REVIEW OF SERIES

0.1. **Introduction.** As vector spaces, $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic; we can think of $z = x + iy$ as a notation for (x, y). (What sets aside $\mathbb{C}$ is the extra structure we endow it with, coming from multiplication.) In particular, we have $|z| = \sqrt{x^2 + y^2}$ the same as the Euclidian norm. Likewise, from a topological point of view we can identify the space of complex functions $f(z)$ with $\mathbb{R}^2$ valued function $u(x, y) + iv(x, y)$ defined on a domain in $\mathbb{R}^2$. A sequence of complex numbers $z_n = x_n + iy_n$ converges to $z = x + iy$ if, by definition, the sequence (x_n, y_n) converges to (x, y).

A sequence of functions $f_n = u_n + iv_n$ defined on a set $S \subset \mathbb{C}$ converges pointwise to $f = u + iv$ if the sequence of vector functions (u_n, v_n) converges pointwise to (u, v).

It converges uniformly on S if $\sup_{\in S} |f_n(z) - f(z)| \to 0$. Continuity of $f(z)$ is defined in the same way as continuity of $(u(x, y), v(x, y))$.

Things become more interesting when we introduce multiplication on $\mathbb{R}^2$ this is a vector valued function of two vector variables $(x, y) \cdot (s, t) = (xs - yt, xt + ys)$. Of course, this is the same operation but certainly less intuitive than saying that zw is calculated by "usual" multiplication of $x + iy$ and $s + it$ with the convention $i^2 = -1$. Direct calculation shows that

$$|zw| = |z||w| \tag{1}$$

It is clear from (1) that multiplication is a continuous function from $\mathbb{C} \times \mathbb{C}$ into $\mathbb{C}$. Inductively, the function z^n is shown to be continuous, and so is any polynomial.

At this stage, we can introduce complex conjugation $\overline{z} = \overline{x + iy} = x - iy$ and then $|z|^2 = z\overline{z}$.

A sequence $\{z_n\}_{n\in\mathbb{N}}$ is convergent **iff** it is Cauchy. We know this from the $\mathbb{R}^2$ convergence. A finite power series is a polynomial

$$S = \sum_{k=0}^{n} c_k z^k; \quad \text{or, also,} \quad \sum_{k=0}^{n} c_k (z - z_0)^k; \tag{2}$$

and a "true" power series is an expression of the form

$$S = \sum_{k=0}^{\infty} c_k z^k = \lim_{m\to\infty} \sum_{k=0}^{m} c_k z^k \tag{3}$$

is the limit exists. If the limit does not exist, or we have not yet been proven to exist, $\sum_{k=0}^{\infty} c_k z^k$ is viewed as an abstract notation and we most often start from it and say that $\sum_{k=0}^{\infty} c_k z^k$ converges if the limit in (3) exists. By the

Cauchy criterion,

$$S \text{ is convergent if } \lim_{n_1,n_2\to\infty}\left|\sum_{n_1}^{n_2} c_k z^k\right| = 0 \tag{4}$$

uniformly in n_1, n_2 as they go to infinity.

Note 1. *if* the series of nonnegative numbers

$$\sum_{k=1}^{\infty} |c_n||z|^n \tag{5}$$

converges, then it is Cauchy,

$$\lim_{n_1,n_2\to\infty}\sum_{n_1}^{n_2} |c_k||z|^k = 0 \tag{6}$$

uniformly in n_1, n_2 as they go to infinity, and then by the triangle inequality, $\lim_{n_1,n_2}\left|\sum_{n_1}^{n_2} c_k z^k\right| = 0$ and thus $\sum_{k=0}^{\infty} c_k z^k$ converges as well. A series for which (6) holds is called *absolutely convergent*.

We have thus shown:

Proposition 2. *If the series S converges absolutely, then it converges.*

We know that, for series of real numbers of the form

$$\sum_{k=1}^{\infty} c_n t^n, \quad c_n \in \mathbb{R} \tag{7}$$

The domain of convergence is a symmetric interval around 0. This can be $\langle -a, a\rangle$ where $\langle , \rangle$ can be closed or open parentheses. Here the maximal $a \in [0,\infty]$ is the *radius of convergence*; it can be anything from 0 ($[0,0] = \{0\}$) or to ∞, $(-\infty,\infty) = \mathbb{R}$. We know from series over $\mathbb{R}$ that if $a \in (0,\infty)$, then the series (7) converges absolutely and uniformly for $|t| < a$ while for $|t| > a$ the sequence $|c_k||t^k|$ is unbounded, in particular S cannot be Cauchy nor therefore convergent.

0.2. **Uniform convergence.** There are several equivalent ways to define uniform convergence. We'll use the following: the function sequence $\{f_n\}_{n\in\mathbb{N}}$, $f_n : S \to \mathbb{R}^n$ converges uniformly to f, $f_n \stackrel{u}{\to} f$ if

$$\lim_{n\to\infty}\sup_{x\in S}|f_n(x) - f(x)| = 0 \tag{8}$$

Proposition 3. *Assume f_n are continuous in S and $f_n \stackrel{u}{\to} f$. Then f is continuous in S.*

Proof. Let a be any point in S. To show continuity at a, let $\varepsilon > 0$ and choose n s.t. $\sup_{x\in S}|f_n(x) - f(x)| < \varepsilon/3$. Since f_n is continuous, let δ be s.t. $|f_n(x) - f_n(a)| < \varepsilon$ for all x with $|x - a| < \delta$. Then, for $|x - a| < \delta$ we have

$$|f(x) - f(a)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| < \varepsilon \tag{9}$$

∎

Proposition 4. *Assume $f_n : S \to \mathbb{R}^m$ are continuous and the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy in S, that is*

$$\lim_{n\to\infty} \sup_{m\ge n, x\in S} |f_n(x) - f_m(x)| = 0 \tag{10}$$

Then there is a (unique) function f s.t. $f_n \overset{u}{\to} f$, and f is continuous.

