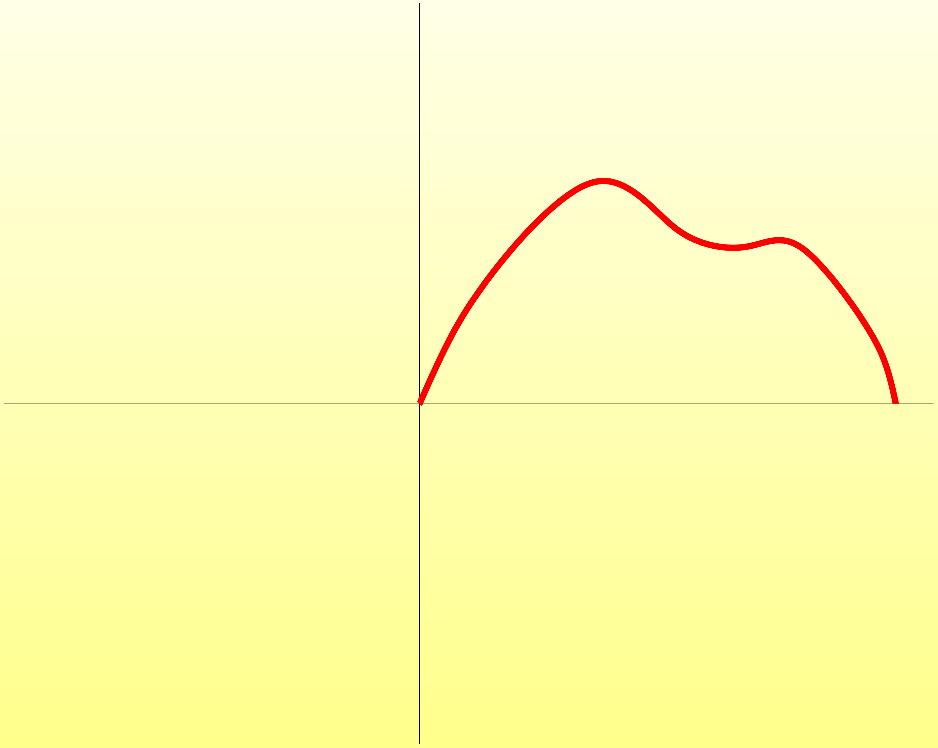
§10.4, end; Partial differential equations (beginning)

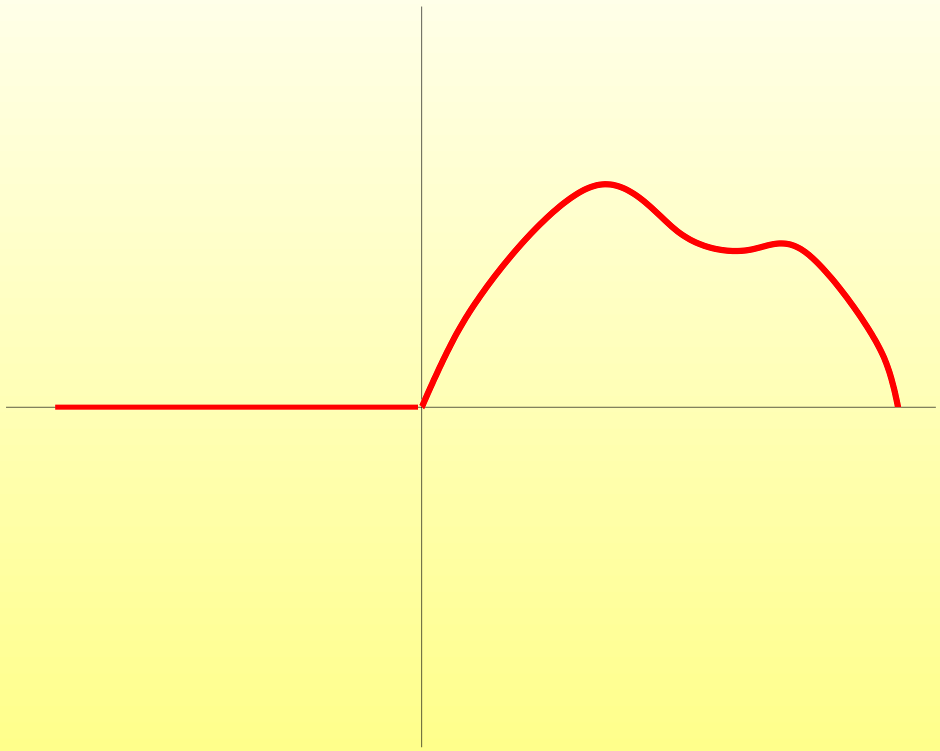
§10.4, end; Partial differential equations (beginning)

