ITERATED INTEGRALS: THE CONTINUOUS CASE

Proposition 1. *If f is continuous on $R = [a,b] \times [c,d]$ then*

$$(1) \qquad \iint\limits_R f(x,y)dxdy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

Proof. The function $f(x,y)$ is continuous in y for every fixed x, and thus the integral $g(x) = \int_c^d f(x,y)dy$ is well defined. By Theorem 4.5 p. 189 that we proved today for $\mathbb{R} \times \mathbb{R}$, $g(x)$ is a continuous function. Thus, the iterated integrals

$$(2) \qquad \int_a^b \left(\int_c^d f(x,y)dy \right) dx; \ \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

are well defined (the second one by a similar argument). Certainly, it suffices to prove the first equality in (1). Let $R' \subset R$ be any rectangle. By the Mean Value Theorem for Integrals 4.24, we have, for some $(x', y') \in R'$,

$$(3) \qquad \iint\limits_{R'} f(x,y)dxdy = f(x',y')\text{Area}(R')$$

Take any rectangular partition $P = \{x_1, ..., x_N; y_1, ..., y_K\}$ of R, and denote the rectangles $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ by R_{ij}. We have, using Corollary 4.23 b,
(4)

$$J_R := \iint\limits_R f(x,y)dxdy = \sum_{i=1}^N \sum_{j=1}^K \iint\limits_{R_{ij}} f(x,y)dxdy = \sum_{i=1}^N \sum_{j=1}^K f(x_i', y_j')\Delta x_i \Delta y_j$$

for some $(x_i', y_j') \in R_{ij}$. On the other hand, by the same mean value theorem, we have

$$(5) \quad J_I := \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} \left(\int_c^d f(x,y)dy \right) dx$$

$$= \sum_{i=1}^N \Delta x_i \int_c^d f(x_i'', y)dy = \sum_{i=1}^N \sum_{j=1}^K \Delta x_i \int_{y_{j-1}}^{y_j} f(x_i'', y)dy = \sum_{i=1}^N \sum_{j=1}^K f(x_i'', y_j'')\Delta x_i \Delta y_j$$

for some $(x_i'', y_j'') \in R_{ij}$.

We now choose δ so that, by uniform continuity, whenever two points in R satisfy $|(x_1,y_1) - (x_2,y_2)| < \delta$, we have $|f(x_1,y_1) - f(x_2,y_2)| < \epsilon/\text{Area}(R)$. We choose a partition of R into rectangles small enough so that $|\Delta x_i| +$

$|\Delta y_j| < \delta$. In particular, $|(x_i', y_j') - (x_i'', y_j'')| < \delta$. We then have

$$|J_R - J_I| < \sum_{i=1}^{N} \sum_{j=1}^{K} \frac{\epsilon}{\text{Area}(R)} \Delta x_i \Delta y_j = \epsilon \qquad \Box \tag{6}$$

Since $\epsilon > 0$ is arbitrary, and by definition J_R and J_I do not depend on ϵ, it follows that $J_R = J_I$.