§10.5-10.6 Partial differential equations, separation of variables.

§10.5-10.6 Partial differential equations, separation of variables.

Partial differential equations, distinctive features; simple examples.

Possibly the simplest PDE that we can imagine is

$$\frac{\partial u(x,t)}{\partial t} = 0$$

The general solution is

Partial differential equations, distinctive features; simple examples.

Possibly the simplest PDE that we can imagine is

$$\frac{\partial u(x,t)}{\partial t} = 0$$

The general solution is

$$u(x,t) = f(x)$$

for *any* function f!

Solutions of PDEs always have "functional degree of freedom" (as opposed to free constants in the case of ODEs).

Another simple example:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} \tag{1}$$

Claim: the general solution is

Another simple example:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} \tag{1}$$

Claim: the general solution is $f(x+t)$, for any differentiable function f. Indeed, by the chain rule

$$\frac{\partial f(x+t)}{\partial t} = f'(x+t) = \frac{\partial f(x+t)}{\partial x}$$

Another simple example:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} \tag{1}$$

Claim: the general solution is $f(x+t)$, for any differentiable function f. Indeed, by the chain rule

$$\frac{\partial f(x+t)}{\partial t} = f'(x+t) = \frac{\partial f(x+t)}{\partial x}$$

Exercise: Show that there are no other solutions!

Another simple example:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} \tag{1}$$

Claim: the general solution is $f(x+t)$, for any differentiable function f. Indeed, by the chain rule

$$\frac{\partial f(x+t)}{\partial t} = f'(x+t) = \frac{\partial f(x+t)}{\partial x}$$

Exercise: Show that there are no other solutions!What do we need to specify to determine the particular solution we are interested in? Since we have functional degree of freedom, we need to specify, say an initial **function**.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x}$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} \tag{2}$$

For instance (2) can be solved uniquely if we specify $u(x,0) = U(x)$.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} \tag{2}$$

For instance (2) can be solved uniquely if we specify $u(x,0) = U(x)$. Say, we give $u(x,0) = \sin^3(x)$. Then indeed, the *general solution* $f(x+t)$ must satisfy the condition $f(x+0) = \sin^3(x)$,

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} \tag{2}$$

For instance (2) can be solved uniquely if we specify $u(x,0) = U(x)$. Say, we give $u(x,0) = \sin^3(x)$. Then indeed, the *general solution* $f(x+t)$ must satisfy the condition $f(x+0) = \sin^3(x)$, that is $f = \sin^3$,

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} \tag{2}$$

For instance (2) can be solved uniquely if we specify $u(x,0) = U(x)$. Say, we give $u(x,0) = \sin^3(x)$. Then indeed, the *general solution* $f(x+t)$ must satisfy the condition $f(x+0) = \sin^3(x)$, that is $f = \sin^3$, that is $u(x,t) = \sin^3(x+t)$.

$$u_t = \alpha^2 u_{xx}$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} \tag{2}$$

For instance (2) can be solved uniquely if we specify $u(x,0) = U(x)$. Say, we give $u(x,0) = \sin^3(x)$. Then indeed, the *general solution* $f(x+t)$ must satisfy the condition $f(x+0) = \sin^3(x)$, that is $f = \sin^3$, that is $u(x,t) = \sin^3(x+t)$.

$$u_t = \alpha^2 u_{xx}$$

This involves now two derivatives in x, more information is needed. For instance

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} \tag{2}$$

For instance (2) can be solved uniquely if we specify $u(x,0) = U(x)$. Say, we give $u(x,0) = \sin^3(x)$. Then indeed, the *general solution* $f(x+t)$ must satisfy the condition $f(x+0) = \sin^3(x)$, that is $f = \sin^3$, that is $u(x,t) = \sin^3(x+t)$.

$$u_t = \alpha^2 u_{xx}$$

This involves now two derivatives in x, more information is needed. For instance

$$u(x,0); \quad \text{and} \quad u(0,t) \quad \text{and} \quad u(0,L)$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} \tag{2}$$

For instance (2) can be solved uniquely if we specify $u(x,0) = U(x)$. Say, we give $u(x,0) = \sin^3(x)$. Then indeed, the *general solution* $f(x+t)$ must satisfy the condition $f(x+0) = \sin^3(x)$, that is $f = \sin^3$, that is $u(x,t) = \sin^3(x+t)$.

$$u_t = \alpha^2 u_{xx}$$

This involves now two derivatives in x, more information is needed. For instance

$$u(x,0); \quad \text{and} \quad u(0,t) \quad \text{and} \quad u(0,L)$$

For instance

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, u(x,0) = f(x)$$

For instance

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, u(x,0) = f(x)$$

$$u_n(x,t) = c_n \exp(-n^2\alpha^2\pi^2 t/L^2) \sin(n\pi x/L)$$

is a *particular* solution for any n, and any constant c_n.

