Sec. 2.7, further examples and exercises

April 14, 2012

Most of the formulas below are generated by Maple 15, but I edited them a bit to make them more understandable by humans. There is a small chance of typos. If you think you found one, please let me know!
Multinomial formula (fifth power, three variables):

$$(z_1+z_2+z_3)^5 = z_1^5 + 5z_1^4z_2 + 5z_1^4z_3 + 10z_1^3z_2^2 + 20z_1^3z_2z_3 + 10z_1^3z_3^2 + 10z_1^2z_2^3 + 30z_1^2z_2^2z_3 + 30z_1^2z_2z_3^2 + 10z_1^2z_3^3 + 5z_1z_2^4 + 20z_1z_2^3z_3 + 30z_1z_2^2z_3^2 + 20z_1z_2z_3^3 + 5z_1z_3^4 + z_2^5 + 5z_2^4z_3 + 10z_2^3z_3^2 + 10z_2^2z_3^3 + 5z_2z_3^4 + z_3^5 \quad (1)$$

Exercise. Calculate by hand a few coefficients, say those of $z_1^3z_2^2$, $z_1^2z_2^2z_3$ and $z_2^2z_3^3$, and compare with (1).

The multi-index notation. Third order **Taylor's formula** in two variables using the (expanded out) multi-index notation (I omitted (x_1, x_2) in the partial derivatives, to save space):

$$f(x_1+h_1, x_2+h_2) = f(x_1,x_2) + h_1\partial^{1,0}f + h_2\partial^{0,1}f + \tfrac{1}{2}h_1^2\partial^{2,0}f + h_1h_2\partial^{1,1}f + \tfrac{1}{2}h_2^2\partial^{0,2}f + \tfrac{1}{6}h_1^3\partial^{3,0}f + \tfrac{1}{2}h_1^2h_2\partial^{2,1}f + \tfrac{1}{2}h_1h_2^2\partial^{1,2}f + \tfrac{1}{6}h_2^3\partial^{0,3}f + R_{\mathbf{x},3}(\mathbf{h}) \quad (2)$$

Taylor's formula, partial derivative notation. Here is the third order Taylor formula in the standard partial derivatives notation:

$$f(x+h, y+k) = f(x,y) + \frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}h^2 + \frac{\partial^2 f}{\partial y \partial x}hk + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}k^2 + \frac{1}{6}\frac{\partial^3 f}{\partial x^3}h^3 + \frac{1}{2}\frac{\partial^3 f}{\partial y \partial x^2}h^2k + \frac{1}{2}\frac{\partial^3 f}{\partial y^2 \partial x}hk^2 + \frac{1}{6}\frac{\partial^3 f}{\partial y^3}k^3 + R_{\mathbf{x},3}(\mathbf{h}) \quad (3)$$

Exercise. Compare these expressions with (2.69) in the book; calculate (2) yourselves.

Examples. For those of you who have access to Maple and know how to use it, the command:

$$\text{mtaylor}\left(\sqrt{1+x^2+y^2}, [x,y], 8\right)$$

instructs Maple to output all the terms of total degree strictly less than 8 in the Taylor polynomial ; the result is:

$$1+\tfrac{1}{2}\,x^2+\tfrac{1}{2}\,y^2-\tfrac{1}{8}\,x^4-\tfrac{1}{4}\,y^2x^2-\tfrac{1}{8}\,y^4+\tfrac{1}{16}\,x^6+\tfrac{3}{16}\,y^2x^4+\tfrac{3}{16}\,y^4x^2+\tfrac{1}{16}\,y^6 \quad (4)$$

Likewise, the command

$$\text{mtaylor}\left(\mathrm{e}^x \sin(x+y), [x,y], 5\right)$$

produces the output

$$x+y+x^2+yx+\tfrac{1}{3}x^3-\tfrac{1}{2}y^2x-\tfrac{1}{6}y^3-\tfrac{1}{3}yx^3-\tfrac{1}{2}y^2x^2-\tfrac{1}{6}y^3x \qquad (5)$$

Remark. On Monday we will see that, for a given function, the Taylor polynomial is unique. This is also shown in the textbook. As a consequence, if you obtain by other means a polynomial for $\mathrm{e}^x \sin(x+y)$ that differs from $\mathrm{e}^x \sin(x+y)$ by $o\left(|(x,y)|^4\right)$ then it coincides with the one obtained using the several variables Taylor formula (here, as usual, $|(x,y)| = \sqrt{x^2+y^2}$).

Exercise. In the two concrete examples above, try to use combinations of Taylor formulas *in one variable* to obtain (4) and (5).