with nonvanishing derivative at the relevant point, and it now can be shown by elementary complex analysis means:

$$f^{-1}(z) = f^{-1}(z_0) + \sum_{n=1}^{\infty} \frac{d^{n-1}}{dw^{n-1}} \left(\frac{w - f^{-1}(z_0)}{f(w) - z_0} \right)^n \Big|_{w=f^{-1}(z_0)} \frac{(z-z_0)^n}{n!} \quad (3.54)$$

3.5 Oscillatory integrals and the stationary phase method

In this setting, an integral of a function against a rapidly oscillating exponential becomes small as the frequency of oscillation increases. Again we first look at the case where there is minimal regularity; the following is a version of the Riemann-Lebesgue lemma.

Proposition 3.55 *Assume $f \in L^1[0, 2\pi]$. Then $\int_0^{2\pi} e^{ixt} f(t)dt \to 0$ as $x \to \pm\infty$. A similar statement holds in $L^1(\mathbb{R})$.*

It is enough to show the result on a set which is dense[1] in L^1. Since trigonometric polynomials are dense in the continuous functions on a compact set[2], say in $C[0, 2\pi]$ in the sup norm, and thus in $L^1[0, 2\pi]$, it suffices to look at trigonometric polynomials, thus (by linearity), at e^{ikx} for fixed k; for the latter we just calculate explicitly the integral; we have

$$\int_0^{2\pi} e^{ixs} e^{iks} ds = O(x^{-1}) \text{ for large } x. \qquad \square$$

No rate of decay of the integral in Proposition 3.55 follows without further knowledge about the regularity of f. With some regularity we have the following characterization.

Proposition 3.56 *For $\eta \in (0, 1]$ let the $C^\eta[0, 1]$ be the Hölder continuous functions of order η on $[0, 1]$, i.e., the functions with the property that there is some constant $a > 0$ such that for all $x, x' \in [0, 1]$ we have $|f(x) - f(x')| \le a|x - x'|^\eta$.*

(i) We have

$$f \in C^\eta[0, 1] \Rightarrow \left| \int_0^1 f(s) e^{ixs} ds \right| \le \frac{1}{2} a \pi^\eta x^{-\eta} + O(x^{-1}) \quad \text{as} \quad x \to \infty \quad (3.57)$$

---

[1] A set of functions f_n which, collectively, are arbitrarily close to any function in L^1. Using such a set we can write

$$\int_0^{2\pi} e^{ixt} f(t)dt = \int_0^{2\pi} e^{ixt}(f(t) - f_n(t))dt + \int_0^{2\pi} e^{ixt} f_n(t)dt$$

and the last two integrals can be made arbitrarily small.

[2] One can associate the density of trigonometric polynomials with approximation of functions by Fourier series.

(ii) If $f \in L^1(\mathbb{R})$ and $|x|^\eta f(x) \in L^1(\mathbb{R})$ with $\eta \in (0,1]$, then its Fourier transform $\hat{f} = \int_{-\infty}^{\infty} f(s)e^{-ixs}ds$ is in $C^\eta(\mathbb{R})$.

(iii) Let $f \in L^1(\mathbb{R})$. If $x^n f \in L^1(\mathbb{R})$ with $n-1 \in \mathbb{N}$ then $\hat{f}$ is n times differentiable, with the $n-1$th derivative Lipschitz continuous. If $e^{|Ax|}f \in L^1(\mathbb{R})$ then $\hat{f}$ extends analytically in a strip of width $|A|$ centered on $\mathbb{R}$.

PROOF (i) We have as $x \to \infty$ ($\lfloor \cdot \rfloor$ denotes the integer part)

$$\left|\int_0^1 f(s)e^{ixs}ds\right| =$$
$$\left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor}\left(\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} f(s)e^{ixs}ds + \int_{(2j+1)\pi x^{-1}}^{(2j+2)\pi x^{-1}} f(s)e^{ixs}ds\right)\right| + O(x^{-1})$$
$$= \left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor}\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} \big(f(s)-f(s+\pi/x)\big)e^{ixs}ds\right| + O(x^{-1})$$
$$\le \sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor} a\left(\frac{\pi}{x}\right)^\eta \frac{\pi}{x} \le \frac{1}{2}a\pi^\eta x^{-\eta} + O(x^{-1}) \quad (3.58)$$