For instance

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, u(x,0) = f(x)$$

$$u_n(x,t) = c_n \exp(-n^2\alpha^2\pi^2 t/L^2) \sin(n\pi x/L)$$

is a *particular* solution for any n, and any constant c_n. What we mean by u_n being a solution of the PDE is precisely that $(u_n(x,t))_t = \alpha^2 (u_n(x,t))_{xx}$ for **any** t **and** x.

1. $u_t = -(n^2\alpha^2\pi^2/L^2) \exp(-n^2\alpha^2\pi^2 t/L^2) c_n \sin(n\pi x/L)$

For instance

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, u(x,0) = f(x)$$

$$u_n(x,t) = c_n \exp(-n^2\alpha^2\pi^2 t/L^2) \sin(n\pi x/L)$$

is a *particular* solution for any n, and any constant c_n. What we mean by u_n being a solution of the PDE is precisely that $(u_n(x,t))_t = \alpha^2 (u_n(x,t))_{xx}$ for **any** t **and** x.

1. $u_t = -(n^2\alpha^2\pi^2/L^2)\exp(-n^2\alpha^2\pi^2 t/L^2) c_n \sin(n\pi x/L)$

2. $\alpha^2 u_{xx} = c_n \alpha^2 (-(n\pi/L)^2 \sin(n\pi x/L)) \exp(-n^2\alpha^2\pi^2 t/L^2)$

For instance

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, u(x,0) = f(x)$$

$$u_n(x,t) = c_n \exp(-n^2\alpha^2\pi^2 t/L^2)\sin(n\pi x/L)$$

is a *particular* solution for any n, and any constant c_n. What we mean by u_n being a solution of the PDE is precisely that $(u_n(x,t))_t = \alpha^2(u_n(x,t))_{xx}$ for **any** t **and** x.

1. $u_t = -(n^2\alpha^2\pi^2/L^2)\exp(-n^2\alpha^2\pi^2 t/L^2)c_n \sin(n\pi x/L)$

2. $\alpha^2 u_{xx} = c_n\alpha^2(-(n\pi/L)^2 \sin(n\pi x/L))\exp(-n^2\alpha^2\pi^2 t/L^2)$

3. So: $u_t = \alpha^2 u_{xx}$ for all x and t.

$$u_n(x,t) = c_n \exp(-n^2\alpha^2\pi^2 t/L^2) \sin(n\pi x/L)$$

4. Boundary conditions? Check: $u(0,t) = u(L,t) = 0$.

5. Initial condition? $u(x,0) = c_n \sin(n\pi x/L)$. This is not general enough. We need more solutions.

Separation of variables.

Separation of variables.

$$u_t = \alpha^2 u_{xx}$$

Find all solutions of the form

$$u(x, t) = X(x)T(t):$$

Separation of variables.

$$u_t = \alpha^2 u_{xx}$$

Find all solutions of the form

$$u(x,t) = X(x)T(t):$$

$u_t = X(x)T'(t);\ u_{xx} = X''(x)T(t).$

Separation of variables.

$$u_t = \alpha^2 u_{xx}$$

Find all solutions of the form

$$u(x,t) = X(x)T(t):$$

$u_t = X(x)T'(t)$; $u_{xx} = X''(x)T(t)$. Thus

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

Separation of variables.

$$u_t = \alpha^2 u_{xx}$$

Find all solutions of the form

$$u(x,t) = X(x)T(t):$$

$u_t = X(x)T'(t)$; $u_{xx} = X''(x)T(t)$. Thus

$$X(x)T'(t) = \alpha^2 X''(x)T(t) \text{ so } \underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

$$\underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

Thus $\dfrac{X''(x)}{X(x)}$ is simply a constant, say $-\lambda$.

$$\underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

Thus $\dfrac{X''(x)}{X(x)}$ is simply a constant, say $-\lambda$. Then $\dfrac{T'(t)}{\alpha^2 T(t)}$ is equal to the same constant.

$$\underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

Thus $\dfrac{X''(x)}{X(x)}$ is simply a constant, say $-\lambda$. Then $\dfrac{T'(t)}{\alpha^2 T(t)}$ is equal to the same constant.

