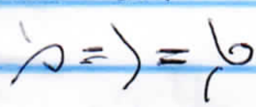
Invitation to Math Dec. 5, 2017, 4:05–5:35

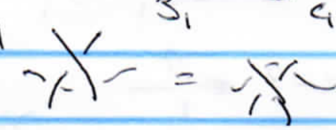
# Khovanov homology of knots

$3_1$ $4_1$

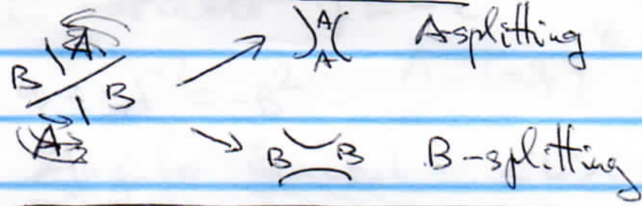
## Reidemeister moves

RI 

RII 

RIII 

## Kauffman bracket




$$\boxed{[L](A,B,d) := \sum_S A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1}}$$

**Def** A state is a chose of a splitting @ every crossing

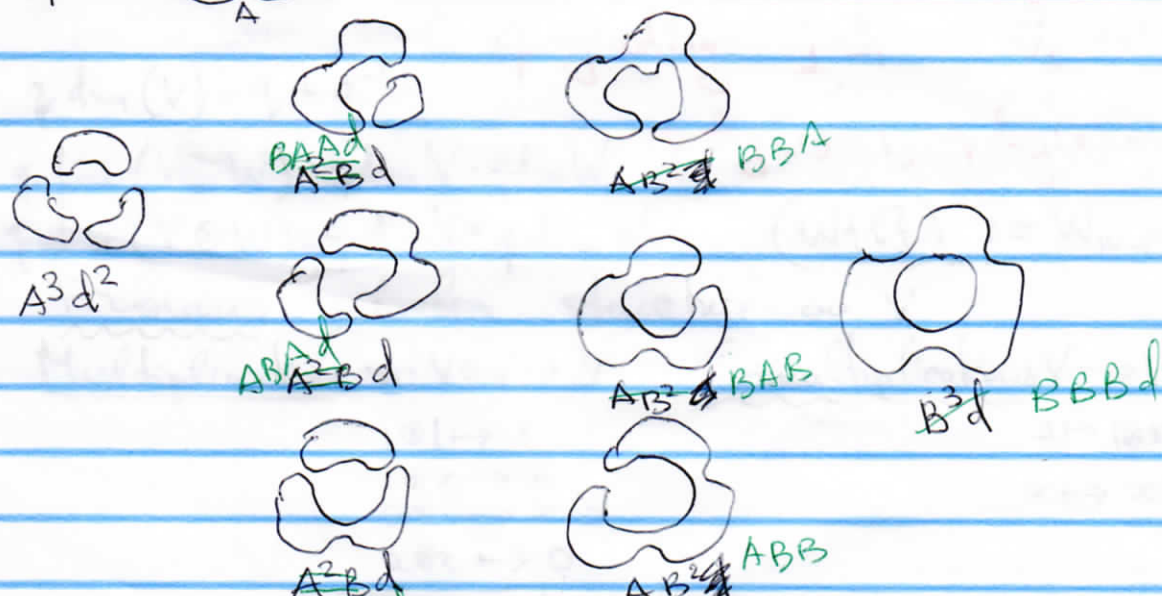
$\alpha(S) = \#$ A-splittings

$\beta(S) = \#$ B-splittings

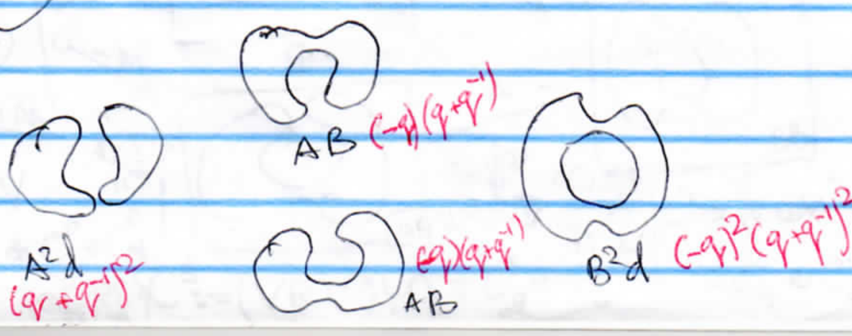
$\delta(S) = \#$ circles

## Examples

(1) $3_1$


(2) $L_1$ $w=2$


$\langle L\rangle(q) = q^2 + \cancel{2} + q^{-2}$
$\cancel{-2q^2 - 2}$
$+q^4 + 2q^2 + 1$
$= q^4 + q^2 + 1 + q^{-2}$
$= (1-q^{-2})(q^4+1)$

Jones polynomial  $A = t^{-1/4}$, $B = t^{1/4}$, $d = -t^{1/2} - t^{-1/2}$

$$J_L(t) := (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