(ii) We see that

$$\left|\frac{\hat{f}(s)-\hat{f}(s')}{(s-s')^\eta}\right| = \left|\int_{-\infty}^{\infty}\frac{e^{-ixs}-e^{-ixs'}}{x^\eta(s-s')^\eta}x^\eta f(x)dx\right| \le \int_{-\infty}^{\infty}\left|\frac{e^{-ixs}-e^{-ixs'}}{(xs-xs')^\eta}\right|\left|x^\eta f(x)\right|dx \quad (3.59)$$

is bounded. Indeed, by elementary geometry we see that for $|\phi_1-\phi_2|<1$ we have

$$|\exp(i\phi_1)-\exp(i\phi_2)| \le |\phi_1-\phi_2| \le |\phi_1-\phi_2|^\eta \quad (3.60)$$

while for $|\phi_1-\phi_2|\ge 1$ we see that

$$|\exp(i\phi_1)-\exp(i\phi_2)| \le 2 \le 2|\phi_1-\phi_2|^\eta$$

(iii) Follows in the same way as (ii), using dominated convergence. ◻

Exercise 3.61 *Complete the details of this proof. Show that for any $\eta \in (0,1]$ and all $\phi_{1,2} \in \mathbb{R}$ we have $|\exp(i\phi_1)-\exp(i\phi_2)| \le 2|\phi_1-\phi_2|^\eta$.*

Note. In Laplace type integrals Watson's lemma implies that it suffices for a function to be continuous to ensure an $O(x^{-1})$ decay of the integral, whereas in Fourier-like integrals, the considerably weaker decay (3.57) is optimal as seen in the exercise below.

Exercise 3.62 (*) (a) Consider the function f given by the *lacunary trigonometric series* $f(z) = \sum_{k=2^n, n\in\mathbb{N}} k^{-\eta} e^{ikz}$, $\eta \in (0,1)$. Show that $f \in C^\eta[0, 2\pi]$. One way is to write $\phi_{1,2}$ as $a_{1,2}2^{-p}$, use the first inequality in (3.60) to estimate the terms in $f(\phi_1) - f(\phi_2)$ with $n < p$ and the simple bound $2/k^\eta$ for $n \geq p$. Then it is seen that $\int_0^{2\pi} e^{-ijs} f(s)ds = 2\pi j^{-\eta}$ (if $j = 2^m$ and zero otherwise) and the decay of the Fourier transform is exactly given by (3.57).

(b) Use Proposition 3.56 and the result in Exercise 3.62 to show that the function $f(t) = \sum_{k=2^n, n\in\mathbb{N}} k^{-\eta} t^k$, analytic in the open unit disk, has no analytic continuation across the unit circle, that is, the unit circle is a *barrier of singularities for* f.

Note 3.63 *Dense non-differentiability is essentially the only way one can get poor decay; see also Exercise 3.71.*

Note. In part (i), compactness of the interval is crucial. In fact, the Fourier transform of an $L^2(\mathbb{R})$ entire function may not necessarily decrease pointwise. Indeed, the function $\hat{f}(x) = 1$ on the interval $[n, n + e^{-n^2}]$ for $n \in \mathbb{N}$ and zero otherwise is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and further has the property that $e^{|Ax|}\hat{f} \in L^1(\mathbb{R})$ for any $A \in \mathbb{R}$, and thus $\mathcal{F}^{-1}\hat{f}$ is entire. Thus $\hat{f}$ is the Fourier transform of an entire function, it equals $\mathcal{F}^{-1}\hat{f}$ a.e., and nevertheless it does not decay pointwise as $x \to \infty$. Evidently the issue here is poor behavior of f at infinity, otherwise integration by parts would be possible, implying decay.

Proposition 3.64 *Assume $f \in C^n[a, b]$. Then we have*

$$\int_a^b e^{ixt} f(t)dt = e^{ixa} \sum_{k=1}^n c_k x^{-k} + e^{ixb} \sum_{k=1}^n d_k x^{-k} + o(x^{-n})$$
$$= e^{ixt} \left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + \dots + (-1)^{n-1} \frac{f^{(n-1)}(t)}{(ix)^n} \right) \Bigg|_a^b + o(x^{-n}) \quad (3.65)$$

PROOF This follows by integration by parts and the Riemann-Lebesgue lemma since

$$\int_a^b e^{ixt} f(t)dt = e^{ixt} \left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + \dots + (-1)^{n-1} \frac{f^{(n-1)}(t)}{(ix)^n} \right) \Bigg|_a^b + \frac{(-1)^n}{(ix)^n} \int_a^b f^{(n)}(t) e^{ixt} dt \quad (3.66)$$

∎