We arrive at a pair of ODEs:

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda$$

$$\underbrace{\frac{T'(t)}{\alpha^2 T(t)}}_{\text{depends on } t \text{ alone}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{depends on } x \text{ alone}}$$

Thus $\dfrac{X''(x)}{X(x)}$ is simply a constant, say $-\lambda$. Then $\dfrac{T'(t)}{\alpha^2 T(t)}$ is equal to the same constant.

We arrive at a pair of ODEs:

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda \tag{3}$$

$$\frac{X''(x)}{X(x)} = -\lambda \tag{4}$$

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda$$

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda \tag{5}$$

$$\frac{X''(x)}{X(x)} = -\lambda$$

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda \tag{5}$$

$$\frac{X''(x)}{X(x)} = -\lambda \tag{6}$$

The **boundary value problem** (6) is an eigenvalue problem

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda \tag{5}$$

$$\frac{X''(x)}{X(x)} = -\lambda \tag{6}$$

The **boundary value problem** (6) is an eigenvalue problem

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \; X(L) = 0$$

$$\frac{T'(t)}{\alpha^2 T(t)} = -\lambda \tag{5}$$

$$\frac{X''(x)}{X(x)} = -\lambda \tag{6}$$

The **boundary value problem** (6) is an eigenvalue problem

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \; X(L) = 0 \tag{7}$$

where we seek **nonzero solutions**.

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \; X(L) = 0$$

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \; X(L) = 0 \tag{8}$$

We studied (8) before.

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \ X(L) = 0 \tag{8}$$

We studied (8) before. The eigenvalues for this problem are

$$\lambda_n = n^2\pi^2/L^2, n = 1, 2, 3, ...$$

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \; X(L) = 0 \tag{8}$$

We studied (8) before. The eigenvalues for this problem are

$$\lambda_n = n^2\pi^2/L^2, n = 1, 2, 3, \ldots$$

and the eigenfunctions are

$$X_n = (c_n)\sin(n\pi x/L)$$

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \; X(L) = 0 \tag{8}$$

We studied (8) before. The eigenvalues for this problem are

$$\lambda_n = n^2\pi^2/L^2, n = 1, 2, 3, \ldots$$

and the eigenfunctions are

$$X_n = (c_n) \sin(n\pi x/L)$$

For each of them, we have the $T(t)$ equation,

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \; X(L) = 0 \tag{8}$$

We studied (8) before. The eigenvalues for this problem are

$$\lambda_n = n^2\pi^2/L^2, n = 1, 2, 3, \ldots$$

and the eigenfunctions are

$$X_n = (c_n)\sin(n\pi x/L)$$

For each of them, we have the $T(t)$ equation,

$$\frac{T_n'(t)}{\alpha^2 T_n(t)} = -\lambda_n \text{ that is } T_n'(t) = (-n^2\pi^2/L^2)\alpha^2 T_n(t), n = 1, 2, 3, \ldots$$

$$X''(x) = -\lambda X(x); \quad X(0) = 0, \; X(L) = 0 \tag{8}$$

We studied (8) before. The eigenvalues for this problem are

$$\lambda_n = n^2\pi^2/L^2, n = 1, 2, 3, ...$$

and the eigenfunctions are

$$X_n = (c_n)\sin(n\pi x/L)$$

For each of them, we have the $T(t)$ equation,

$$\frac{T_n'(t)}{\alpha^2 T_n(t)} = -\lambda_n \quad \text{that is} \quad T_n'(t) = (-n^2\pi^2/L^2)\alpha^2 T_n(t), n = 1, 2, 3, ...$$

which gives immediately

$$T_n(t) = \exp(-n^2\alpha^2\pi^2 t/L^2)$$

which gives immediately

$$T_n(t) = \exp(-n^2\alpha^2\pi^2 t/L^2)$$

Putting X_n and T_n together –remember,

$$u_n(x,t) = X_n(x)T_n(t) \quad \text{we have:}$$

$$u_n(x,t) = c_n \exp(-n^2\alpha^2\pi^2 t/L^2)\sin(n\pi x/L)$$

Now we really have many solutions, as desired.

Now we really have many solutions, as desired. Then, by the linearity and homogeneity of the equation

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L)$$

Now we really have many solutions, as desired. Then, by the linearity and homogeneity of the equation

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L) \tag{9}$$

is also a solution of the problem. (We'll deal with convergence: later.)

