PRACTICE PROBLEMS FOR MIDTERM 1

Justify your answers. You can use any theorem you learned as well as facts proved in homework problems. Many problems admit simple solutions, so if the calculations get very involved, there is probably a better approach. Some problems may even admit solutions which do not use results from the book; such solutions are perfectly acceptable as long as they are rigorous.

A "typical Midterm 1" would consist of about 4-5 problems, for instance problems 5, 8, 10, and 13 below. (You may want to save these problems for a self-test.) I might give you a bonus problem that would increase your score up to a total of about 110 points for an exam worth 100 points.

(1) Can the intersection of a sequence of nonempty closed sets $C_1 \supset C_2 \supset \cdots$ be empty?
(2) Is the set of rational numbers $\mathbb{Q}$ a connected subset of $\mathbb{R}$?
(3) Is the function $f(x) = 1/\ln x$ uniformly continuous on $(0, 1/2)$?
(4) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function.
  (a) Show that L is continuous on $\mathbb{R}^n$.
  (b) Decide whether the statements below are true or false. If a statement is true, provide a proof; if not, give a counterexample.
    (i) A nonempty compact set in $\mathbb{R}^n$ is never open.
    (ii) If A is an open set, then $L(A)$ is open.
    (iii) If A is a closed set, then $L(A)$ is closed.
    (iv) If A is a compact set, then $L(A)$ is compact.
    (v) If A is convex, then $L(A)$ is convex.
(5) Show that the function given by $f(0) = 0$ and $f(x, y) = xy^2/(x^2 + y^2)$ is continuous in $\mathbb{R}^2$.
(6) Find the minimal and maximal distance between the points on the curve given by $x^4 + y^4 = 1$ in $\mathbb{R}^2$ and the origin.
(7) Show that the following functions are continuous from $\mathbb{R}^n$ to $\mathbb{R}$:
  (a) $f(x_1, x_2, ..., x_j, ..., x_n) = x_j$.
  (b) $f(\mathbf{x}) = |\mathbf{x}|$.
  (c) $f(\mathbf{x}) = x_2/(1 + |\mathbf{x}|)$

(8) Consider the polynomial

$$P(\mathbf{x}) = \sum_{|\alpha| \leq k} c_\alpha \mathbf{x}^\alpha$$

Show that $P \in C^k$ for all k.

(9) Find the norm of the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(10) Find the minimum and the maximum of the function $f(x, y, z) = xyz$ in the closed ball of radius one centered at zero.

(11) Suppose the partial derivatives of a function f are given by $\frac{\partial f}{\partial x} = e^{2x}(2 \sin x + \cos x)$ and $\frac{\partial f}{\partial y} = 1$. Is this function C^2? Is it C^∞?

Is there a C^2 function g with partial derivatives $\frac{\partial g}{\partial y} = e^{2x}(2 \sin x + \cos x)$ and $\frac{\partial g}{\partial y} = x$?

(12) Find the Taylor polynomial of degree three about zero of the function $f(x, y) = e^{x^2} \cos(x+y) \sin(x+y)$. Justify your answer.

(13) Does the function $f(x, y) = (x - y)e^{-x^2-y^2}$ have an absolute minimum, an absolute maximum or both on $\mathbb{R}^2$? Explain, and if extremum points exist, find them. Same questions for $f(x, y) = xye^{-x^2-y}$.

(14) Assume the function $f : \mathbb{R}^n \to \mathbb{R}$ is C^1. Is the function $g(x) = |f(x)|$ necessarily C^1 as well? If not, find a sufficient condition on f for g to be C^1.