Proof. The fact that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy in S implies in particular that $\{f_n(x)\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$ is Cauchy for any $x \in S$ and thus $f_n(x) \to f(x)$ for any x. Choose any $\varepsilon > 0$ and let n be large enough s.t. for any $m > n$ we have $\sup_{x\in S} |f_n(x) - f_m(x)| < \varepsilon$. Now

$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| < \varepsilon \tag{11}$$

and the rest follows from Proposition 3. ∎

0.3. **Convergence of integrals.**

Proposition 5. *Assume $S \in \mathbb{R}^n$ is Jordan measurable (in particular, bounded) and let $\{f_k(x)\}_{k\in\mathbb{N}}$ be a sequence of integrable functions converging uniformly in S to f, assumed to be integrable. Then*

$$\lim_{k\to\infty} \int\cdots\int_S f_k(s)d^n s = \int\cdots\int_S f(s)d^n s \tag{12}$$

Proof. Choose any $\varepsilon > 0$ and let k_0 be large enough s.t. $\sup_{x\in S} |f_j(x) - f(x)| < \varepsilon/\mathrm{Vol}_n(S)$ for all $j > k$. Then,

$$\left|\int\cdots\int_S f_k(s)d^n s - \int\cdots\int_S f(s)d^n s\right| = \int\cdots\int_S |f_k(s) - f(s)|d^n s \le \varepsilon \tag{13}$$

∎

Proposition 6. *Assume $f_n : [a,b] \to \mathbb{R}^m$ are C^1, $f_n' \overset{u}{\to} g$ and for some $c \in [a,b]$ we have $f_n(c) \to a$. Then $f_n \to^u f$ where $f \in C^1$ and $f' = g$.*

Proof. The proof can be done component by component and thus we can assume without loss of generality that $m = 1$. We have, using Proposition 5,

$$f_n(x) = f_n(c) + \int_c^x f_n'(s)ds \to a + \int_c^x g(s)ds \text{ as } n \to \infty \tag{14}$$

and the rest is straightforward. ∎

Proposition 7. *Assume that the series $f = \sum_{k=1}^\infty c_k z^k$ with $z \in \mathbb{C}$ converges for some $z_0 \ne 0 \in \mathbb{C}$. Then f, and, for any m, the series of derivatives $f^{(m)} = \sum_{k=1}^\infty c_k (z^k)^{(m)}$, converge uniformly in any disk of the form $\mathbb{D}_a := \{z : |z| < a\}$ if $a < z_0$.*

Proof. Since $\sum_{k=1}^{\infty} c_k z_0^k$ is a convergent series, the partial sums $S_m := \sum_{k=1}^{m} c_k z_0^k$ converge (by definition) and thus $\{S_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence. In particular, $S_m - S_{m-1} \to 0$ as $m \to \infty$ and thus

$$\lim_{m\to\infty} |c_m||z_0^m| = 0 \Rightarrow \sup_{m\geq 1} |c_m||z_0^m| = M < \infty \tag{15}$$

Let $\rho = a/|z_0| < 1$; for $|z| < a$ we have

$$\sum_{j=k}^{l} |c_j||z|^j < \sum_{j=k}^{l} |c_j||z_0|^j \rho^j \leq M \sum_{j=k}^{\infty} \rho^j = \frac{M\rho^k}{1-\rho} \to 0 \tag{16}$$

and thus $\sum_{j=0}^{\infty} |c_j||z|^j$ is uniformly Cauchy; the rest of the proof for f is straightforward. As for $f^{(m)}$, it suffices to prove the result for $m = 1$ since the result for general m follows by induction. Since $\rho < 1$ there is an $\varepsilon > 0$ s.t $\rho_1 = \rho(1+\varepsilon) < 1$. Clearly $j(1+\varepsilon)^{-j} \to 0$ as $j \to \infty$ and thus $\sup_j j(1+\varepsilon)^{-j} = M_1 < \infty$. Then, for some $M_3 > 0$ and $|z| < a$ we have

$$\sum_{j=k}^{l} |c_j||j||z|^{j-1} \leq M_3 \sum_{j=k}^{\infty} \rho_1^j = \frac{M_3 \rho_1^k}{1-\rho_1} \to 0 \tag{17}$$

∎

Corollary 8. *There are three possibilities for a series $f(z) = \sum_{j=0}^{\infty} |c_j||z|^j$: (a) f converges for all $z \in \mathbb{C}$; (b) f converges only for $z = 0$; (c) there is an $r \in (0,\infty)$ s.t. f converges for z in any disk $\mathbb{D}_a, a < r$ and it diverges if $|z| > R$; furthermore the sequence $|c_k R^k|$ is unbounded for any $R > r$.*

Proof. Assume f converges for some $z_0 > 0$. Let $r = \sup\{r' > 0 : \sup_k |c_k|{r'}^k < \infty\}$. As in the proof of Proposition 7, we must have $r \geq |z_0|$. The same proof shows that f converges uniformly and absolutely in any disk $\mathbb{D}_a$ if $a < r'$. The rest is immediate. ∎

Example 9. Check that

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!}, \ \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots, \ \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \tag{18}$$

are entire, and the function

$$\frac{1}{1+z^2} = \sum_{k=0}^{\infty} (-1)^k z^{2k} \tag{19}$$

is analytic in $\mathbb{D}_1$ and the series diverges at $z = \pm i$. This example shows how the function $1/(1+x^2)$ which is smooth on $\mathbb{R}$ fails to have a convergent power series in $\mathbb{R}$.