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L)$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2) \sin(n\pi x/L) \tag{10}$$

Initial condition We have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) \tag{11}$$

Can this be now made to fit any initial temperature distribution, $u(x,0) = U(x)$? Yes, by the Fourier theorem.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2) \sin(n\pi x/L) \tag{10}$$

Initial condition We have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) \tag{11}$$

Can this be now made to fit any initial temperature distribution, $u(x,0) = U(x)$? Yes, by the Fourier theorem. We are looking here for a **Fourier sine** decomposition of U on $[0, L]$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L) \tag{10}$$

Initial condition We have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) \tag{11}$$

Can this be now made to fit any initial temperature distribution, $u(x,0) = U(x)$? Yes, by the Fourier theorem. We are looking here for a **Fourier sine** decomposition of U on $[0,L]$ which means we have to take **the odd extension of** U**, on** $[-L,L]$.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L) \tag{10}$$

Initial condition We have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) \tag{11}$$

Can this be now made to fit any initial temperature distribution, $u(x,0) = U(x)$? Yes, by the Fourier theorem. We are looking here for a **Fourier sine** decomposition of U on $[0,L]$ which means we have to take **the odd extension of** U**, on** $[-L,L]$.That is, define $U_1(x) = -U(-x)$ if $x < 0$ and $U_1(x) = U(x)$ if $x \geq 0$.

$U_1(x) = -U(-x)$ if $x < 0$ and $U_1(x) = U(x)$ if $x \geq 0$

That ensures that

$U_1(x) = -U(-x)$ if $x < 0$ and $U_1(x) = U(x)$ if $x \geq 0$

That ensures that (1) U_1 has a pure sine FS.

$U_1(x) = -U(-x)$ if $x < 0$ and $U_1(x) = U(x)$ if $x \geq 0$

That ensures that (1) U_1 has a pure sine FS. (2) $U_1 = U$ on the interval of interest, $[0, L]$.

$$f(x) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 0/L^2)\sin(n\pi x/L) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

$$f(x)=\sum_{n=1}^{\infty}c_n\exp(-n^2\pi^2\alpha^2 0/L^2)\sin(n\pi x/L)=\sum_{n=1}^{\infty}c_n\sin(n\pi x/L)$$

In our example $U(x)=20$, thus the function to be worked with is

$$f(x) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 0/L^2)\sin(n\pi x/L) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

In our example $U(x) = 20$, thus the function to be worked with is

$$U_1(x) = \begin{cases} -20 \text{ for } x \in (-L, 0) \\ 20 \text{ for } x \in (0, L) \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 0/L^2) \sin(n\pi x/L) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

In our example $U(x) = 20$, thus the function to be worked with is

$$U_1(x) = \begin{cases} -20 \text{ for } x \in (-L, 0) \\ 20 \text{ for } x \in (0, L) \end{cases} \tag{12}$$

Since this is indeed an odd function, the coefficients c_n are given by

$$f(x) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 0/L^2) \sin(n\pi x/L) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

In our example $U(x) = 20$, thus the function to be worked with is

$$U_1(x) = \begin{cases} -20 \text{ for } x \in (-L,0) \\ 20 \text{ for } x \in (0,L) \end{cases} \tag{12}$$

Since this is indeed an odd function, the coefficients c_n are given by

$$\frac{1}{L}\int_0^L U_1 \sin(n\pi x/L)dx = \frac{1}{L}\int_0^L 20 \sin(n\pi x/L)dx = 40\,\frac{1-(-1)^n}{n\,\pi}$$

$$f(x) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 0/L^2)\sin(n\pi x/L) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

In our example $U(x) = 20$, thus the function to be worked with is

$$U_1(x) = \begin{cases} -20 \text{ for } x \in (-L, 0) \\ 20 \text{ for } x \in (0, L) \end{cases} \tag{12}$$

Since this is indeed an odd function, the coefficients c_n are given by

$$\frac{1}{L}\int_0^L U_1 \sin(n\pi x/L)dx = \frac{1}{L}\int_0^L 20 \sin(n\pi x/L)dx = 40\,\frac{1-(-1)^n}{n\,\pi}$$

