LINEAR CHANGES OF VARIABLES IN MULTIPLE INTEGRALS

1. Some linear algebra notes about matrices

This is basically a crash course in the algebra of matrices starting essentially from scratch, knowing only what multiplication of matrices is.

What is discussed below for 2×2 matrices can be generalized in a straightforward way to $m\times m$ matrices. Think about this when you finish the work in this section.

Remember that the square matrix M is invertible if, by definition, there is another matrix, denoted by M^{-1} s.t. $MM^{-1}=I$ (or, equivalently, $M^{-1}M=I$) where

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Let

$$M:=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

and

$$\Delta := \det(M)=ad-bc$$

Check that M is invertible **iff** $\Delta\ne 0$ and

$$M^{-1}:=\begin{pmatrix}\frac{d}{\Delta} & \frac{-b}{\Delta}\\ \frac{-c}{\Delta} & \frac{a}{\Delta}\end{pmatrix}=\frac{1}{\Delta}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

and then

$$M^{-1}M=MM^{-1}=I$$

(a) Define the following elementary matrices:

$$A:=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix};\ T:=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};\ S_{1,\alpha}=\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\tag{1}$$

We will also denote these matrices by B_i, $i=1,2,3$.

(b) Check that

$$\det(A)=1,\ \det(T)=-1,\ \det(S_{1,\alpha})=\alpha\tag{2}$$

Note that

$$TS_{1,\beta}T=:S_{2,\beta}=\begin{pmatrix}1 & 0\\ 0 & \beta\end{pmatrix}\tag{3}$$

Exercise 1. *Show that $B_i^{-1}, i=1,2,3$ exist and they can be written as products of $B_i, i=1,2,3$. For instance,*

$$A^{-1}=S_{2,-1}AS_{2,-1}=TS_{1,-1}TATS_{1,-1}T\tag{4}$$

where, for the last equality we used (3)*.*

Exercise 2. *Show that for $i=1,2,3$ and M an arbitrary 2×2 matrix we have*

$$\det(B_iM)=\det(MB_i)=\det(M)\det(B_i)\tag{5}$$

(c) Note that

$$AM=\begin{pmatrix}a+c & b+d\\ c & d\end{pmatrix};\ TM=\begin{pmatrix}c & d\\ a & b\end{pmatrix};\ MT=\begin{pmatrix}b & a\\ d & c\end{pmatrix};\ S_{1,\alpha}M=\begin{pmatrix}\alpha\, a & \alpha\, b\\ c & d\end{pmatrix}\tag{6}$$

Exercise 3. Show that the following operations can be performed on an arbitrary matrix M, solely through multiplications with $B_i, i=1,2,3$:

(i) Switching two rows or two columns of M.
(ii) Multiplying a row or a column of M by a number.
(iii) Adding a multiple of a row to another row, for instance

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+\lambda c & b+\lambda d \\ c & d \end{pmatrix} \tag{7}$$

Exercise 4. (A) Take a matrix M with nonzero determinant. This implies that at least one matrix element is nonzero. Show that, using a sequence of operations from the ones listed in Exercise 3, M can be brought to I, that is, for some choice of indices i_k, j_l we have

$$B_{i_1} B_{i_2} \cdots B_{i_n} M B_{j_1} B_{j_2} \cdots B_{j_k} = I \tag{8}$$

If you remember this concept, it means that you can bring M to the **reduced row-echelon form** by the elementary transformations B_i.

(B) Show, using Exercise 1 on p.1. that this implies that any invertible matrix can be written as a product of $B_i, i = 1, 2, 3$, that is, for some choice of indices $l_1, ..., l_k$

$$M = B_{l_1} B_{i_2} \cdots B_{l_k} \tag{9}$$

(C) Show that this and Exercise 2 implies that for arbitrary invertible matrices M_1 and M_2 we have

$$\det(M_1 M_2) = \det(M_2 M_1) = \det(M_1)\det(M_2) \tag{10}$$

(D) Show that, in fact, (10) holds even if $\det(M_i)$ are not necessarily nonzero.

(d) Now, as discussed, you may want to think how to generalize these results to the $m \times m$ case. The only thing is, you may not know what the determinant of an $m \times m$ matrix is. But this can be done using the B_i decomposition, defining $\det B_i$ and of I as a natural generalization from the 2×2 case, and showing that the product of determinants is independent of the way you write the decomposition. There is an easy way to show this independence!

2. Geometrical interpretation

Exercise 5. (i) Two vectors $\mathbf{v}, \mathbf{w}$ determine a parallelogram $P_{\mathbf{v},\mathbf{w}}$. Denote by $A(P_{\mathbf{v},\mathbf{w}})$ its area (understood as a positive number). A matrix M transforms $P_{\mathbf{v},\mathbf{w}}$ into $P_{M\mathbf{v},M\mathbf{w}}$. Convince yourself that, since the transformation M is linear,

$$\mu(M) = \frac{A(P_{M\mathbf{v},M\mathbf{w}})}{A(P_{\mathbf{v},\mathbf{w}})}$$

does not depend on $\mathbf{v}, \mathbf{w}$. (Don't use the formula $A(P_{\mathbf{v},\mathbf{w}}) = \det(\mathbf{v}, \mathbf{w})$ since we are proving this from first principles and want to generalize to $\mathbb{R}^n$.)

Explain why $\mu(T) = 1$ and $\mu(A) = 1$ whereas $\mu(S_{1,\alpha}) = |\alpha|$.

(ii) Use (i) above and formula (10) to show that $\mu(M) = |\det M|$.

(iii) Think of the $\mathbb{R}^n$ generalization of this.