where writhe of L  $w(L) := \sum_{X} \varepsilon_X$   $\nearrow$ $\varepsilon = +1$, $\varepsilon = -1$

in variable $q = -t^{1/2}$

$q = -A^{-2} = -B^2$   $A = (-q)^{-1/2}$, $B = (-q)^{1/2}$, $d = q + q^{-1}$

Angular bracket  $\langle L \rangle(q) := (-q)^{n/2} (q + q^{-1}) [L](q)$

$$:= \sum_s (-q)^{\beta(s)} (q + q^{-1})^{\delta(s)}$$

$$\boxed{J_L(q) = (-1)^{n_-} q^{n_+ - 2n_-} (q + q^{-1})^{-1} \langle L \rangle}$$

Khovanov chain complex

a circle

$\bigcirc \rightsquigarrow V := \mathbb{R}[x]/(x^2) = \langle 1, x^2 \rangle = \langle 1 \rangle \oplus \langle x \rangle$

q-grading  $1, -1$   $V_1$   $V_{-1}$

$q\dim(V) = q + q^{-1}$   $q\deg(1) = 1$, $q\deg(x) = -1$

$q\dim(V \oplus W) = q\dim V + q\dim W$

$q\dim(V \otimes W) = q\dim V * q\dim W$   $(W\{l\})_m := W_{m-l}$

Frobenius algebra structure on V

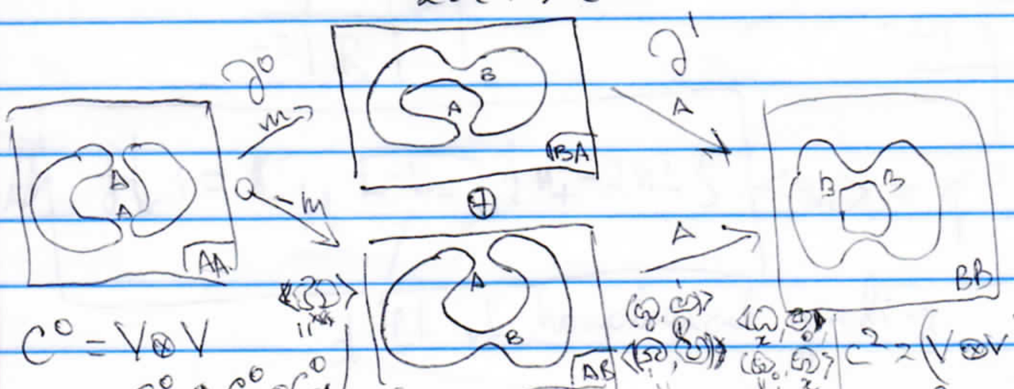
Multiplication $m: V \otimes V \to V$   Comultiplication $\Delta: V \to V \otimes V$

