SOME PROPERTIES OF ANALYTIC FUNCTIONS

0.1. **Introduction.** The multidimensional analysis we have studied so far allows us to obtain quite easily a number of basic properties of analytic functions. We'll take a short trip to the analytic world, as it also provides us with an opportunity to review many of the notions and theorems we have developed.

0.2. **Complex integrals.** Here too, before we introduce properties specific to analytic function theory, the definitions, constructions and theorems are very similar to those on $\mathbb{R}^2$. In particular a smooth curve in $\mathbb{C}$ is simply the image of a smooth function $g : [a, b] \subset \mathbb{R} \to \mathbb{C}$. A path integral of a continuous complex valued function $f(z) = u(x, y) + iv(x, y)$ of one complex variable $z = x + iy$ is simply

$$\begin{aligned}(1)\qquad \int_C f(z)dz &:= \int_a^b f(g(t))g'(t)dt = \int_a^b [u(g(t)) + iv(g(t))][g_1'(t) + ig_2'(t)]dt \\ &= \int_a^b u(g(t))g_1'(t) - v(g(t))g_2'(t)dt + i\int_a^b v(g(t))g_2'(t) + u(g(t))g_1'(t)dt \\ &= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy = \int_C \mathbf{F}_1 d\mathbf{x} + i\int_C \mathbf{F}_2 d\mathbf{x} \\ &\qquad \mathbf{F}_1 = (u, -v);\ \mathbf{F}_2 = (v, u)\end{aligned}$$

Proposition 1. Assume u, v are C^1 in a regular domain S with piecewise smooth boundary ∂S in $\mathbb{C}$ as in Theorem 5.12. Then,

$$(2)\qquad \int_{\partial S} f(z)dz = -\iint_S \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$

Proof. This is simply Green's theorem, Theorem 5.12., applied to (1). ∎

Definition 2. *The equations*

$$(3)\qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$(4)\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are called the Cauchy-Riemann equations (C-R).

Corollary 3. *(a) If f is C^1 and satisfies the CR equations in S then*

$$(5)\qquad \int_{\partial S} f(z)dz = 0$$

(b) Assume S is a convex domain. Then the integral

$$(6)\qquad \int_C f(z)dz$$

is **path-independent** *(the integral only depends on the endpoints of C) iff the CR equations are satisfied.*

(c) Assume S is a convex domain and C is a smooth closed curve in S. Then

$$\int_C f(z)dz = 0 \tag{7}$$

Proof. (a) is once more Green's theorem. (b) and (c) are immediate consequences of Theorems 5.60 and 5.62. ∎

The following definition is a natural adaptation of the general definition of derivative we used in this course:

Definition 4. *A function is differentiable* **in** z *at* $z = z_0 = a + ib \in \mathbb{C}$ *if there is an* $A \in \mathbb{C}$ *s.t.*

$$f(a+\varepsilon) - f(a) = A\varepsilon + o(\varepsilon) \quad as \quad |\varepsilon| \to 0, \quad \varepsilon \in \mathbb{C} \tag{8}$$

A function which is differentiable in an open set $\mathcal{O}$ is called **analytic in** $\mathcal{O}$.

Proposition 5. *Assume f is C^1 in an open domain $\mathcal{O} \in \mathbb{C}$. Then f is differentiable in $\mathcal{O}$ iff it satisfies the CR equations in $\mathcal{O}$.*

Proof. Let $\varepsilon = h + ik$. We have

$$\begin{aligned} &u(a+h, b+k) - u(a,b) + iv(a+h, b+k) - iv(a,b) \\ &\qquad = u_x h + u_y k + i(v_x h + v_y k) = A(h+ik) + o(\varepsilon) \end{aligned} \tag{9}$$

Taking first $[k = 0,\ h \to 0]$ and then $[h = 0,\ k \to 0]$ and dividing by h and then k respectively, we get

$$u_x + iv_x = A; \quad u_y + iv_x = iA = i(u_x + iv_x) \tag{10}$$

The last pair of equalities is equivalent to CR. ∎

Proposition 6. *Assume f is continuous in a convex domain S, $c \in S$ and the integral of S around any triangle is zero. Then $F(z) := \int_c^z f(s)ds$ is differentiable in S and $F'(z) = f$.*

Proof. Same as that of Theorem 5.60. ∎

Exercise 7. Show that if f, g are differentiable in a region S then so are $f+g, fg$, $f \circ g$ (when the composition makes sense) etc. Check that if $g : [a,b] \to \mathbb{C}$ is C^1 and $f(z)$ is continuously differentiable (w.r.t. z) in a region containing the curve $C = g([a,b])$, then $\frac{d}{dt} f(g(t)) = f'(g(t))g'(t)$. Check also that

$$\int_{z_1}^{z_2} f'(z)dz = f(z_2) - f(z_1) \tag{11}$$

where $z_1 = g(a), z_2 = g(b)$ and the integral is taken along C (in fact, now we know that the particular curve does not matter).