Extension of functions defined on $[0, L]$

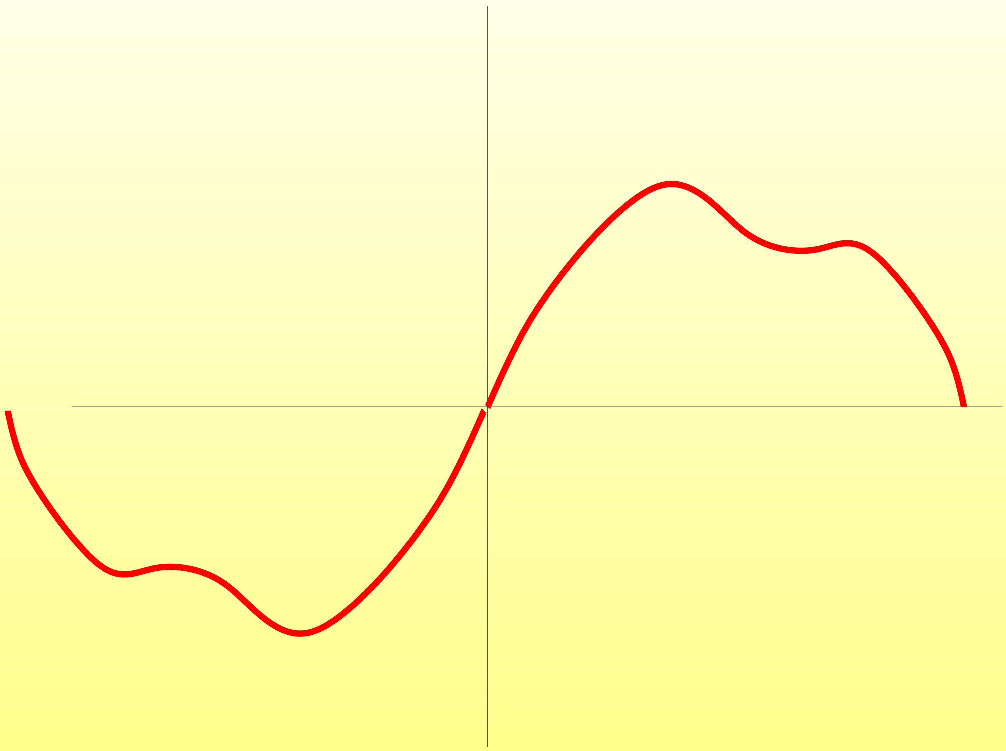
Extension of functions defined on $[0, L]$

