Thompson's group links

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Thompson's group F. Definition.

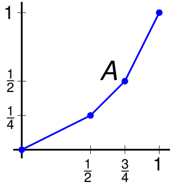
Introduced by Richard Thompson in 1965.
Definition. Elements of F are piecewise linear homeomorphisms of $[0, 1]$ to itself satisfying the conditions

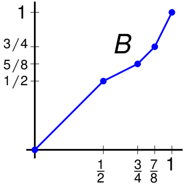
- linear except at finitely many dyadic rational numbers;
- fixing 0 and 1;
- on intervals of linearity the derivatives are powers of 2.

Examples.

$$A(x) := \begin{cases} \frac{x}{2}, & 0 \leqslant x \leqslant \frac{1}{2} \\ x - \frac{1}{4}, & \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ 2x - 1, & \frac{3}{4} \leqslant x \leqslant 1 \end{cases}$$

$$B(x) := \begin{cases} x, & 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ x - \frac{1}{8}, & \frac{3}{4} \leqslant x \leqslant \frac{7}{8} \\ 2x - 1, & \frac{7}{8} \leqslant x \leqslant 1 \end{cases}$$

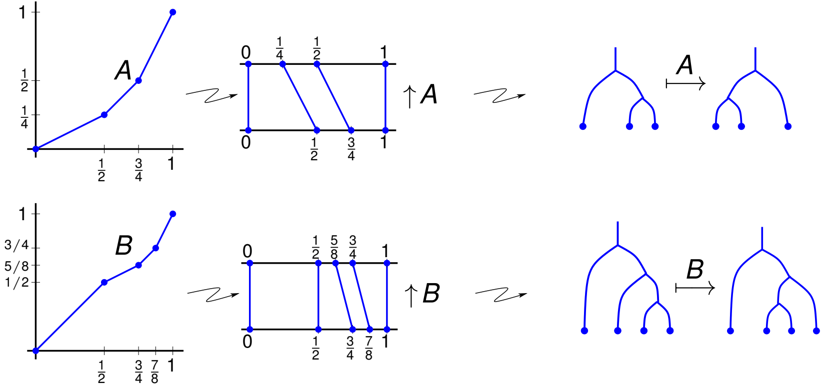




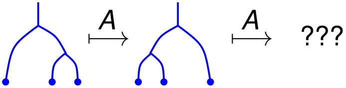
J. W. Cannon, W. J. Floyd, W. R. Parry, *Introductory notes on Richard Thompson's groups*, L'Enseignement Mathématique **42** (1996) 215–256.

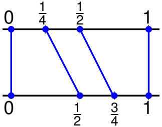
Theorem. *F has the finite presentation*

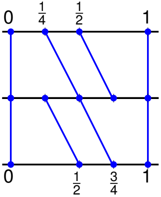
$$\langle A, B \mid [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2] \rangle$$

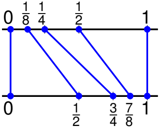


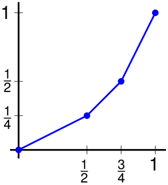
Thompson's group F. Calculations.

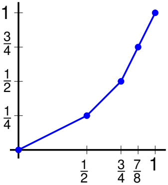


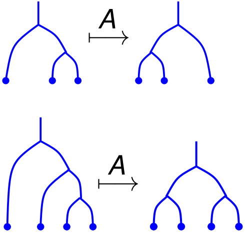
$A =$ 

,
$A^2 =$ 

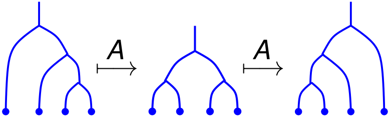
$=$ 

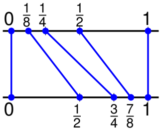
$A =$ 

$=$ 

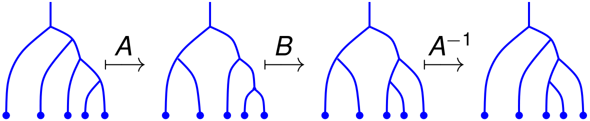


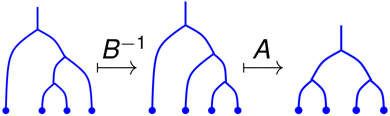
Adding a *caret*, , to the corresponding vertices of both trees does not change the element of the Thompson group.



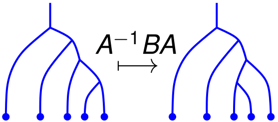
$A^2 =$ 

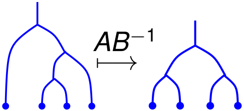
Now we can do calculations.

$A^{-1}BA$: 

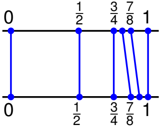
AB^{-1} : 

Thompson's group F. Calculations.

