LEBESGUE MEASURE, INTEGRAL, MEASURE THEORY: A QUICK INTRO

This note is meant to give an overview of some general constructions and results, certainly not meant to be complete, but with your knowledge of Riemann integration you should be able to get the general idea. For a more detailed survey with no proofs but with good references see [1]. A self-contained manageably short intro can be found in [2]. Please read Sec. 4.8 in the book too.

1.

A natural starting point is to first clarify what we understand by length, area, volume, or, more generally by a *positive measure of sets* μ (there are negative, complex-valued or even more general measures, but we are not going that far). What properties should a positive measure have? What should we allow for as a measurable set?

Clearly, if A and B are measurable and disjoint, it is natural to try to arrange that $A \cup B =: A \oplus B$ is measurable. Furthermore, we should have

(1) $\mu(A \oplus B) = \mu(A) + \mu(B)$; more generally, $\mu(A_1 \oplus \cdots \oplus A_n) = \sum_{j=1}^{n} \mu(A_j)$

Why stop with finitely many sets? Assume $\{A_j\}_{j\in\mathbb{N}}$ is a family of measurable sets which are disjoint. Then we define

$$\mu(\oplus_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j) \tag{2}$$

The infinite sum on the right always makes sense in $[0, \infty]$: a sum of positive numbers converges by the monotone convergence theorem to a number or to $+\infty$. Thus, $\mu(A) \in [0, \infty]$. Allowing for countable unions is a fundamental improvement over the Jordan measure, making it possible to measure limits of sets. We can define limits of sets precisely by saying that

$$A_j \to A \text{ if by definition } \chi(A_j) \to \chi(A) \tag{3}$$

where as usual χ is the characteristic function.

Our experience with Jordan measures indicates that sets are measurable if their boundary is reasonable. Since $\partial A = \partial A^c$ we should allow for A_c to be measurable if A is.

We now proceed with a precise definition of the family of measurable sets: A $\sigma-$*algebra of sets*, subsets of a bigger space, say $X = \mathbb{R}^n$ is a family of sets $\mathcal{F} \subset 2^X$ (2^X is the set of *all subsets* of X) with the properties:

(1) $X \in \mathcal{F}$ (more generally, $\mathcal{F}$ is non-empty).

(2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.
(3) $\forall j \in \mathbb{N}\ A_j \in \mathcal{F} \Rightarrow \cup_{j=1}^{\infty} A_j \in \mathcal{F}$

(of course, the union can also be finite, just take $A_{j+1} = A_j$ for all $j \geq j_0$).

Exercise 1. *Show that the definition of a $\sigma-$ algebra implies that*

$$\cap_{j=1}^{\infty} A_j \in \mathcal{F}; \qquad A_i \setminus A_j \in \mathcal{F}; \qquad \emptyset \in \mathcal{F} \tag{4}$$

1.1. **Borel sigma algebras $\mathcal{B}$ (in $\mathbb{R}^n$).** It would be nice to be able to provide a good measure for every set. This is not possible in the standard axiomatization of math (ZFC). See however Note 4. Open and closed sets, and many more however can be arranged to be measurable; again, see Note 4.

The Borel sigma algebra $\mathcal{B}$ in $\mathbb{R}^n$ is defined as the smallest $\mathcal{F} \subset 2^{\mathbb{R}^n}$ such that every open ball in $\mathbb{R}^n$ is in $\mathcal{B}$.

It is quite easy to see that $\mathcal{B}$ indeed exists. We leave it as a guided exercise:

Exercise 2. Check that $2^{\mathbb{R}^n}$ is a $\sigma-$algebra that contains every open ball in $\mathbb{R}^n$. Show that if $\mathcal{F}_\alpha$ are $\sigma-$algebras then their intersection $\mathcal{F} = \cap_\alpha \mathcal{F}_\alpha$ is a $\sigma-$algebra. Here the intersection $\cap_\alpha \mathcal{F}_\alpha$ can be finite or infinite, even uncountably infinite. Hint: if $A, B \in \mathcal{F}$ then $A, B \in \mathcal{F}$ for *every* α and thus $A \cup B \in \mathcal{F}$ as well, for every α implying $A \cup B \in \mathcal{F}$.