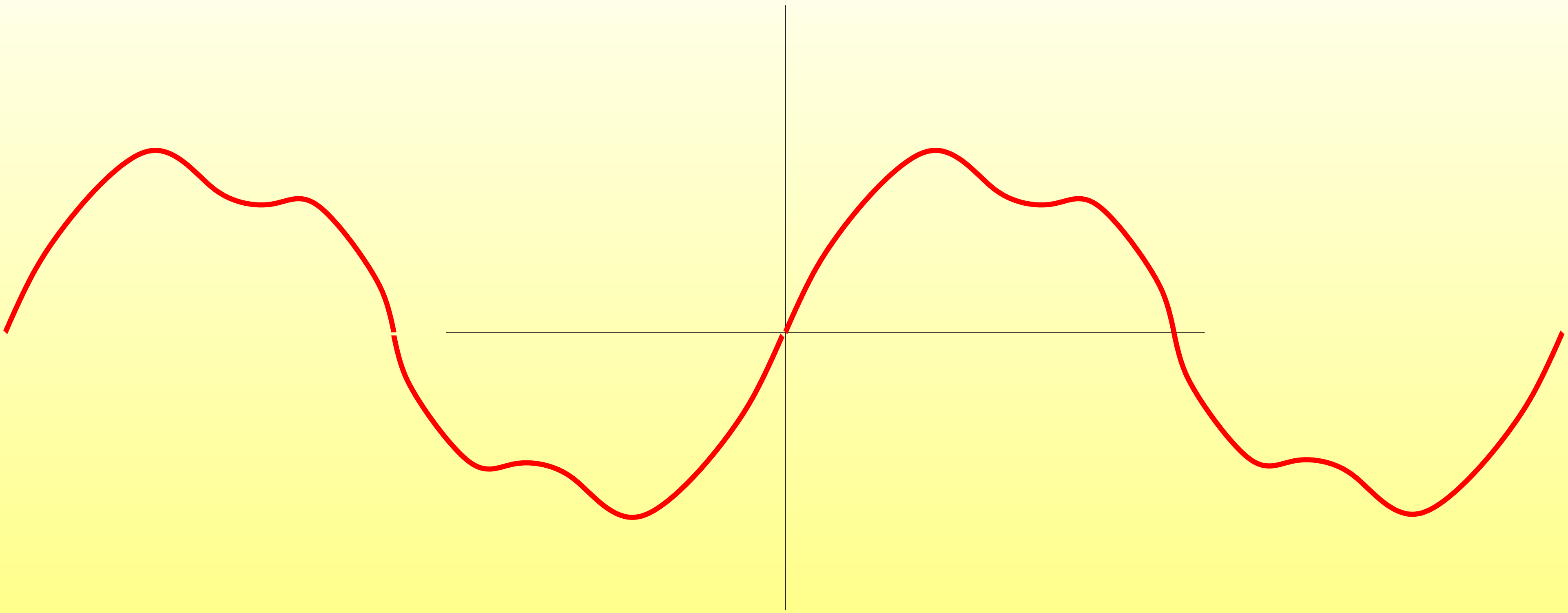
If we only want to calculate a sufficiently nice function on, say $[0, L]$, it does not have to be periodic. We can just *extend* it periodically. Eg, we extend it by zero on $[-L, 0]$ and then repeat it periodically.

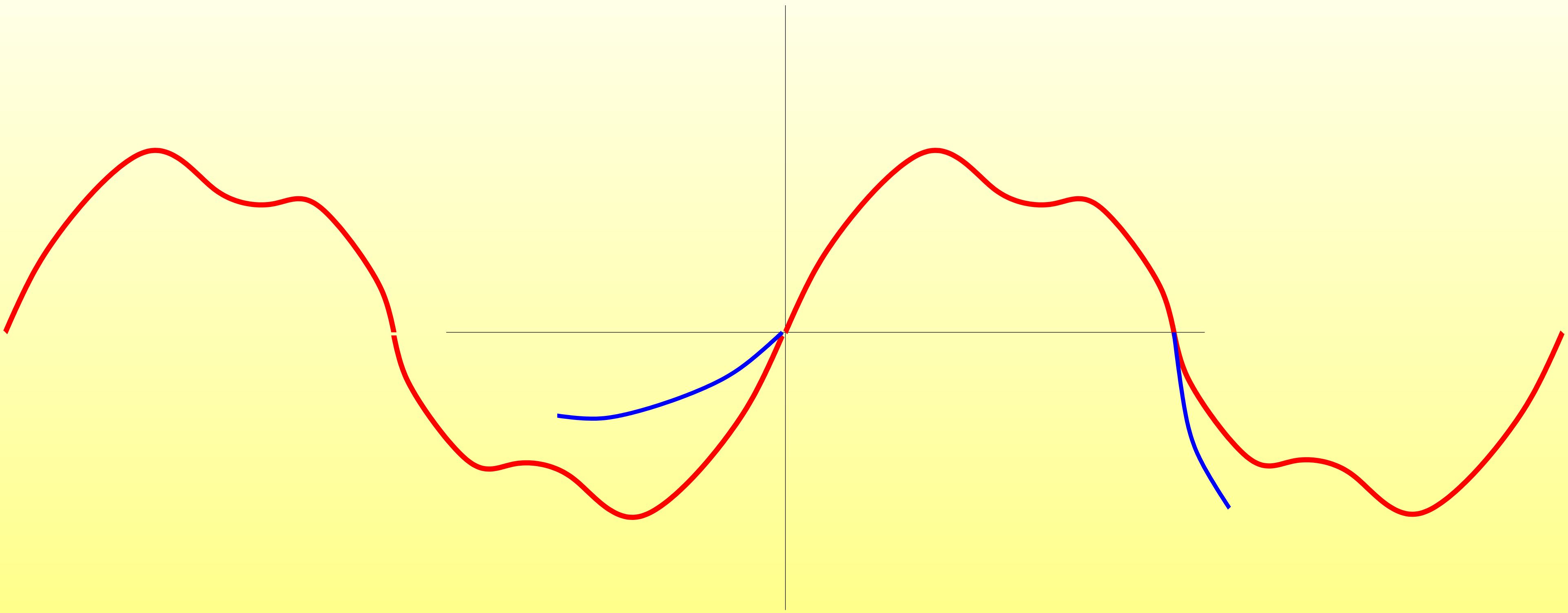




Better suited, especially if we want a pure sine decomposition is the odd extension:



