Stanley's chromatic symmetric function

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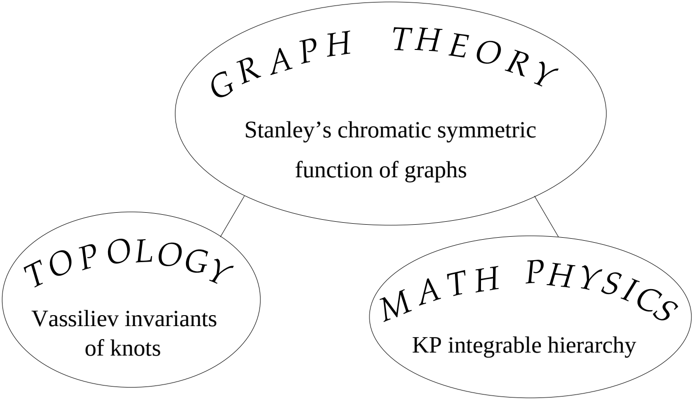
2:00 — 2:50

Overview.


GRAPH THEORY
Stanley's chromatic symmetric
function of graphs
TOPOLOGY
Vassiliev invariants
of knots
MATH PHYSICS
KP integrable hierarchy

Stanley's chromatic symmetric function.

R. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, Advances in Math. **111**(1) 166–194 (1995).

$$X_G(x_1, x_2, \ldots) := \sum_{\substack{\kappa : V(G)\to\mathbb{N} \\ \text{proper}}} \prod_{v\in V(G)} x_{\kappa(v)}$$

Power function basis. $\quad p_m := \sum_{i=1}^{\infty} x_i^m.$

Example. X 

$$\begin{aligned} = \; & \widehat{x_1 x_1} + x_1 x_2 + x_1 x_3 + \ldots \\ & x_2 x_1 + \widehat{x_2 x_2} + x_2 x_3 + \ldots \\ & x_3 x_1 + x_3 x_2 + \widehat{x_3 x_3} + \ldots \\ & \vdots \qquad\quad \vdots \qquad\qquad\quad \ddots \\ = \; & p_1^2 - p_2 . \end{aligned}$$

James Enouen, Eric Fawcett, Rushil Raghavan, Ishaan Shah: Su'18

$$X_G(x_1,x_2,\dots) = \sum_{\substack{\kappa:V(G)\to\mathbb{N}\\ \text{all}}} \prod_{v\in V(G)} x_{\kappa(v)} \prod_{e=(v_1,v_2)\in E(G)} (1-\delta_{\kappa(v_1),\kappa(v_2)})$$
$$= \sum_{\substack{\kappa:V(G)\to\mathbb{N}\\ \text{all}}} \prod_{v\in V(G)} x_{\kappa(v)} \sum_{S\subseteq E_G} (-1)^{|S|} \prod_{e\in S} \delta_{\kappa(v_1),\kappa(v_2)}$$

$$\prod_{e\in S} \delta_{\kappa(v_1),\kappa(v_2)} = \begin{cases} 1 & \text{all vertices of a connected component of the spanning subgraph with } S \text{ edges are colored by } \kappa \text{ into the same color} \\ 0 & \text{otherwise} \end{cases}$$

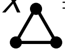
$$\boxed{X_G = \sum_{S\subseteq E_G} (-1)^{|S|} p_{\lambda(S)}}$$

, where $\lambda(S) \vdash |V(G)|$ is a partition of the number of verticies according to the connected components of the spanning subgraph S, and for $\lambda(S) = (\lambda_1,\dots,\lambda_k)$, $p_{\lambda(S)} := p_{\lambda_1}p_{\lambda_2}\dots p_{\lambda_k}$.

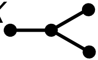
Chromatic symmetric function. Examples.

$$X_G = \sum_{S \subseteq E_G} (-1)^{|S|} p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_k}$$

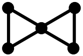
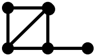
Examples. X  $= p_1^2 - p_2,$

X  $= p_1^3 - 2p_1p_2 + p_3,$ X  $= p_1^3 - 3p_1p_2 + 2p_3.$

X  $= p_1^4 - 3p_1^2p_2 + p_2^2 + 2p_1p_3 - p_4,$

X  $= p_1^4 - 3p_1^2p_2 + 3p_1p_3 - p_4.$

Two graphs with the same chromatic symmetric function:

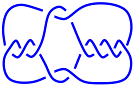
X  $= X$ 

Tree conjecture.
X_G distingushes trees.

