PRACTICE PROBLEMS FOR THE FINAL EXAM

Justify your answers. You can use any theorem you learned as well as facts proved in homework problems. Many problems admit simple solutions, so if the calculations get very involved, there is probably a better approach. Some problems may even admit solutions which do not use results from the book; such solutions are perfectly acceptable as long as they are rigorous.

A "typical final exam" would consist of about 6 problems from the material up to (and including) Sec. 5.8 + 2 bonus ones.

(1) Let $K_1 \subset \mathcal{O} \subset K_2$ be nonempty sets in $\mathbb{R}^n$. Assume $K_{1,2}$ are compact and $\mathcal{O}$ is open. Show that $\partial K_1 \cap \partial K_2 = \emptyset$.

(2) Find the absolute maximum and minimum of $\sin A + \sin B + \sin C$ where A, B, C are the angles of a (possibly degenerate) triangle. (That is, $A, B, C \in [0, \pi], A + B + C = \pi$.)

(3) Find the area of the region $\{(x, y) : |2x - 3y| \leq 1, |x + 3y| \leq 1\}$.

(4) Use the vector field $z\mathbf{k}$ and the divergence theorem to show that the volume of a right cylinder of height h having as a base a region S with smooth boundary equals $h \times \text{Area}(S)$.

(5) Show that the function $f(x, y) = x^4 - y^4 : \mathbb{R}^2 \to \mathbb{R}$ has no interior point of minimum or maximum.

(6) Is there a vector field $\mathbf{E}$ with the property $\nabla \times \mathbf{E} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$?

(7) Let $\mathbf{r} = (x, y, z)$ and $\mathbf{F}(x, y, z) = \dfrac{\mathbf{r}}{r^3}$. Let B be any ball containing the origin in its interior. Show that the integral

$$J = \iint_{\partial B} \mathbf{F} \cdot \mathbf{n} dA$$

does not depend on the choice of B (subject to the conditions above). Choose a suitable B and calculate the value of J.

(8) Is the graph of the function $f(x) = \sin(1/x), x \in (0, 1)$ a rectifiable curve? Explain.

(9) Calculate the surface area of the sphere $\{(x,y,z) : x^2+y^2+z^2=1\}$ above the plane $z=1/2$. (For a parametrization $\mathbf{x}=\mathbf{G}(u,v), (u,v)\in W$, the surface area is given by $\iint_W |\frac{\partial\mathbf{G}}{\partial u}\times\frac{\partial\mathbf{G}}{\partial v}|\,du\,dv$.)

(10) Show that the boundary of the region $x^4+y^4\leqslant 1$ is a smooth closed curve.

(11) Is there a C^1 function V defined on $\mathbb{R}^2\setminus\{0\}$ with the property $\nabla V=\mathbf{F}=\left(\dfrac{-y}{x^2+y^2},\dfrac{x}{x^2+y^2}\right)$?

(12) Calculate the integral
$$\iiint\limits_S z^2\,dV$$
where S is the ellipsoid $\{(x,y,z) : 4x^2+y^2+z^2\leqslant 1\}$.

For the bonus part.

(1) The Fourier coefficients c_k of a function f have the property $\sup_{k\in\mathbb{Z}}|k^{5/2}c_k|<\infty$. Show that there is only one function with these coefficients c_k, and that the function is differentiable.

(2) Find the Fourier coefficients of $\cos(t)^4$.