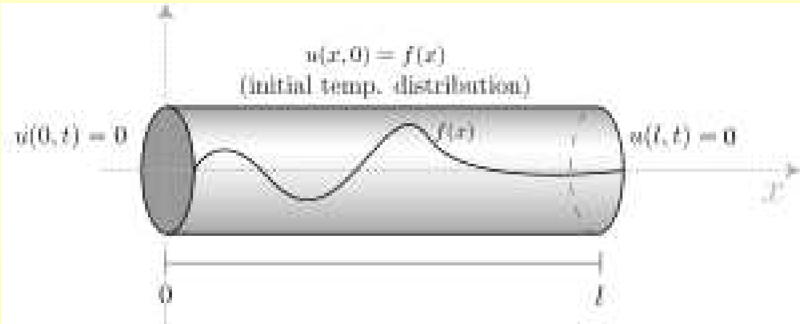




But maybe your function, in reality, followed the blue path instead. The Fourier series, calculated by this method, will give the red function, nonetheless.

PDEs

The Heat Equation.



(picture from Wikipedia) We start by considering the following physical problem: a rod of length L is placed between two ice cubes, so that the temperature u at the endpoints is zero.

At $t = 0$ $u(x,0) = f(x)$ in the rod, on $(0, L)$ Say the whole rod was at 20°C. What is the temperature distribution at time t?

Note that now there are two variables, t and x. Whatever equation is applicable, it has to involve both x and t. It is a differential equation, and since there are two independent variables, it involves partial derivatives.

Note that now there are two variables, t and x. Whatever equation is applicable, it has to involve both x and t. It is a differential equation, and since there are two independent variables, it involves partial derivatives. It is thus a PDE.

Note that now there are two variables, t and x. Whatever equation is applicable, it has to involve both x and t. It is a differential equation, and since there are two independent variables, it involves partial derivatives. It is thus a PDE.

The applicable PDE is the heat conduction equation, in short the heat equation,

$$u_t = \alpha^2 u_{xx}$$

The whole problem is

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, u(x,0) = f(x)$$

The whole problem is

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, u(x,0) = f(x)$$

Note that there are three specifications, the analog of initial conditions for ODEs.

The whole problem is

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, u(x,0) = f(x)$$

Note that there are three specifications, the analog of initial conditions for ODEs. These are the constraints written in blue.

The whole problem is

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, u(x,0) = f(x)$$

Note that there are three specifications, the analog of initial conditions for ODEs. These are the constraints written in blue. This is a **boundary value problem** ($u(0,t) = u(L,t) = 0$) for the heat equation, with an initial condition: $u(x,0) = f(x)$.

The whole problem is

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, u(x,0) = f(x)$$

Note that there are three specifications, the analog of initial conditions for ODEs. These are the constraints written in blue. This is a **boundary value problem** $(u(0,t) = u(L,t) = 0)$ for the heat equation, with an initial condition: $u(x,0) = f(x)$.

α^2 is a constant, depending only on the material of the rod, and it is called **thermal diffusivity**. See textbook for common values of α.

The whole problem is

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, u(x,0) = f(x)$$

Note that there are three specifications, the analog of initial conditions for ODEs. These are the constraints written in blue. This is a **boundary value problem** ($u(0,t) = u(L,t) = 0$) for the heat equation, with an initial condition: $u(x,0) = f(x)$.

α^2 is a constant, depending only on the material of the rod, and it is called **thermal diffusivity**. See textbook for common values of α. This is a **linear PDE.**

One simple way to solve **really simple, linear** PDEs is separation of variables.

One simple way to solve **really simple, linear** PDEs is separation of variables. This is a different from the same named method in ordinary differential equations.

One simple way to solve **really simple, linear** PDEs is separation of variables. This is a different from the same named method in ordinary differential equations.

It consists in seeking solutions in the form

$$u(x,t) = X(x)T(t)$$

One simple way to solve **really simple, linear** PDEs is separation of variables. This is a different from the same named method in ordinary differential equations.

It consists in seeking solutions in the form

$$u(x,t) = X(x)T(t)$$

that is in **a product form**, product of two functions each solely depending on **one variable**.

