PRACTICE PROBLEMS FOR THE SECOND MIDTERM

Justify your answers. You can use any theorem you learned as well as facts proved in homework problems. Many problems admit simple solutions, so if the calculations get very involved, there is probably a better approach. Some problems may even admit solutions which do not use results from the book; such solutions are perfectly acceptable as long as they are rigorous.

A "typical Midterm 2" would consist of about 4 problems, 25 p. each. As in the first one, I might give you a bonus problem that would increase your score up to a total of about 110 points for an exam worth 100 points. Some of the problem below have many parts, cases, etc. These are meant to help you practice. You would not expect more than say 2 cases on an actual exam.

(1) Find the equation of the tangent plane to the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$ at the point $(1, 1, 1)$.

(2) Is the curve $\mathbf{g}(t) = \{(t^3, t^2) : t \in (-1, 1)\}$ smooth? Explain carefully. Is $\mathbf{g}([-\frac{1}{2}, \frac{1}{2}])$ a set of zero content?

(3) Show that the intersection between the sphere $x^2 + y^2 + z^2 = 1$ and the ellipsoid $10(x^2 + y^2) + z^2 = 6$ consists of two smooth curves.

(4) Consider the region $S = \{(x, y) : x \in [0, 1], y \in [0, f(x)]\}$ where f is positive and continuous. Is S a Jordan measurable subset of $\mathbb{R}^2$? Explain.

(5) Consider the double integral

$$\iint_S f(x, y)dA$$

where S is the parallelogram with vertices $(0, 0), (2, 1), (3, 3), (1, 2)$ and the transformation $x_1 = 2x + y, y_1 = x + 2y$. Write the transformed integral (that is the integral in (x_1, y_1); remember to calculate what S becomes).

(6) Which of the following improper integrals exist? Explain. (*You are* **not** *required to calculate the integrals*). Below, D_1 is the unit disk.

$$\iint_{D_1} \frac{x^2y^2}{(x^2+y^2)^2}dxdy$$
$$\iint_{D_1} \frac{x^2y^2}{(x^2+y^2)^3}dxdy$$
$$\iint_{\mathbb{R}^2\setminus D_1} \frac{x^2y^2}{(x^2+y^2)^3}dxdy$$
$$\iint_{\mathbb{R}^2} e^{-x^2-y^2+xy}dxdy$$
$$\iint_{\mathbb{R}^2} e^{-x^2-y^2+2xy}dxdy$$

(7) Let
$$g(x) = \int_0^1 e^{xy+y^2}dy$$
Show that $g'(0) = \frac{1}{2}(e-1)$. Justify your steps.

(8) Calculate the integral
$$\int_0^a \left(\int_y^a e^{x^2}dx\right)dy$$
in terms of elementary functions.

(9) Let S be the solid obtained by intersecting the ball $\{(x,y,z) : x^2+y^2+z^2 \leq 4\}$ and the cylindrical domain $\{(x,y,z) : x^2+z^2 \leq 1\}$. Show that S is Jordan measurable, and find its volume.