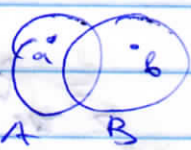
Lecture 4: Delta-matroids and ribbon graphs. June 2

Matroids $M=(E,\mathcal{B})$, $\mathcal{B}\subset 2^E$ bases

1) $\forall B\in\mathcal{B}$, $B' \subsetneq B \Rightarrow B' \in \mathcal{B}$ (proper)

2) $\forall A,B\in\mathcal{B}$, $\forall a\in A\setminus B$ $\exists b\in B\setminus A$ | **Exchange axiom**

$A\setminus\{a\}\cup\{b\}\in\mathcal{B}$



Example

Graph $G=(V,E)$ connected $E=E(G)$

$\{$edges of a spanning tree$\}\in\mathcal{B}$

Dual matroid: $M^* := (E,\mathcal{B}^*)$ | Rank

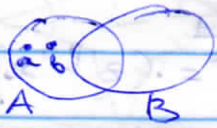
$B\in\mathcal{B}^* \Leftrightarrow E\setminus B\in\mathcal{B}$

Δ-Matroids $(E,\mathcal{F})$, $\mathcal{F}\subset 2^E$ feasible sets.

André Bouchet '1987

Symmetric exchange axiom

$\forall A,B\in\mathcal{F}$, $\forall a\in A\Delta B$, $\exists b\in A\Delta B \mid A\Delta\{ab\}\in\mathcal{F}$



Def

A quasi-tree = a ribbon graph G with $bc(G)=1$

Theorem [Bouchet '89] Let $G=(V,E)$ be a ribbon graph. Then $(E,\{$spanning quasi-trees$\})$ is a Δ-matroid.

Def Let $D=(E,\mathcal{F})$ be a Δ-matroid

$e\in E$ is a loop $\Leftrightarrow \forall F\in\mathcal{F}$, $e\notin F$.

$e\in E$ is a coloop $\Leftrightarrow \forall F\in\mathcal{F}$, $e\in F$

if e is not a loop $\Rightarrow D/e := (E\setminus\{e\}, \{F\setminus e \mid F\in\mathcal{F} \text{ and } e\in F\})$

if e is not a coloop $\Rightarrow D\setminus e := (E-\{e\}, \{F \mid F\in\mathcal{F} \text{ and } F\subset E\setminus\{e\}\})$

Minors

$D_{min} := (E, \mathcal{F}_{min})$, $\mathcal{F}_{min} := \{F \in \mathcal{F} \mid F$ ~~has~~ is of minimal cardinality $\}$

$D_{max} := (E, \mathcal{F}_{max})$, $\mathcal{F}_{max} := \{F \in \mathcal{F} \mid F$ — " — maximal — " $\}$

↑ matroids $w(D) := r(D_{max}) - r(D_{min})$

$(D_G)_{min} = M_G$

$(D_G)_{max} = (M_{G^*})^*$

$D_G = M_G \Leftrightarrow$ G is a planar ribbon graph

Twists of Δ-matroids (Bouchet '87)

$D = (E, \mathcal{F})$, $A \subset E$

$D * A = (E, \{F \Delta A \mid F \in \mathcal{F}\})$

Theorem $D_G * A = D_{G^A}$

Bollobás–Riordan polynomial of $D = (E, \mathcal{F})$

$$R_D(x,y,z) := \sum_{F \subseteq E} x^{r_{D_{min}}(E) - r_{D_{min}}(F) + s(F)} \, y^{n_{D_{min}}(F) - s(F)} \, z^{w(F)}$$

$w(F) := w(D|_F) = w(D \setminus \{E \setminus F\})$

Example

$E = \{-1, +1\}$

$\mathcal{F} = \{\emptyset, \{+1\}, \{-1,+1\}\}$

$\mathcal{F}_{min} = \emptyset$, $\mathcal{F}_{max} = \{\{-1,+1\}\}$

$r_{D_{min}}(F) = 0$ $r_{D_{max}}(F) = |F|$

$n_{D_{min}}(F) = |F|$

F	$\emptyset$	$\{-1\}$
$w(F)$	0	0
	$x^{-1/2} y^{1/2}$	$x^{1/2} y^{1/2}$

F	$\{+1\}$	$\{-1, 1\}$
$w(F)$	1	2
	$x^{-1/2} y^{3/2} z$	$x^{1/2} y^{3/2} z^2$