^{n-j}$$

$$(x-a)_q^n = (x-a)(x-qa)\dots(x-q^{n-1}a)$$

Derivative $\quad f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

q-derivative

$$(D_q f) := \frac{f(qx) - f(x)}{qx - x} = \frac{f(qx)-f(x)}{(q-1)x} \Rightarrow D_q(\text{const}) = 0, \quad D_q(x) = 1$$

$$D_q x^n = [n]x^{n-1}$$

Product rule:

$$D_q(f(x)\cdot g(x)) = f(qx)D_q(g(x)) + g(x)D_q(f(x))$$

$$= f(x)D_q(g(x)) + g(qx)D_q(f(x))$$

$$D_q\left((x-a)_q^n\right) = D_q\left((x-a)_q^{n-1}(x-q^{n-1}a)\right) =$$

$$= (qx-a)_q^{n-1} D_q(x - q^{n-1}a) + D_q\left((x-a)_q^{n-1}\right)(x-q^{n-1}a)$$

$$= (qx-a)q^{n-2}(x-a)_q^{n-2} + \underbrace{D_q\left((x-a)_q^{n-1}\right)}_{[n-1](x-a)_q^{n-2}}(x-q^{n-1}a)$$

$$[n-1]\underbrace{(x-a)_q^{n-2}(x-q^{n-1}a)}$$

$$= (x-a)_q^{n-2}\left(q^{n-1}x - q^{n-2}a + [n-1](x-q^{n-1}a)\right)$$

$(x-a)_q^{n-1}$

$$= (x-a)_q^{n-2}\left(\frac{q^n x - \cancel{q^{n-1}a} - \cancel{q^{n-1}x} + q^{n-2}a + \cancel{q^{n-1}x} - q^{2n-2}a - x + \cancel{q^{n-1}a}}{q-1}\right)$$

$$= (x-a)_q^{n-2}\cdot\frac{(q^n-1)\,x - q^{n-2}a\,(q^n-1)}{q-1} = [n]\cdot(x-a)_q^{n-1}$$

Taylor's formula $\underset{\text{pol. of deg } N}{f(x)} = \sum_{j=0}^{N} \frac{D_q^j f(c)}{[j]!}(x-c)_q^j$

q-exp function

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]!}\,, \qquad E_q^x = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}}\frac{x^j}{[j]!} = e_{1/q}^x$$

$$D_q e_q^x = e_q^x \qquad\qquad D_q E_q^x = E_q^{qx}$$

q-trig functions

$$\sin_q x = \frac{e_q^{ix} - e_q^{-ix}}{2i}\,, \qquad \mathrm{Sin}_q x = \frac{E_q^{ix} - E_q^{-ix}}{2i} = \sin_{1/q} x$$

$$\cos_q x = \frac{e_q^{ix} + e_q^{-ix}}{2}\,, \qquad \mathrm{Cos}_q x = \frac{E_q^{ix} + E_q^{-ix}}{2} = \cos_{1/q} x$$

Trig. Fundamental Identity

$$D_q \sin_q x = \cos_q x$$
$$D_q \cos_q x = -\sin_q x$$

$$\sin_q x\,\mathrm{Sin}_q x + \cos_q x\,\mathrm{Cos}_q x = 1$$

q-antiderivative $F(x) = \int f(x)\,d_q x$

$$D_q F(x) = f(x)$$