Partial duality of hypermaps

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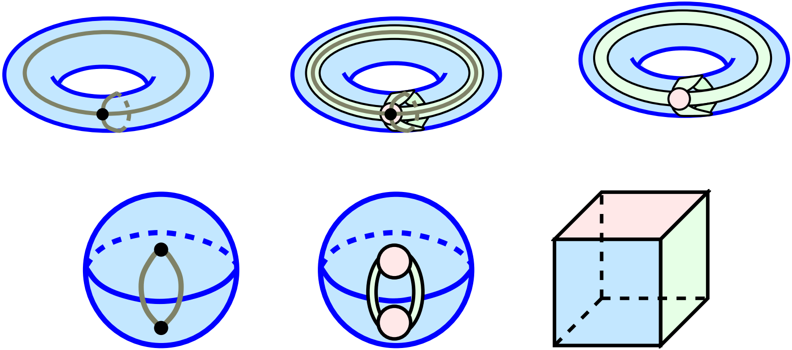
Conference **Legacy of Vladimir Arnold**, Fields Institute, Toronto.

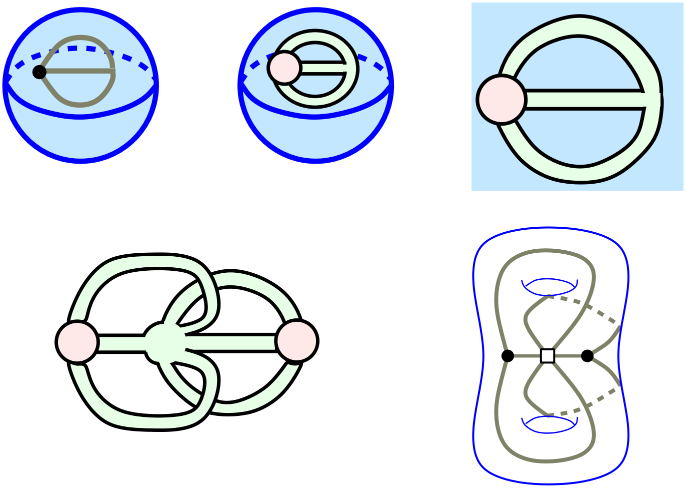
Joint with Fabien Vignes-Tourneret
`arXiv:1409.0632 [math.CO]`

Tuesday, November 25, 2014
9:00–9:30am

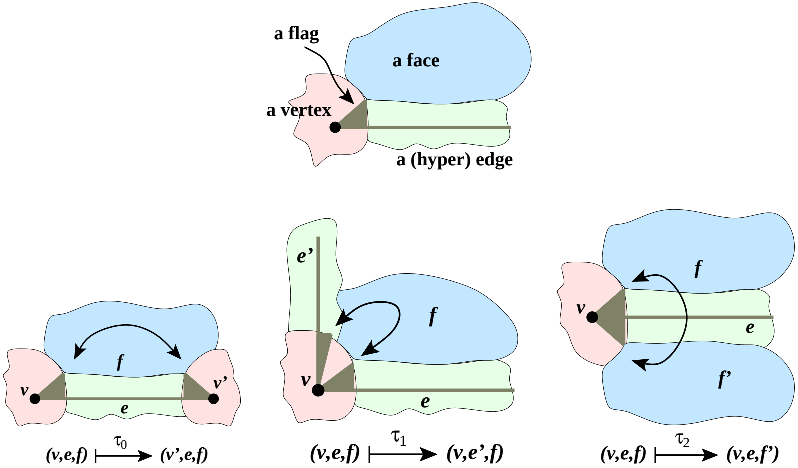
Maps (Graphs on surfaces)



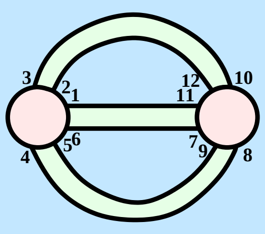
Hypermaps



τ-model for hypermaps



τ-model. Example.

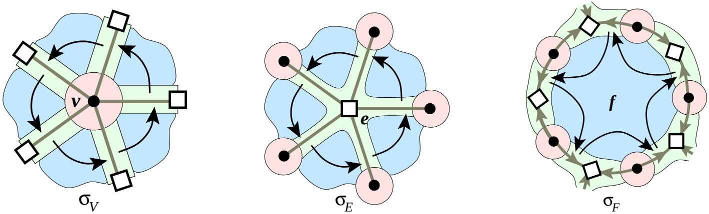


$$\tau_0 = (1, 11)(2, 12)(3, 10)(4, 8)(5, 9)(6, 7)$$

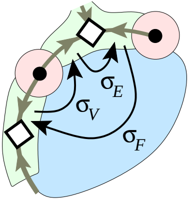
$$\tau_1 = (1, 2)(3, 4)(5, 6)(7, 9)(8, 10)(11, 12)$$

$$\tau_2 = (1, 6)(2, 3)(4, 5)(7, 11)(8, 9)(10, 12)$$

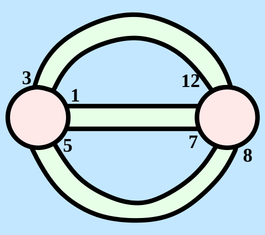
σ-model for oriented hypermaps



$\sigma_F \sigma_E \sigma_V = 1$:



σ-model. Example.