A $(3+1)$ poset is the disjoint union of a 3-element chain and 1-element chain.
A poset P is $(3+1)$-*free* if it contains no induced $(3+1)$ posets.
Incomparability graph $inc(P)$ of P: vertices are elements of P; (uv) is an edge if neither $u \leqslant v$ nor $v \leqslant u$.

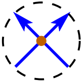
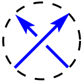
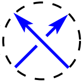
***e*-positivity conjecture.**
The expansion of $X_{inc(P)}$ in terms of elementary symmetric functions has positive coefficients for $(3+1)$-free posets P.

Vassiliev knot invariants.

A knot $K =$ , let $\mathcal{K} \ni K$ be a set of all knots.

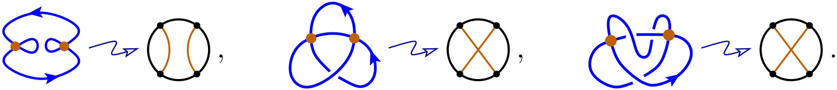
A knot invariant $v : \mathcal{K} \to \mathbb{C}$.

Definition.
A knot invariant is said to be a *Vassiliev invariant* of order (or degree) $\leqslant n$ if its extension to the knots with double points according to the rule

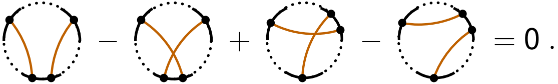
$v($  $) := v($  $) - v($  $)$.

vanishes on all singular knots with more than n double points.

The value of v on a singular knot K with n double points does not depend on the specific knotedness of K. It depends only on the combinatorial arrangement of double points along the knot, which can be encoded by a *chord diagram* of K.

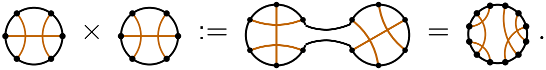


Algebra of chord diagrams.
$\mathcal{A}_n$ is a $\mathbb{C}$-vector space spanned by chord diagrams modulo four term relations:



Vassiliev knot invariants. Bialgebra of chord diagrams.

The vector space $\mathcal{A} := \bigoplus_{n\geq 0} \mathcal{A}_n$ has a natural bialgebra structure.

Multiplication: 

Comultiplication: $\delta : \mathcal{A}_n \to \bigoplus_{k+l=n} \mathcal{A}_k \otimes \mathcal{A}_l$ is defined on chord diagrams by the sum of all ways to split the set of chords into two disjoint parts: $\delta(D) := \sum_{J\subseteq[D]} D_J \otimes D_{\overline{J}}$.

Primitive space $\mathcal{P}(\mathcal{A})$ is the space of elements $D \in \mathcal{A}$ with the property $\delta(D) = 1 \otimes D + D \otimes 1$.
$\mathcal{P}(\mathcal{A})$ is also a graded vector space $\mathcal{P}(\mathcal{A}) = \bigoplus_{n\geq 1} \mathcal{P}_n$.

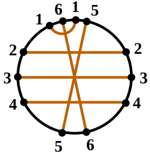
The classical **Milnor—Moore** theorem: *any commutative and cocommutative bialgebra $\mathcal{A}$ is isomorphic to the symmetric tensor algebra of the primitive space, $\mathcal{A} \cong \mathcal{S}(\mathcal{P}(\mathcal{A}))$.*

Let $p_1, p_2, \dots$ be a basis for the primitive space $\mathcal{P}(\mathcal{A})$ then any element of $\mathcal{A}$ can be uniquely represented as a polynomial in commuting variables $p_1, p_2, \dots$.
The dimensions of $\mathcal{P}_n$:

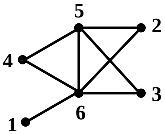
n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{P}_n$	1	1	1	2	3	5	8	12	18	27	39	55

Vassiliev knot invariants. Weighted graphs.

S. Chmutov, S. Duzhin, S. Lando, *Vassiliev knot invariants III. Forest algebra and weighted graphs*, Advances in Soviet Mathematics **21** 135–145 (1994).



A chord diagram



The intersection graph

Definition. A *weighted graph* is a graph G without loops and multiple edges given together with a *weight* $w : V(G) \to \mathbb{N}$ that assigns a positive integer to each vertex of the graph.

Ordinary simple graphs can be treated as weighted graphs with the weights of all vertices equal to 1.

Bialgebra of weighted graphs.