$1 \otimes 1 \mapsto 1$   $1 \mapsto 1 \otimes x + x \otimes 1$

$1 \otimes x \mapsto x$   $x \mapsto x \otimes x$

$x \otimes 1 \mapsto x$

$x \otimes x \mapsto 0$




$C^0 = V \otimes V$

$= C^0_2 \oplus C^0_0 \oplus C^0_{-2}$

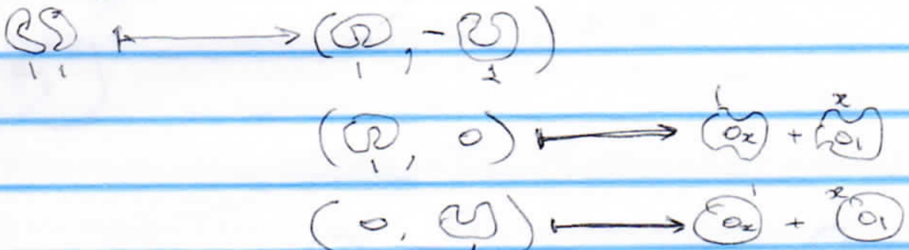
$C^1 := (V \oplus V)\{1\} = C^1_2 \oplus C^1_0$

$C^2 = (V \otimes V)\{2\}$

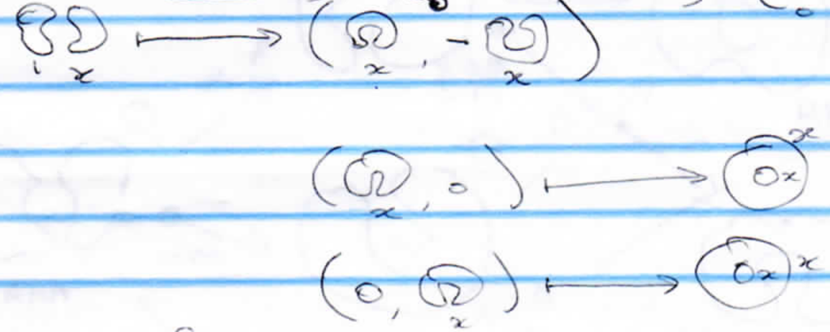
$= C^2_4 \oplus C^2_2 \oplus C^2_0$

$\mathrm{qdim}\, C^0 = (q+q^{-1})^2, \quad \mathrm{qdim}\, C^1 = 2q(q+q^{-1}), \quad \mathrm{qdim}\, C^2 = q^2(q+q^{-1})^2$

$$C_2^0 \xrightarrow{\partial_2^0} C_2^1 \xrightarrow{\partial_2^1} C_2^2$$



$$C_0^0 \xrightarrow{\partial_0^0} C_0^1 \xrightarrow{\partial_0^1} C_0^2$$



$$\mathcal{C}_{Kh}: 0 \to C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \to 0$$

$$\chi(\mathcal{C}) = \sum_i (-1)^i \,\mathrm{qdim}(\mathcal{C}^i), \quad f(L_1) = q^4+q^2+1+q^{-2}$$

$$H^{i,j}(L) = \dim \mathrm{Ker}(\partial_j^i) / \dim \mathrm{Im}(\partial_j^{i-1})$$

$$\chi(\mathcal{C}) = \sum_i (-1)^i \,\mathrm{qdim}\, H^{i,\bullet}(L)$$

$H^{i,j}(L_1)$

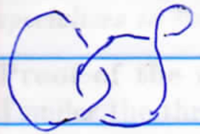
| $j \backslash i$ | 0 | 1 | 2 |
|---|---|---|---|
| 4 | | | $\mathbb{R}$ |
| 2 | | | $\mathbb{R}$ |
| 0 | $\mathbb{R}$ | | |
| −2 | $\mathbb{R}$ | | |

