Université Lyon 1. May 12, 2016

Lecture 1 Graphs and their polynomials

Graphs $G=(V,E)$

Coloring: $c: V \to \{1, \dots, q\}$ in q colors, $q \in \mathbb{N}$

proper coloring $c(v_1) \neq c(v_2)$ if $(v_1, v_2) \in E$

Chromatic polynomial

$\chi_G(q) :=$ # proper colorings of G in q colors.

Properties

- $\chi_G = \chi_{G \setminus e} - \chi_{G/e}$ Example $\triangle$ $q(q-1)(q-2)$
- $\chi_{G_1 \sqcup G_2} = \chi_{G_1} \cdot \chi_{G_2}$
- $\chi_\bullet = q$
- $\chi_G(q) = \sum_{F \subseteq E(G)} (-1)^{|F|} q^{k(F)}$

$k(F)$: # connected components of the spanning subgraph with edge set F.

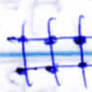
Dichromatic polynomial

$Z_G(q,v) := \sum_{c \in Col_q(G)} (1+v)^{\text{\# edge colored not properly by } c}$

Properties

- $Z_G(q,-1) = \chi_G(q)$
- $Z_G = Z_{G \setminus e} + v Z_{G/e}$
- $Z_{G_1 \sqcup G_2} = Z_{G_1} \cdot Z_{G_2}$
- $Z_\bullet = q$
- $Z_G(q,v) = \sum_{F \subseteq E(G)} q^{k(F)} v^{|F|}$

Potts model



spin $\in \{1, \dots, q\}$, $q=2$ Ising model

ferromagnetic $J_e > 0$

antiferromagnetic $J_e < 0$

Energy of interaction along e $= -J_e$ (coupling constant)

A state = an assignment of spins to all particles = $c \in Col_q(G)$

Energy of state = Hamiltonian = $H(c) = -\sum_{(a,b) \in E(G)} J_e \delta(c(a), c(b))$

Boltzmann weight of c: $e^{-\beta H(c)} = \prod_{(a,b) \in E(G)} e^{J_e \beta \delta(c(a),c(b))}$

inverse temperature $\beta = \frac{1}{\varkappa T}$

$= \prod_{(a,b) \in E(G)} \left(1 + (e^{J_e \beta} - 1)\delta(c(a),c(b))\right)$

$\varkappa$ = Boltzmann constant

Potts partition function $(v_e = e^{J_e\beta} - 1)$

$$Z_G(q, v_e) := \sum_{c\in Col_q(G)} e^{-\beta H(c)} = \sum_{c\in Col_q(G)} \prod_{e\in E(G)} \left(1 + v_e\,\delta(c(a), c(b))\right)$$

if $v_e = v$

$Z_G(q,v)$ dichromatic polynomial of G

Fortuin – Kasteleyn' 1972: $Z_G(q,v_e) = \sum_{F\subseteq E(G)} q^{k(G)} \prod_{e\in F} v_e$

Probability of a state c. $P(c) := e^{-\beta H(c)}/Z_G$

Expected value of a function $f(c)$

$$\langle f\rangle := \sum_c f(c)P(c) = \sum_c f(c)\,\frac{e^{-\beta H(c)}}{Z_G}$$

of the energy

$$\langle H\rangle = \sum_c H(c)\cdot \frac{e^{-\beta H(c)}}{Z_G} = -\frac{d}{d\beta}\ln Z_G$$

Tutte polynomial

$$T_G(x,y) := \sum_{F\subseteq E(G)} (x-1)^{r(G)-r(F)}\,(y-1)^{n(F)}$$

where $r(F) = |V(G)| - k(F)$, $n(F) = |F| - r(F)$

- $T_G(x,y) = (x-1)^{-k(G)}(y-1)^{-|V(G)|} Z_G\left((x-1)(y-1), y-1\right)$
- $Z_G(q,v) = q^{k(G)} v^{r(G)} T_G(1+qv^{-1}, 1+v)$

$\chi_G(q) = q^{k(G)}(-1)^{r(G)} T_G(1-q, 0)$

Doubly weighted Tutte polynomial $\{(u_e, v_e)\}_{e\in E(G)}$

$$T_G(x,y,\{u_e,v_e\}) := \sum_{F\subseteq E(G)} \left(\prod_{e\notin F} u_e \prod_{e\in F} v_e\right)(x-1)^{r(G)-r(F)}(y-1)^{n(F)}$$

Properties • $T_G = u_e T_{G\setminus e} + v_e T_{G/e}$ if e is neither a bridge nor a loop

- $T_G = (u_e(x-1)+v_e)\, T_{G/e}$, e is a bridge
- $T_G = (u_e + (y-1)v_e)\, T_{G-e}$, e is a loop
- $T_{G_1\sqcup G_2} = T_{G_1\cdot G_2} = T_{G_1}\, T_{G_2}$ | For planar G: $T_G(x,y) = T_{G^*}(y,x)$ | by Kauffman
- $T_\bullet = 1$

for signed graphs $u_+ = v_+ = 1$

$u_- = \sqrt{\frac{y-1}{x-1}}$ $v_- = \sqrt{\frac{y-1}{x+1}}$

Godsil, Royle

$u_+ = \alpha$, $v_+ = \beta$, $u_- = \beta$, $u_+ = \alpha$