One simple way to solve **really simple, linear** PDEs is separation of variables. This is a different from the same named method in ordinary differential equations.

It consists in seeking solutions in the form

$$u(x,t) = X(x)T(t)$$

that is in **a product form**, product of two functions each solely depending on **one variable**. In this sense the variables are separated. But we cannot hope to find the solution to the whole problem in exactly this form.

One simple way to solve **really simple, linear** PDEs is separation of variables. This is a different from the same named method in ordinary differential equations.

It consists in seeking solutions in the form

$$u(x, t) = X(x)T(t)$$

that is in **a product form**, product of two functions each solely depending on **one variable**. In this sense the variables are separated. But we cannot hope to find the solution to the whole problem in exactly this form. Why should the variation in temperature not depend on x?

One simple way to solve **really simple, linear** PDEs is separation of variables. This is a different from the same named method in ordinary differential equations.

It consists in seeking solutions in the form

$$u(x, t) = X(x)T(t)$$

that is in **a product form**, product of two functions each solely depending on **one variable**. In this sense the variables are separated. But we cannot hope to find the solution to the whole problem in exactly this form. Why should the variation in temperature not depend on x?It must be faster near the endpoints and slower in the middle, farther from the ice cubes.

But the problem

$$u_t = \alpha^2 u_{xx}, u(0,t) = u(L,t) = 0$$

But the problem

$$u_t = \alpha^2 u_{xx}, u(0,t) = u(L,t) = 0$$

is **linear homogeneous**.

But the problem

$$u_t = \alpha^2 u_{xx}, u(0,t) = u(L,t) = 0$$

is **linear homogeneous**. Thus, like in ODEs, if u_1, u_2 are solutions, then $u_1 + u_2$ is a solution too (**check!**)

But the problem

$$u_t = \alpha^2 u_{xx}, u(0,t) = u(L,t) = 0$$

is **linear homogeneous**. Thus, like in ODEs, if u_1, u_2 are solutions, then $u_1 + u_2$ is a solution too (**check!**) Here again, **homogeneity is essential**.

But the problem

$$u_t = \alpha^2 u_{xx}, u(0,t) = u(L,t) = 0$$

is **linear homogeneous**. Thus, like in ODEs, if u_1, u_2 are solutions, then $u_1 + u_2$ is a solution too (**check!**) Here again, **homogeneity is essential**. We cannot simply add up solutions in nonlinear or nonhomogeneous equations.

But the problem

$$u_t = \alpha^2 u_{xx}, u(0,t) = u(L,t) = 0$$

is **linear homogeneous**. Thus, like in ODEs, if u_1, u_2 are solutions, then $u_1 + u_2$ is a solution too (**check!**) Here again, **homogeneity is essential**. We cannot simply add up solutions in nonlinear or nonhomogeneous equations.

Now, we can hope to find sufficiently many solutions u_1, u_2, *etc.* so that, when we add $u_1 + u_2 + u_3 + \ldots$ we get the actual solution.

But the problem

$$u_t = \alpha^2 u_{xx}, u(0,t) = u(L,t) = 0$$

is **linear homogeneous**. Thus, like in ODEs, if u_1, u_2 are solutions, then $u_1 + u_2$ is a solution too (**check!**) Here again, **homogeneity is essential**. We cannot simply add up solutions in nonlinear or nonhomogeneous equations.

Now, we can hope to find sufficiently many solutions u_1, u_2, $etc.$ so that, when we add $u_1 + u_2 + u_3 + ...$ we get the actual solution.

This really works for the heat equation and other simple linear problems and it is known as the method of separation of variables.

Now back to work.

Now back to work.

$$u_t = \alpha^2 u_{xx}$$

try

$$u(x, t) = X(x)T(t)$$

Now back to work.

$$u_t = \alpha^2 u_{xx}$$

try

$$u(x,t) = X(x)T(t)$$

Then $u_t = X(x)T'(t)$; $u_{xx} = X''(x)T(t)$.