According to Exercise 2, you can convince yourself that

$$\mathcal{B} = \cap\{\mathcal{F} \ \in 2^{\mathbb{R}^n} : \text{every open ball in } \mathbb{R}^n \text{ is in } \mathcal{F}\} \tag{5}$$

Exercise 3. Show that if $\mathcal{O} \subset \mathbb{R}^n$ is open then $\mathcal{O} \in \mathcal{B}$. Hint: $\mathbb{Q}$ is countable and so is $\mathbb{Q}^n$. Every open set contains any x together with an open ball. Any such open ball contains a smaller ball centered on a $a \in \mathbb{Q}^n$...

Now show that every *closed* set is in $\mathcal{B}$, every countable intersection of open sets is also in $\mathcal{B}$ as is every countable union of closed sets. In particular $\mathbb{Q}^n$ is measurable.

Note 4. In fact every set S that you can in one way or another describe concretely, construct algorithmically or for which you can decide logically (and not by a "random choice") whether a given x is an element of S or not is automatically measurable. The existence of non-measurable sets relies on the *Axiom of choice*, which in some vague sense states that completely arbitrary choices exist, or slightly more precisely that for any family of unordered sets of objects there is a choice function which associates to set of objects exactly one of its elements. In simple cases, a definite choice can be made. For instance if we have to choose between a pair of real numbers, we choose the largest. It is easy to choose between two polynomials. We can even come up with a definite choice between two continuous functions, but it is trickier. Any idea? But to choose an element from a completely arbitrary set, we need the axiom of choice.

The measure generated by balls is the Lebesgue measure. For instance in $\mathbb{R}^2$ we define $\lambda(B_r) = area(B_r) = \pi r^2$ for every ball $B_r \in \mathbb{R}^2$ of radius r. Let's go into more detail for $\mathbb{R}$; here, the balls are the intervals and we define $\lambda([a,b]) = b-a$. A point has measure $\lambda(\{a\}) = \lambda([a,a]) = 0$; thus $\lambda((a,b)) = \lambda([a,b))$ etc. (We can take this as a definition at this point.) Now, for an open set $\mathcal{O}$ we take a countable set of intervals B_j s.t. $\mathcal{O} = \cup_{j\in\mathbb{N}} B_j$. We can arrange every finite union $\cup_{j\le N} B_j$ to be disjoint, by breaking the intervals further into $B'_j, j = 1...N_1$ if needed. Then, we have $\cup_{j\le N} B_j \to \mathcal{O}$ by construction, in the sense (3) and we define

$$\lambda(\mathcal{O}) = \lim_{N\to\infty} \lambda(\cup_{j\le N} B_j) =: \lim_{N'\to\infty} \lambda(\oplus_{j\le N'} B'_j) = \lim_{N'\to\infty} \sum_{j=1}^{N'} \lambda(B'_j) \tag{6}$$

It can be shown that this is well defined and does not depend on the way we write $\mathcal{O}$ as union of B_j. Similarly we can define the measure of a compact set $\mathcal{C}$. Now on $\mathcal{B}$ we define λ by

$$A \in \mathcal{B} \Rightarrow \lambda(A) = \inf\{\lambda(\mathcal{O}) : \mathcal{O} \supset A\} = \sup\{\lambda(\mathcal{C}) : \mathcal{C} \text{ compact}, \mathcal{C} \subset A\} \tag{7}$$

The fact that we have equality here requires a proof, of course. It it not sufficiently simple to be left as an exercise, so you have to look say at [2] for how to show this.

Next, one allows for *any* subset E of a zero measure set to be measurable, and assigns zero measure E. One defines $\mathcal{L}$, the sigma-algebra of Lebesgue measurable sets to be the sigma-algebra generated by $\mathcal{B}$ together with all these zero measure sets. Every set A^* in $\mathcal{L}$ differs from a set $\mathcal{B}$ by a set of zero measure; we assign a measure on $\mathcal{L}$ by defining $\lambda(A^*) = \lambda(A)$. *These are the Lebesgue measurable sets, and λ is the Lebesgue measure. It has the property*

$$\lambda(\oplus_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \lambda(A_j) \tag{8}$$

Exercise: show that $\lambda(\mathbb{Q}) = \lambda(\mathbb{Q}^n) = 0$.

1.2. **Measurable functions and the Lebesgue integral.** A function is measurable if $f^{-1}(\mathcal{O}) \in \mathcal{L}$ for every open $\mathcal{O}$. Equivalently, $f^{-1}(A) \in \mathcal{L}$ for every $A \in \mathcal{B}$. All continuous functions and many more are measurable:

Note 5. Essentially equivalently to Note 4: every function that you can concretely define, or calculate, or even approximate with arbitrary precision is measurable.