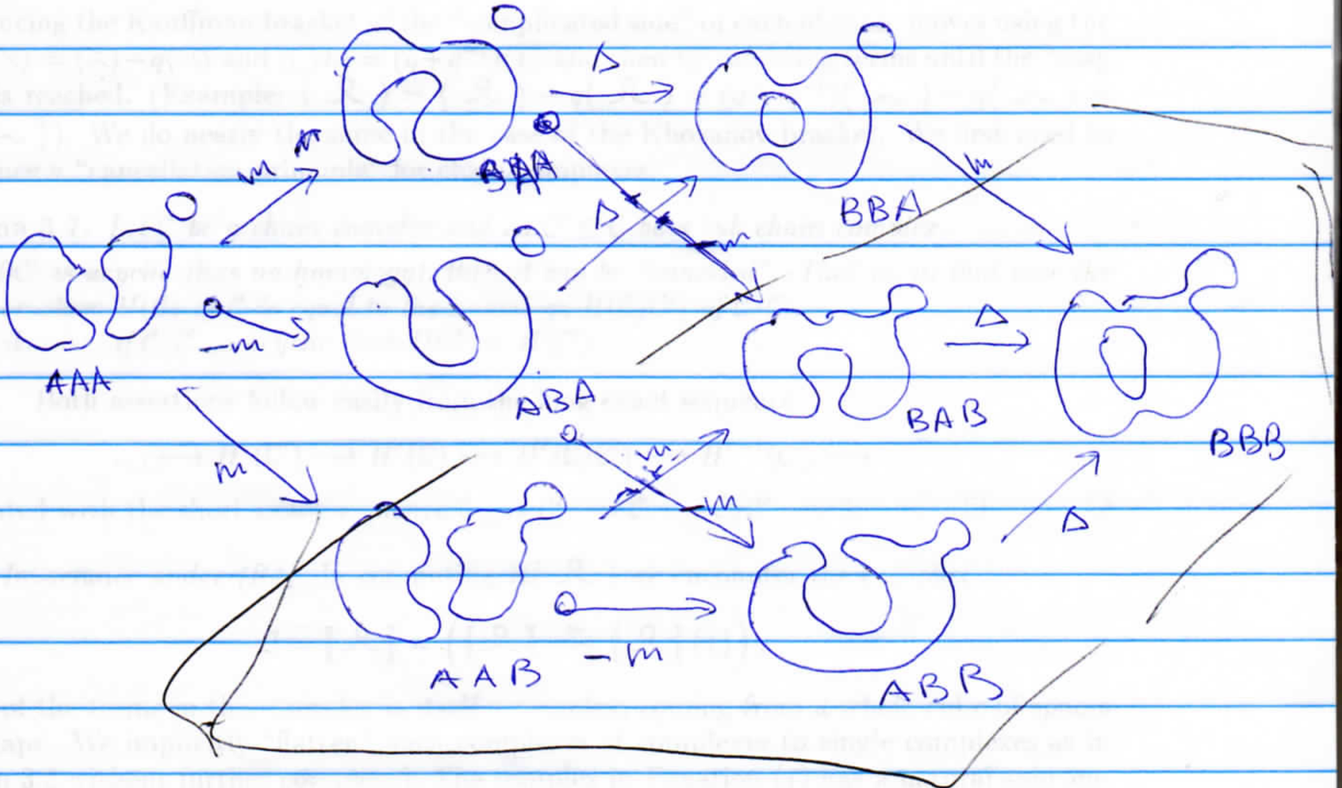
$H^{i,j}(3_1)$

| $j \backslash i$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 5 | | | | $\mathbb{R}$ |
| 3 | | | | $\mathbb{R}$ |
| 1 | | $\mathbb{R}$ | | |
| −1 | | | | |
| −3 | $\mathbb{R}$ | | | |

$$\boxed{\mathcal{H}_K := \mathcal{C}_{Kh}[-n_-]\{n_+ - 2n_-\}} \qquad \langle 3_1 \rangle = q^{-3} - q - q^3 - q^5$$

shift of homological grading

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