Now back to work.

$$u_t = \alpha^2 u_{xx}$$

try

$$u(x, t) = X(x)T(t)$$

Then $u_t = X(x)T'(t)$; $u_{xx} = X''(x)T(t)$. Thus

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

Now back to work.

$$u_t = \alpha^2 u_{xx}$$

try

$$u(x,t) = X(x)T(t)$$

Then $u_t = X(x)T'(t)$; $u_{xx} = X''(x)T(t)$. Thus

$$X(x)T'(t) = \alpha^2 X''(x)T(t) \text{ so } \underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

$X(x)T'(t) = \alpha^2 X''(x)T(t)$ so $$\underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

$$X(x)T'(t) = \alpha^2 X''(x)T(t) \text{ so } \underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

How can a function of x exactly match a function of t?

$$X(x)T'(t) = \alpha^2 X''(x)T(t) \text{ so } \underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

How can a function of x exactly match a function of t? These are independent variables.

$$X(x)T'(t) = \alpha^2 X''(x)T(t) \text{ so } \underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

How can a function of x exactly match a function of t? These are independent variables. Thus they can be changed independently.

$$X(x)T'(t) = \alpha^2 X''(x)T(t) \text{ so } \underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

How can a function of x exactly match a function of t? These are independent variables. Thus they can be changed independently. One is fixed, say t and we change x.

$$X(x)T'(t) = \alpha^2 X''(x)T(t) \text{ so } \underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

How can a function of x exactly match a function of t? These are independent variables. Thus they can be changed independently. One is fixed, say t and we change x. If $\frac{X''(x)}{X(x)}$ changes, then we have a contradiction, since $\frac{T'(t)}{\alpha^2 T(t)}$ does not change, since it does not depend on x.

Thus $\dfrac{X''(x)}{X(x)}$ is simply a constant, say $-\lambda$.

Thus $\dfrac{X''(x)}{X(x)}$ is simply a constant, say $-\lambda$. But then $\dfrac{T'(t)}{\alpha^2 T(t)}$ is equal to the same constant, and it is a constant too.

Thus $\dfrac{X''(x)}{X(x)}$ is simply a constant, say $-\lambda$. But then $\dfrac{T'(t)}{\alpha^2 T(t)}$ is equal to the same constant, and it is a constant too.

We arrive at a pair of ODEs:

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda$$

Thus $\dfrac{X''(x)}{X(x)}$ is simply a constant, say $-\lambda$. But then $\dfrac{T'(t)}{\alpha^2 T(t)}$ is equal to the same constant, and it is a constant too.

We arrive at a pair of ODEs:

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda \tag{1}$$

$$\frac{X''(x)}{X(x)} = -\lambda$$

Thus $\dfrac{X''(x)}{X(x)}$ is simply a constant, say $-\lambda$. But then $\dfrac{T'(t)}{\alpha^2 T(t)}$ is equal to the same constant, and it is a constant too.

We arrive at a pair of ODEs:

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda \tag{1}$$

$$\frac{X''(x)}{X(x)} = -\lambda \tag{2}$$

Now, (1) is an initial value problem (since $T(0)$ is given), while (2) is a **boundary value problem** since it is subject to the conditions $X(0) = 0, \ X(L) = 0$ (where the ice cubes lie).

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda$$

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda \tag{3}$$

$$\frac{X''(x)}{X(x)} = -\lambda$$

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda \tag{3}$$

$$\frac{X''(x)}{X(x)} = -\lambda \tag{4}$$

Note that the **boundary value problem** (4) is an eigenvalue problem!

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda \tag{3}$$

$$\frac{X''(x)}{X(x)} = -\lambda \tag{4}$$

Note that the **boundary value problem** (4) is an eigenvalue problem! Indeed, it is

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \ X(L) = 0$$

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda \tag{3}$$

$$\frac{X''(x)}{X(x)} = -\lambda \tag{4}$$

Note that the **boundary value problem** (4) is an eigenvalue problem! Indeed, it is

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \ X(L) = 0 \tag{5}$$

where we seek **nonzero solutions!** (a zero solution would not help much here).

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \ X(L) = 0$$

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \; X(L) = 0 \tag{6}$$

We studied (6) before.

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \ X(L) = 0 \tag{6}$$

We studied (6) before. Look at that section.