Let $\mathcal{H}_n$ be a vector space spanned by all weighted graphs of the total weight n modulo the *weighted contraction/deletion relation* $G = (G \setminus e) + (G/e)$,where the graph $G \setminus e$ is obtained from G by removing the edge e and G/e is obtained from G by a contraction of e such that if a multiple edge arises, it is reduced to a single edge and the weight $w(v)$ of the new vertex v is set up to be equal to the sum of the weights of the two ends of the edge e.

$$\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$$

Multiplication: disjoint union of graphs;
Comultiplication: splitting the vertex set into two subsets.
The primitive space $P(\mathcal{H}_n)$ is of dimension 1 and spanned by a single vertex of weight n.
The bialgebra $\mathcal{H}$ has a one-dimensional primitive space in each grading and thus is isomorphic to $\mathbb{C}[q_1, q_2, \dots]$.

Weighted chromatic polynomial.

The image of an ordinary graph G (considered as a weighted graph with weights of all vertices equal to 1) in $\mathcal{H}$ can be represented by a polynomial $W_G(q_1, q_2, \ldots)$ in the variables q_n.

S. Noble, D. Welsh, *A weighted graph polynomial from chromatic invariants of knots*, Annales de l'institut Fourier **49**(3) 1057–1087 (1999):

$$(-1)^{|V(G)|} W_G\Big|_{q_j=-p_j} = X_G(p_1, p_2, \ldots).$$

Examples. $W_{\bullet\!-\!\bullet} = (\bullet\ \bullet) + \underset{2}{\bullet} = q_1^2 + q_2$

$$\begin{aligned} W_{\bullet\!-\!\bullet\!-\!\bullet} &= (\bullet \quad \bullet\!-\!\bullet) + \underset{2}{\bullet}\!-\!\bullet = (\bullet \quad \bullet \quad \bullet) + 2(\ \bullet\ \underset{2}{\bullet}\ ) + (\ \underset{3}{\bullet}\ ) \\ &= q_1^3 + 2q_1 q_2 + q_3 \end{aligned}$$

Kadomtsev–Petviashvili (KP) hierarchy.

The KP hierarchy is an infinite system of nonlinear partial differential equations for a function $F(p_1, p_2, \dots)$ of infinitely many variables.

$$\frac{\partial^2 F}{\partial p_2^2} = \frac{\partial^2 F}{\partial p_1 \partial p_3} - \frac{1}{2}\left(\frac{\partial^2 F}{\partial p_1^2}\right)^2 - \frac{1}{12}\frac{\partial^4 F}{\partial p_1^4}$$

$$\frac{\partial^2 F}{\partial p_2 \partial p_3} = \frac{\partial^2 F}{\partial p_1 \partial p_4} - \frac{\partial^2 F}{\partial p_1^2} \cdot \frac{\partial^2 F}{\partial p_1 \partial p_2} - \frac{1}{6}\frac{\partial^4 F}{\partial p_1^3 \partial p_2}.$$

The left hand side of the equations correspond to partitions of $n \geq 4$ into two parts none of which is 1, while the terms on the right hand sides correspond to partitions of the same number n involving parts equal to 1. The first two equations above correspond to partitions of 4 and 5. For $n = 6$, there are two equations, which correspond to the partitions $2 + 4 = 6$ and $3 + 3 = 6$, and so on.

Generating function of weighted chromatic polynomial.

S. Chmutov, M. Kazarian, S. Lando, *Polynomial graph invariants and the KP hierarchy*, arXiv:1803.09800

$$\mathcal{W}(q_1, q_2, \dots) := \sum_{\substack{G \text{ connected} \\ \text{non-empty}}} \frac{W_G(q_1, q_2, \dots)}{|\mathrm{Aut}(G)|}$$

$$= \frac{1}{1!}q_1 + \frac{1}{2!}\left(q_1^2 + q_2\right) + \frac{1}{3!}\left(4q_1^3 + 9q_1q_2 + 5q_3\right)$$
$$+\frac{1}{4!}\left(38q_1^4 + 144q_1^2q_2 + 45q_2^2 + 140q_1q_3 + 79q_4\right) + \dots,$$

Theorem. $F(p_1, p_2, \dots) := \mathcal{W}(\alpha_1 p_1, \alpha_2 p_2, \alpha_3 p_3, \alpha_4 p_4, \dots)$ *is a solution of the KP hierarchy of PDEs*,
where $\alpha_n = \frac{2^{n(n-1)/2}(n-1)!}{c_n}$ and $c_1 = 1$, $c_2 = 1$, $c_3 = 5$, $c_4 = 79$, $c_5 = 3377$, ... is the [A134531] sequence from Sloane's Encyclopedia of Integer Sequences.