Proposition 8. *Let C be a circle of radius r around $z = 0$. Then*

$$\int_C \frac{dz}{z} = 2\pi i \tag{12}$$

Proof. We parametrize the circle: $s = r(\cos t + i \sin t), t \in [0, 2\pi]$. Then the integral becomes

$$\int_0^{2\pi} \frac{r(-\sin t + i\cos t)}{r(\cos t + i \sin t)} dt = \int_0^{2\pi} i dt = 2\pi i \tag{13}$$

∎

Proposition 9. *Let S be a convex domain containing zero let f be C^1 satisfying the CR equations in S, and take C be a closed smooth curve in S containing zero in its interior. Then,*

$$\int_C \frac{f(s)}{s} ds = 2\pi i f(0) \tag{14}$$

Proof. Using Corollary 3 (a) the integral in (14) is equal to the integral over any small circle centered at 0. Take one such circle C_r and note that for $t < 1 \in \mathbb{R}^+$ close to one $f(tz)$ is also differentiable. Furthermore, by the change of variable formula

$$\int_{C_r} \frac{f(s)}{s} ds = \int_{C_{r/t}} \frac{f(tu)}{u} du = \int_{C_r} \frac{f(tu)}{u} du = \cdots = \int_{C_r} \frac{f(t^n u)}{u} du \\ \underset{n\to\infty}{\longrightarrow} \int_{C_r} \frac{f(0)}{u} du = f(0) \int_{C_r} \frac{du}{u} = 2\pi i f(0) \tag{15}$$

for all n. where we used continuity of f and the bounded convergence theorem ∎

Proposition 10 (Cauchy's integral formula). *Let S be a convex domain; assume f be C^1 satisfies the CR equations in S and take C be a closed smooth curve in S containing z in its interior. Then,*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z} ds \tag{16}$$

Proof. A simple change of variable $s = z + u$, $g(z + u) = h(u)$ brings (16) to (15). ∎

Proposition 11. *In the assumptions of Prop. 10, f is C^∞ in S and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}} ds \tag{17}$$

Proof. By the bounded convergence theorem, we can differentiate inside the integral; the function $1/(s - z)$ is C^∞ in a neighborhood of C. ∎

Proposition 12 (Morera's theorem). *Assume f is continuous in a convex domain S and the integral of S around any triangle is zero. Then f is analytic in S, $f \in C^\infty$ in S and (17) applies.*

Proof. By Proposition 6, $F(z) = \int_{z_0}^z f(s)ds$ is continuously differentiable in $z \in S$. Then $F \in C^\infty$ and since $f = F'$, $f \in C^\infty$ too. ∎

Proposition 13. *Let S be an open convex domain, assume f_n are analytic in S and assume f_n converge uniformly to f in any closed disk in S. Then f is analytic in S, and $f_n^{(k)}$ converge uniformly to $f^{(k)}$ in any closed disk in S.*

Proof. We can pass to the limit in the formula (16). The rest follows as in Proposition 11. ∎

Corollary 14. *Assume the series $f(z) = \sum_{k=0}^{\infty} c_k z^k$ converges in $\mathbb{D}_r$, $r > 0$. Then f is analytic in $\mathbb{D}_r$ and $f'(z) = \sum_{k=0}^{\infty} c_k k z^{k-1}$.*

Proof. An immediate consequence of Proposition 13. ∎

Example 15. The functions e^z, $\cos z$, $\sin z$ defined in the notes to the previous lecture note are analytic in $\mathbb{C}$ that is, they are *entire*. Differentiating term by term we see immediately that

$$(e^z)' = e^z,\ \cos'(z) = -\sin(z),\ \sin'(z) = \cos(z) \tag{18}$$

Then $(e^z e^{-z})' = 0$ implying that $e^{-z} = 1/e^z$; more generally $(e^{-x}e^{a+x})' = 0$ implying $e^{x+a} = e^x e^a$.

Furthermore, check that $[(\cos z + i \sin z)e^{-iz}]' = 0$ and thus $e^{iz} = \cos z + i \sin z$ for all $z \in \mathbb{C}$. This identity easily implies that the addition formulas for $\sin, \cos$ are valid throughout $\mathbb{C}$. In particular, $\sin(z + 2\pi) = \sin(z)$ for all z, and $e^{2\pi i} = 1$ implying that the exponential is periodic, with period $2\pi i$.

Exercise 16 (homework)**.** (a) Show that the function

$$f(z) = \begin{cases} \dfrac{e^z - 1}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases} \tag{19}$$

is (continuously) differentiable as a function of the complex variable z for all $z \in \mathbb{C}$. (Hint: read these notes and look at Example 9 in the May 24 class notes.)

(b) With $z = x + iy, (x, y) \in \mathbb{R}$ write $f(z)$ in the form $u(x, y) + iv(x, y)$. Show that

$$u(x, y) = \begin{cases} \dfrac{(x \cos y + y \sin y)e^x - x}{x^2 + y^2} & \text{if } (x, y) \neq 0 \\ 1 & \text{if } (x, y) = 0 \end{cases} \tag{20}$$

(c) Show that $u(x, y) \in C^\infty(\mathbb{R}^2)$. Probably the easiest way is to base your proof on (a), but any correct proof is fine. (In fact, it is interesting to see if an approach not using (a) can be found.)