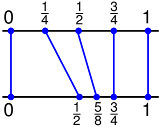




$A^{-1}BA : \uparrow$



$AB^{-1} : \uparrow$



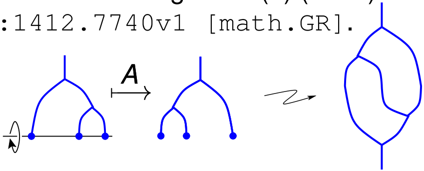
$$[AB^{-1}, A^{-1}BA] = 1$$

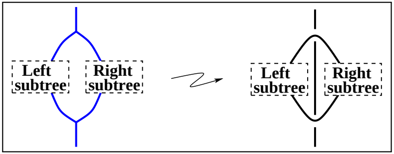
Thompson's group F. Properties.

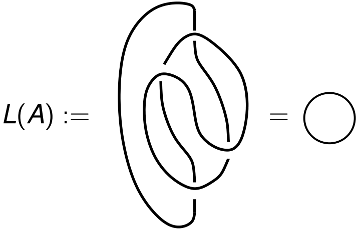
- **Open problem:** Is F amenable?
- It was used to construct finitely-presented groups with unsolvable word problems.
- F does not contain a free group of rank greater than one.
- $F/[F,F] \cong \mathbb{Z} \oplus \mathbb{Z}$
- Every proper quotient group of F is Abelian.
- The commutator subgroup $[F,F]$ of F is a simple group.
- F has exponential growth.
- Every non-Abelian subgroup of F contains a free Abelian subgroup of infinite rank.
- F is a totally ordered group.

Jones' construction of links from elements of F.

V.Jones, *Some unitary representations of Thompson's groups F and T*, Journal of Combinatorial Algebra **1**(1) (2017) 1–44.
Preprint `arXiv:1412.7740v1 [math.GR]`.



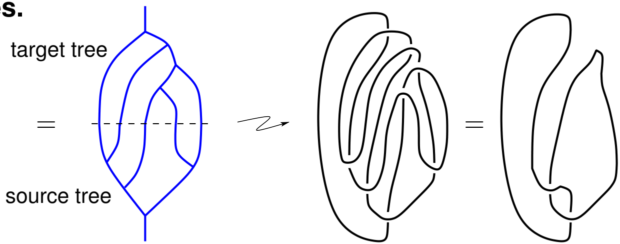




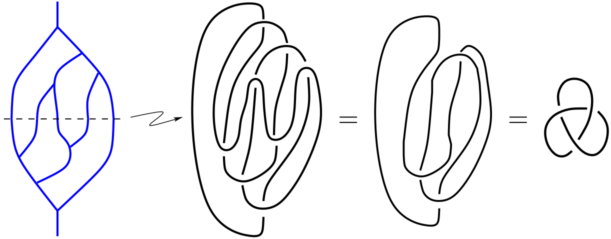
Thompson's group links.

Examples.

$A^{-1}BA^{-1}$ =



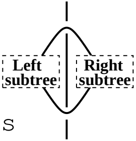
$BA^{-1}B^{-1}A^{-1}$ =



Jones' theorem.

V. F. R. Jones, *On the construction of knots and links from Thompson's groups*, in *Knots, low-dimensional topology and applications*, Springer Proc. Math. Stat. **284**, Springer (2019) 43–66. Preprint `arXiv:1810.06034v1 [math.GT]`.

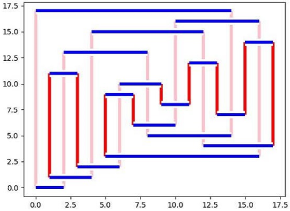
Theorem. *For any link diagram D there is an element $g \in F$, such that the diagram $L(g)$ is isotopic to D.*

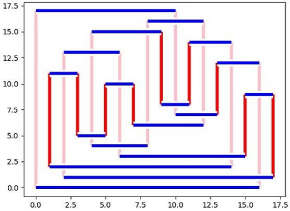


$L(g) = \overline{L(g^{-1})}$

Dennis Sweeney: Borromean rings

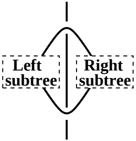
`https://github.com/sweeneyde/thompson_knots`





Regular isotopy.

V. Jones: “Since the proof of the realization of all links as $L(g)$ actually uses a lot of type I Reidemeister moves, one may ask whether all *regular isotopy* classes of link diagrams actually arise as $L(g)$.”



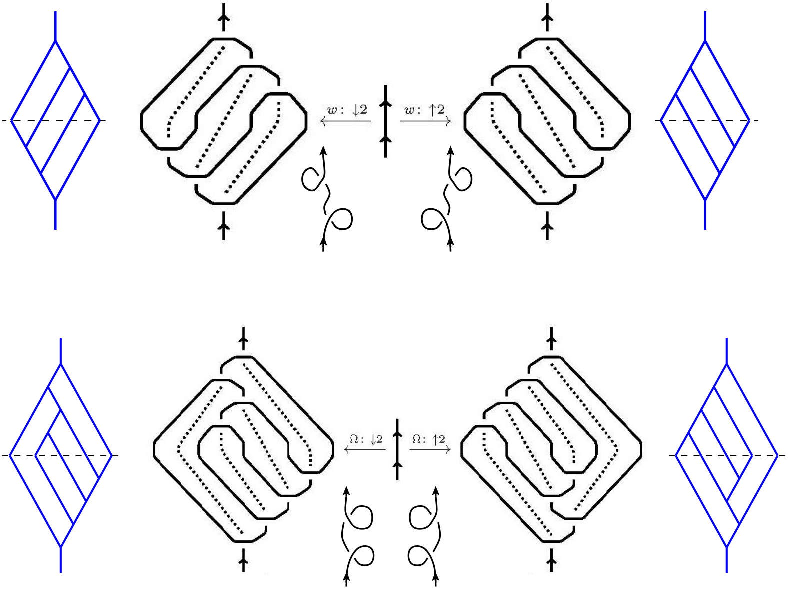
R. Raghavan, D. Sweeney, *Regular Isotopy Classes of Link Diagrams From Thompson's Groups.* Preprint `arXiv:2008.11052 [math.GT]`