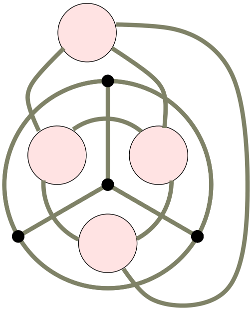


$$\sigma_V = (1,3,5)(7,8,12) = \tau_2\tau_1|_{\{1,3,5,7,8,12\}}$$

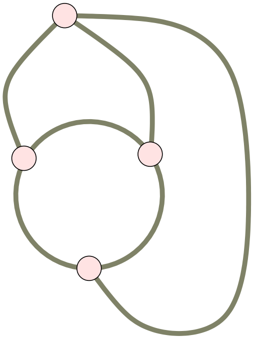
$$\sigma_E = (1,7)(3,12)(5,8) = \tau_0\tau_2|_{\{1,3,5,7,8,12\}}$$

$$\sigma_F = (1,12)(3,8)(5,7) = \tau_1\tau_0|_{\{1,3,5,7,8,12\}}$$

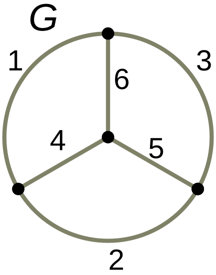
Duality for graphs



$G^* = G^{\{1,2,3,4,5,6\}}$

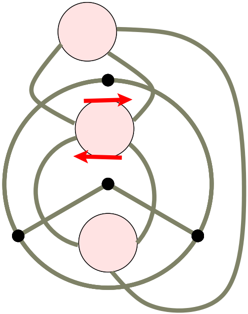


Partial duality for graphs

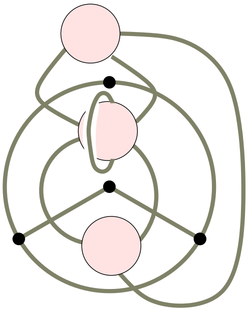


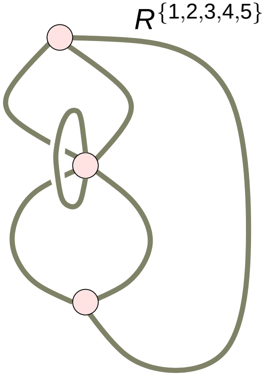
$G^{\{1,2,3,4,5\}} = ???$

Partial duality for graphs (continuation)

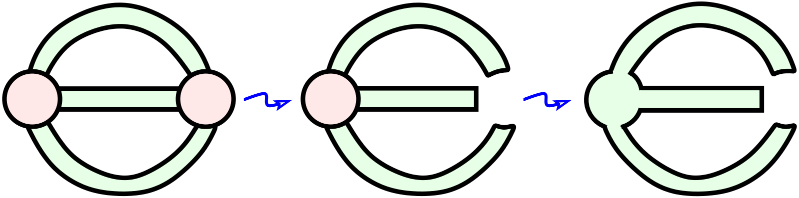


Partial duality for graphs (continuation)

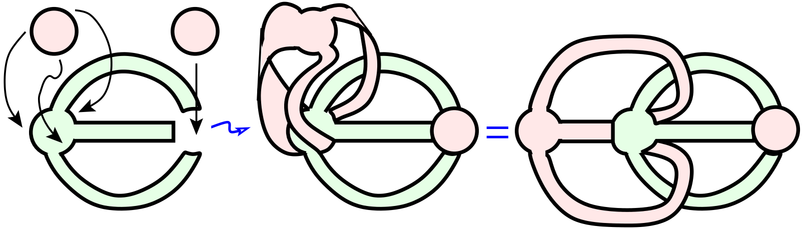




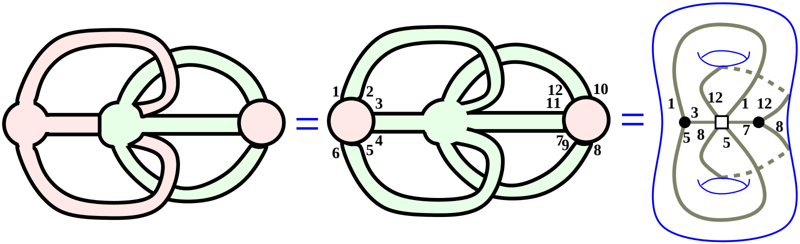
Let S be a subset of the vertex-cells of G.
Choose a different type of cells, say hyperedges.
Step 1. ∂F is the boundary a surface F which is the union of the cells from S and all hyperedge-cells.
Step 2. Glue in a disk to each connected component of ∂F.
These will be the *hyperedge-cells* for G^S.



Step 3. Gluing the vertex-cells.



Step 4. Forming the partial dual hypermap G^S.



Partial duality. Properties.

(a) The resulting hypermap does not depend on the choice of type at the beginning.

(b) $\left(G^S\right)^S = G$.