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \; X(L) = 0 \tag{6}$$

We studied (6) before. Look at that section. The general solution is sine+cosine of $\sqrt{\lambda}$; only sin(0)=0, thus it is a pure sine, but to vanish at L we need $\sqrt{\lambda}L = n\pi$ and thus all the eigenvalues for this problem are

$$\lambda_n = n^2\pi^2/L^2, n = 1, 2, 3, \ldots$$

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \ X(L) = 0 \tag{6}$$

We studied (6) before. Look at that section. The general solution is sine+cosine of $\sqrt{\lambda}$; only sin(0)=0, thus it is a pure sine, but to vanish at L we need $\sqrt{\lambda}L = n\pi$ and thus all the eigenvalues for this problem are

$$\lambda_n = n^2\pi^2/L^2, n = 1, 2, 3, \ldots$$

and the eigenfunctions are

$$X_n = (c_n)\sin(n\pi x/L)$$

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \; X(L) = 0 \tag{6}$$

We studied (6) before. Look at that section. The general solution is sine+cosine of $\sqrt{\lambda}$; only sin(0)=0, thus it is a pure sine, but to vanish at L we need $\sqrt{\lambda}L = n\pi$ and thus all the eigenvalues for this problem are

$$\lambda_n = n^2\pi^2/L^2, n = 1, 2, 3, \ldots$$

and the eigenfunctions are

$$X_n = (c_n)\sin(n\pi x/L)$$

We found infinitely many solutions!

For each of them, we have the $T(t)$ equation,

For each of them, we have the $T(t)$ equation,

$$\frac{T_n'(t)}{\alpha^2 T_n(t)} = -\lambda_n \ \text{ that is } \ T_n'(t) = (-n^2\pi^2/L^2)\alpha^2 T_n(t), n = 1, 2, 3, \ldots$$

For each of them, we have the $T(t)$ equation,

$$\frac{T_n'(t)}{\alpha^2 T_n(t)} = -\lambda_n \text{ that is } T_n'(t) = (-n^2\pi^2/L^2)\alpha^2 T_n(t), n = 1, 2, 3, \dots$$

which gives immediately

$$T_n(t) = \exp(-n^2\alpha^2\pi^2 t/L^2)$$

For each of them, we have the $T(t)$ equation,

$$\frac{T_n'(t)}{\alpha^2 T_n(t)} = -\lambda_n \quad \text{that is} \quad T_n'(t) = (-n^2\pi^2/L^2)\alpha^2 T_n(t), n = 1, 2, 3, \ldots$$

which gives immediately

$$T_n(t) = \exp(-n^2\alpha^2\pi^2 t/L^2)$$

Putting X_n and T_n together –remember,

$$u_n(x,t) = X_n(x)T_n(t) \quad \text{we have:}$$

$$u_n(x,t) = c_n \exp(-n^2\alpha^2\pi^2 t/L^2) \sin(n\pi x/L)$$

Now we really have many solutions, as desired.

Now we really have many solutions, as desired. Then, by the linearity and homogeneity of the equation

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L)$$

Now we really have many solutions, as desired. Then, by the linearity and homogeneity of the equation

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2) \sin(n\pi x/L) \tag{7}$$

is also a solution of the problem.

Now we really have many solutions, as desired. Then, by the linearity and homogeneity of the equation

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L) \tag{7}$$

is also a solution of the problem.

Indeed, $u(0,t) = u(L,t) = 0$,

Now we really have many solutions, as desired. Then, by the linearity and homogeneity of the equation

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2) \sin(n\pi x/L) \tag{7}$$

is also a solution of the problem.

Indeed, $u(0,t) = u(L,t) = 0$,

How about the initial condition, $u(x,0) = f(x) = 20$ on $(0,L)$? Can it be fitted by (9)?

Let's try.

Let's try.

$$f(x) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 0/L^2) \sin(n\pi x/L) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

Let's try.

$$f(x) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 0/L^2) \sin(n\pi x/L) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

But this is a Fourier sine decomposition, on $[-L, L]$ (because of "$n\pi x/L$", the argument of sin.

Let's try.

$$f(x) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 0/L^2) \sin(n\pi x/L) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

But this is a Fourier sine decomposition, on $[-L, L]$ (because of "$n\pi x/L$", the argument of sin. So, we will **extend** f, initially defined on $(0, L)$ as an **odd** function (to be able to get a pure sine Fourier series).

Let's try.

$$f(x) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 0/L^2) \sin(n\pi x/L) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

But this is a Fourier sine decomposition, on $[-L, L]$ (because of "$n\pi x/L$", the argument of sin. So, we will **extend** f, initially defined on $(0, L)$ as an **odd** function (to be able to get a pure sine Fourier series). The function to be worked with is thus:

$$f(x) = \begin{cases} -20 \text{ for } x \in (-L, 0) \\ 20 \text{ for } x \in (0, L) \end{cases}$$

Let's try.

$$f(x) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 0/L^2)\sin(n\pi x/L) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