Theorem. *A link diagram D is regular isotopic to $L(g)$ for some $g \in F$ iff every component of D underpasses even number of times.*

A. Coward. *Ordering the Reidemeister moves of a classical knot*, Algebraic & Geometric Topology, **6**(2) (2006) 659–671.

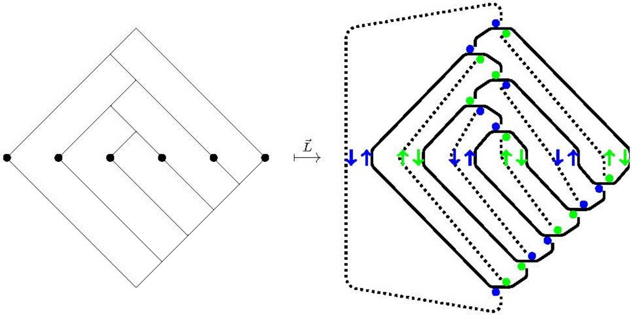
Theorem. *Two isotopic oriented link diagrams are regularly isotopic iff the Whitney rotation numbers and the writhes the corresponding components coincide.*

Idea of proof.



Oriented Thompson group $\vec{F}$.

Definition. The *oriented Thompson group* $\vec{F} \subset F$ is the subgroup of elements $g \in F$ for which the Tait graph of the checkerboard-shading of $L(g)$ is bipartite.



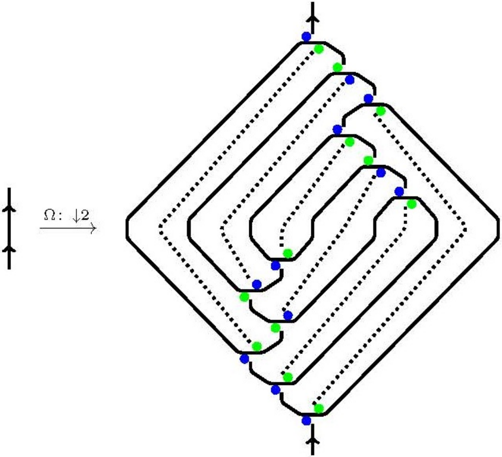
Aiello's theorem.

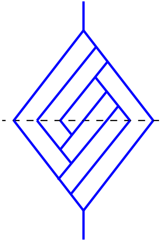
V. Aiello, *On the Alexander theorem for the oriented Thompson group* $\vec{F}$, Algebraic & Geometric Topology, **20** (2020) 429–438.
Preprint `arXiv:1811.08323v3 [math.GT]`.

Theorem. *Given an oriented link diagram D, there is an element* $g \in \vec{F}$ *such that* $\vec{L}(g)$*is isotopic to D.*

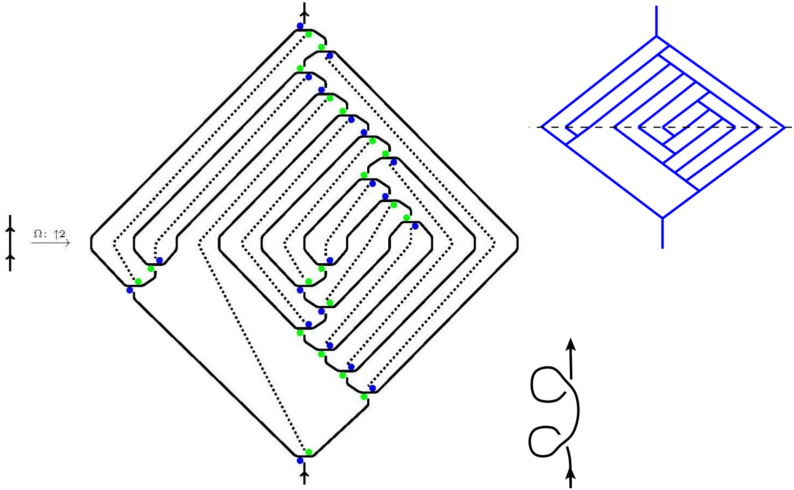
Theorem [R. Raghavan, D. Sweeney]. *An oriented link diagram D is regular isotopic to* $\vec{L}(g)$ *for some* $g \in \vec{F}$ *iff for each component of D the sum of the writhes of all crossings where the component goes under is equal to zero.*

Idea of proof. Whitney down.





Idea of proof. Whitney up.



THANK YOU!!!