(c) There is a bijection between the cells of type S in G and the cells of the same type in G^S. This bijection preserves the valency of cells. The number of cell of other types may change.

(d) Is $s \notin S$ but has the same type as the cells of S, then $G^{S\cup\{s\}} = \left(G^S\right)^{\{s\}}$.

(e) $\left(G^S\right)^{S'} = G^{\Delta(S,S')}$, where $\Delta(S,S') := (S\cup S')\setminus(S\cap S')$ is the symmetric difference of sets.

(f) The partial duality preserves orientability of hypermaps.

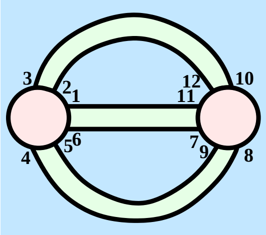
Partial duality in τ-model.

Theorem. *Consider the τ-model for a hypermap G given by the permutations $\tau_0(G) : (v, e, f) \mapsto (v', e, f)$, $\tau_1(G : (v, e, f) \mapsto (v, e', f)$, $\tau_2(G) : (v, e, f) \mapsto (v, e, f')$ of its local flags. Let V' be a subset of its vertices, $\tau_1^{V'}$ be the product of all transpositions in τ_1 for $v \in V'$, and $\tau_2^{V'}$ be the product of all transpositions in τ_2 for $v \in V'$. Then its partial dual $G^{V'}$ is given by the permutations*

$$\tau_0(G^{V'}) = \tau_0, \qquad \tau_1(G^{V'}) = \tau_1\tau_1^{V'}\tau_2^{V'}, \qquad \tau_2(G^{V'}) = \tau_1\tau_1^{V'}\tau_2^{V'}.$$

In other words the permutations τ_1 and τ_2 swap their transpositions of local flags around the vertices in V'. The similar statement hold for partial duality relative to the subset of hyperedges E' and for a subset of faces F'.

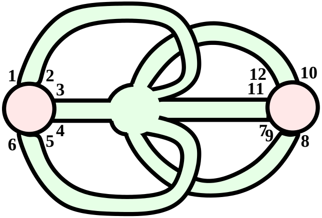
Partial duality in τ-model. Example.



$$\tau_0 = (1,11)(2,12)(3,10)(4,8)(5,9)(6,7)$$

$$\tau_1 = \boxed{(1,2)(3,4)(5,6)}(7,9)(8,10)(11,12)$$

$$\tau_2 = \boxed{(1,6)(2,3)(4,5)}(7,11)(8,9)(10,12)$$



$$\tau_0 = (1,11)(2,12)(3,10)(4,8)(5,9)(6,7)$$

$$\tau_1 = \boxed{(1,6)(2,3)(4,5)}(7,9)(8,10)(11,12)$$

$$\tau_2 = \boxed{(1,2)(3,4)(5,6)}(7,11)(8,9)(10,12)$$

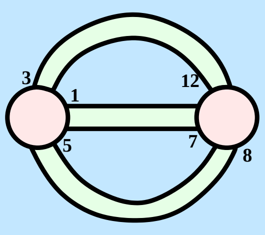
Partial duality in σ-model.

Theorem. *Let S be a subsets $S := V'$ of vertices (resp. subset of hyperedges $S := E'$ and subset of faces $S := F'$) of a hypermap G. Then its partial dual is given by the permutations*

$$
\begin{aligned}
G^{V'} &= (\sigma_{\overline{V'}}\sigma_{V'}^{-1},\ \sigma_E\sigma_{V'},\ \sigma_{V'}\sigma_F) \\
G^{E'} &= (\sigma_{E'}\sigma_V,\ \sigma_{\overline{E'}}\sigma_{E'}^{-1},\ \sigma_F\sigma_{E'}) \\
G^{F'} &= (\sigma_V\sigma_{F'},\ \sigma_{F'}\sigma_E,\ \sigma_{\overline{F'}}\sigma_{F'}^{-1})\ ,
\end{aligned}
$$

where $\sigma_{V'}$, $\sigma_{E'}$, $\sigma_{F'}$ denote the permutations consisting of cycles corresponding to the elements of V', E', F' respectively, and overline means the complementary set of cycles.

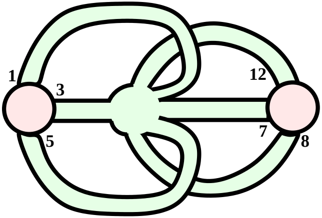
Partial duality in σ-model. Example.



$$\sigma_V = (1,3,5)(7,8,12)$$

$$\sigma_E = (1,7)(3,12)(5,8)$$

$$\sigma_F = (1,12)(3,8)(5,7)$$



$$\sigma_V(G^{\{v\}}) = \sigma_{\overline{V'}}\sigma_{V'}^{-1} = (1,5,3)(7,8,12)$$
$$\sigma_E(G^{\{v\}}) = \sigma_E\sigma_{V'} = (1,12,3,8,5,7)$$
$$\sigma_F(G^{\{v\}}) = \sigma_{V'}\sigma_F = (1,12,3,8,5,7)$$