Given a measurable $f \ge 0 : S \to \mathbb{R}$, with S measurable in $\mathbb{R}^n$ and $f \ge 0$, one can show without much difficulty that the set

$$G_{f;S} := \{(x,y) \in \mathbb{R}^n \times R : x \in S, 0 \le y \le f(x)\} \tag{9}$$

is measurable. Though this is not the most common definition, nor is it suitable for abstract generalization it is perfect for a short presentation: we define

$$\int_S f(x)d\lambda(x) = \lambda(G_{f;S}) \tag{10}$$

where the λ in the integral is the measure on $\mathbb{R}^n$ while the one on the right side, naturally, is the measure on $\mathbb{R}^n \times \mathbb{R}$.

Note 6. It can be checked without much difficulty that if f, g are measurable, then so are $|f|$, $af + bg$ $f \cdot g$ etc., and

$$\int_S [af(x) + bg(x)d\lambda(x)] = a\int_S f(x)d\lambda(x) + b\int_S g(x)d\lambda(x) \tag{11}$$

Also, if $\lambda(S \cap T) = 0$ then

$$\int_{S\cup T} f(x)d\lambda(x) = \int_S f(x)d\lambda(x) + \int_T f(x)d\lambda(x) \tag{12}$$

$$0 \le f \le g \Rightarrow \int_S f(x)d\lambda(x) \le \int_S g(x)d\lambda(x) \tag{13}$$

Also, if $\int_S |f(x)|d\lambda(x) < \infty$ we define

$$\int_S f(x)d\lambda(x) = \int_S |f(x)|d\lambda(x) - \int_S \Big(|f(x)| - f(x)\Big)d\lambda(x) \tag{14}$$

a difference of two integrals of measurable nonnegative functions.

Note 7. *The set of functions for which $\int_S |f|d\lambda < \infty$ is denoted by $L_1(S)$.*

All properties that we have seen for the Riemann integral hold for $\int d\lambda$, but $\int d\lambda$ requires far fewer restrictions. Also, the Lebesgue integral equals the Riemann integral when the latter exists.

1.3. **Some useful theorems.**

1.3.1. *Fubini's theorem–the analog of the iterated integration theorem.* It can be relatively easily shown that if $f(x, y)$ is $A \times B$ measurable then for each x, $g(y) = f(x, y)$ (with fixed x) is B measurable, $h(x) = f(x, y)$ (fixed y) is A measurable, $\int_A |f(x, y)|d\lambda(x)$ is B measurable and $\int_B |f(x, y)|d\lambda(y)$ is A measurable.

Theorem 8 (Fubini)**.** *Suppose $f(x, y)$ is $A \times B$ measurable. If*

$$\int_{A\times B} |f(x, y)|\, d\lambda(x, y) < \infty, \tag{15}$$

then

$$\int_A \left(\int_B f(x, y)\, d\lambda(y)\right)\, dx = \int_B \left(\int_A f(x, y)\, d\lambda(x)\right) = \int_{A\times B} f(x, y)\, d\lambda(x, y) \tag{16}$$

Theorem 9 (Dominated convergence theorem)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued measurable functions. Suppose that $f_n(x) \to f(x)$ for every x in some measurable set S and that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is dominated by some integrable function $g \geq 0$ in the sense that*

$$\forall x \in S \text{ and } n \in \mathbb{N} \text{ we have } |f_n(x)| \leq g(x) \text{ with } \int_S g d\lambda < \infty \tag{17}$$

Then, f is integrable and

$$\lim_{n\to\infty} \int_S f_n \, d\lambda = \int_S f \, d\lambda \tag{18}$$

Theorem 10 (Monotone convergence theorem)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a pointwise non-decreasing sequence of non-negative functions defined on a measurable set S, that is*

$$\forall k \in \mathbb{N}, x \in S \text{ we have } 0 \leq f_k(x) \leq f_{k+1}(x) \tag{19}$$

Let

$$f(x) := \lim_{k\to\infty} f_k(x) \tag{20}$$

Then f is measurable and

$$\lim_{k\to\infty} \int_S f_k \, \mathrm{d}\lambda = \int f \, \mathrm{d}\lambda \tag{21}$$

References

[1] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Academic Press, New York, (1972).
[2] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, (1987).