But this is a Fourier sine decomposition, on $[-L, L]$ (because of "$n\pi x/L$", the argument of sin. So, we will **extend** f, initially defined on $(0, L)$ as an **odd** function (to be able to get a pure sine Fourier series). The function to be worked with is thus:

$$f(x) = \begin{cases} -20 \text{ for } x \in (-L, 0) \\ 20 \text{ for } x \in (0, L) \end{cases} \tag{8}$$

Since this is indeed an odd function, the coefficients c_n are

given by

given by

$$\frac{1}{L}\int_0^L f(x)\sin(n\pi x/L)dx = \frac{1}{L}\int_0^L 20\sin(n\pi x/L)dx = 40\,\frac{1-(-1)^n}{n\,\pi}$$

given by

$$\frac{1}{L}\int_0^L f(x)\sin(n\pi x/L)dx = \frac{1}{L}\int_0^L 20\sin(n\pi x/L)dx = 40\,\frac{1-(-1)^n}{n\,\pi}$$

The complete solution is thus

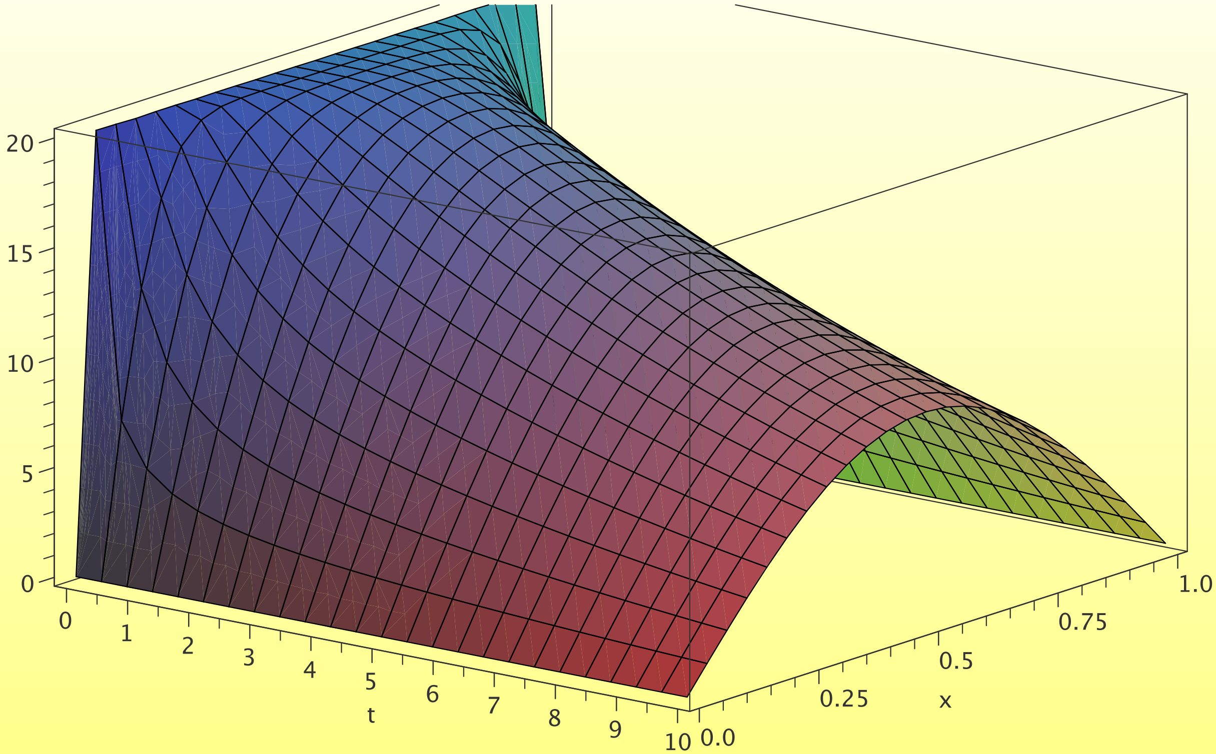
$$u(x,t) = 40\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\,\pi}\exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L)$$

given by

$$\frac{1}{L}\int_0^L f(x)\sin(n\pi x/L)dx = \frac{1}{L}\int_0^L 20\sin(n\pi x/L)dx = 40\,\frac{1-(-1)^n}{n\,\pi}$$

The complete solution is thus

$$u(x,t) = 40\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\,\pi}\exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L) \qquad (9)$$


0
5
10
15
20
0
1
2
3
4
5
6
7
8
9
10
t
0.0
0.25
0.5
0.75
1.0
x