# RECTIFIABLE CURVES

## 1. Discussion of the definition

Assume that the curve $C$ is given by the graph of $\mathbf{g}$, $C = \mathbf{g}([a, b])$ Given a partition $\{t_0, ..., t_J\}$ of $[a, b]$, the length of a polygonal path through $C$ is

$$L_P(C) = \sum_1^J |\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})| \tag{1}$$

Since a straight line gives the shortest distance between two points, the length of $C$, if it exists, is always larger than $L_P(C)$. Note that $L_P$ is increasing in $P$: if $P' \supset P$ is a subpartition of $P$ then, by the triangle inequality (think of the finer polygon), $L_P(C) \le L_{P'}(C)$. Thus, the sup over all partitions $L(C)$ is in a good sense the limit of $L_P$ when the partition becomes finer and finer and it is natural to call $L(C)$ the length of $C$.

## 2. When are curves rectifiable?

Let $f$ be a scalar function defined on $[a, b]$. Its total variation is defined very similarly to $L(C)$:

$$V_b^a(f) = \sup_P \sum_1^J |f(x_j) - f(x_{j-1})| < \infty \tag{2}$$

where $P$ are partitions of $[a, b]$. The space of functions of bounded variation $BV$ is defined as the space of functions for which $V_b^a(f)$ is finite:

$$f \in BV([a, b]) \iff V_b^a(f) < +\infty \tag{3}$$

These notions were introduced by Camille Jordan (the same who introduced what we now call the Jordan measure).

**Note 1.** If $\mathbf{g} = (g_1, ..., g_n)$ it is easy to see that $\mathbf{g}$ is rectifiable on $[a, b]$ **iff** $g_i \in BV([a, b])$ for each $i = 1, 2, ..., n$.

This is because of the equivalence of Euclidian norms, (1.3) on p.6 in the text. Note that $V_b^a(g_i)$ is the length of the graph of $g_i$ on $[a, b]$. It is known that functions in $BV$ are differentiable almost everywhere (that is, except perhaps on a set of zero Lebesgue measure).

If $|\mathbf{g}'|$ is continuous, the construction of its Riemann integral is closely related to that of $L(C)$, and the proof below follows this connection.

**Exercise 2** (Finite differences versus derivatives)**.** *Consider the function*

$$\mathbf{h}(x, y) = \begin{cases} \dfrac{\mathbf{g}(y) - \mathbf{g}(x)}{y - x}; & \text{if } x \neq y \\ \mathbf{g}'(x); & \text{if } x = y \end{cases} \tag{4}$$

*Show that* $\mathbf{h}(x, y) = \int_0^1 \mathbf{g}'\big(tx + (1 - t)y\big)dt$. *Use this to show that* $\mathbf{h}$ *is continuous in* $(x, y) \in [a, b]^2$.

## 3. The length formula when $\mathbf{g} \in C^1$

**Theorem 3.** *If* $\mathbf{g}$ *be* $C^1$ *on* $(a', b') \supset [a, b]$*, then* $L(C)$ *exists and equals* $\int_a^b |\mathbf{g}'(t)|dt$*.*

**Note 4.** Using the exercise, the idea is completely straightforward: with $m_j$ the minimum point of $\mathbf{g}$ between $t_{j-1}$ and $t_j$ we can write

$$\mathbf{g}(t_j) - \mathbf{g}(t_{j-1}) = \mathbf{g}'(m_j)(t_j - t_{j-1}) + \epsilon_j(t_j - t_{j-1}) \tag{5}$$

where $\epsilon_j$ is *uniformly small* on $[a, b]$. Thus for a fine enough partition $P$, $L_P(C)$ is arbitrarily close to the lower Riemann sum $s_P(|\mathbf{g}'|)$ (we could have equally well worked with the maximum point $M_j$ and the upper sum $S_P$).

*Proof of the theorem.* The proof merely consists of writing down the details in the note carefully.

Since $|\mathbf{g}'|$ is continuous, it is integrable on $[a, b]$ and there is a partition $P'$ s.t.

$$\left| s_{P'}(|\mathbf{g}'|) - \int_a^b |\mathbf{g}'(t)|dt \right| = \left| \sum_{1 \leqslant j \leqslant J'} |\mathbf{g}'(m_j)| - \int_a^b |\mathbf{g}'(t)|dt \right| < \epsilon$$

Since $[a, b]^2$ is compact, by the exercise above $\mathbf{h}$ is uniformly continuous on $[a, b]^2$. Thus for any $\epsilon$ there is a $\delta$ such that $|(x, y) - (x', y')| < \delta \Rightarrow |\mathbf{h}(x, y) - \mathbf{h}(x', y')| < \epsilon$. In particular,

$$|(x, y) - (x, y')| < \delta \Rightarrow |\mathbf{h}(x, y) - \mathbf{g}'(x)| < \epsilon \tag{6}$$

Choose any partition $P$ and take a refinement, if necessary, to arrange that $t_j - t_{j-1} < \delta$. For each $j$, choose $m_j \in [t_{j-1}, t_j]$ to be the point where $|\mathbf{g}'(m_j)|$ is minimum. For any partition $P''$ finer than both, $P'' \supset P, P'' \supset P'$, we then have $\left| s_{P''}(|\mathbf{g}'|) - \int_a^b |\mathbf{g}'(t)|dt \right| < \epsilon$. (Review the notions if you forgot them.) By (6) and the choice of $P'$,

$$\mathbf{g}(t_j) - \mathbf{g}(t_{j-1}) - \mathbf{g}'(m_j)(t_j - t_{j-1}) = \epsilon_j(t_j - t_{j-1}); \quad \text{where } |\epsilon_j| < \epsilon \tag{7}$$

Thus, by summing,

(8)

$$L_P(C) \le L_{P''}(C) = \sum_{1 \leqslant j \leqslant J''} |\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})| = \sum_{1 \leqslant j \leqslant J''} |\mathbf{g}'(m_j)|(t_j - t_{j-1}) + \epsilon'(b - a)$$

$$= s_{P''}(|\mathbf{g}'|) + \epsilon'(b - a) = \int_a^b |\mathbf{g}'(t)|dt + \epsilon'' + \epsilon'(b - a); \quad \text{where } |\epsilon'|, |\epsilon''| < \epsilon$$

Since $P$ was arbitrary and we can take $\epsilon$ arbitrarily small, it follows immediately from these inequalities that

$$L(C) = \sup_P L_P(C) = \int_a^b |\mathbf{g}'(t)|dt \tag{9}$$

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