The complete solution is thus

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L)$$

The complete solution is thus

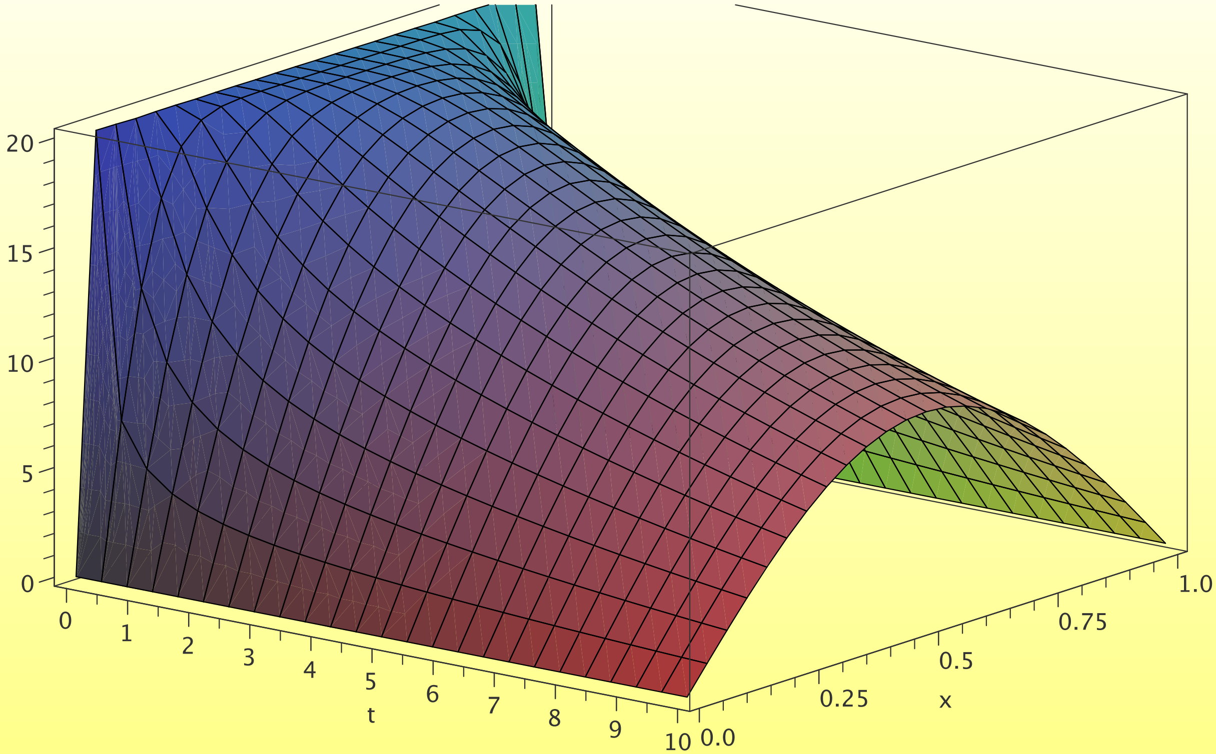
$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2) \sin(n\pi x/L) \tag{13}$$

$$u(x,t) = 40 \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\,\pi} \exp(-n^2\pi^2\alpha^2 t/L^2) \sin(n\pi x/L)$$

The complete solution is thus

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L) \tag{13}$$

$$u(x,t) = 40\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\,\pi} \exp(-n^2\pi^2\alpha^2 t/L^2)\sin(n\pi x/L) \tag{14}$$



20
15
10
5
0
0
1
2
3
4
5
6
7
8
9
10
t
0.0
0.25
0.5
0.75
1.0
x

Other Heat Equation settings

Nonhomogeneous boundary conditions. Here, we seek to solve

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = T_1, u(L,t) = T_2, u(x,0) = f(x)$$

Other Heat Equation settings

Nonhomogeneous boundary conditions. Here, we seek to solve

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = T_1, u(L,t) = T_2, u(x,0) = f(x)$$

that is, we have different temperatures at the endpoints. As in nonhomogeneous ODEs, the solution is essentially **any solution of the nonhomogeneous equation plus the general solution of the homogeneous one**.

Indeed if u_0 satisfies the eq, boundary conditions but not necessarily the initial condition, then if we write $u = u_0 + v$ we have $(u_0)_t + v_t = \alpha^2 (u_0)_{xx} + v_{xx}$ or $v_t + \underbrace{((u_0)_t - (u_0)_{xx})}_{=0,\text{by construction}} = \alpha^2 v_{xx}$

We need $v(0,t) + u_0(0,t) = T_1$ but $u_0(0,t) = T_1$, by construction, so: $v(0,t) = 0$. Likewise, $v(L,t) = 0$. v satisfies the same problem, with homogeneous boundary values, and initial condition

$$v(x,0) + u_0(x,0) = f(x) \ \Rightarrow$$

Indeed if u_0 satisfies the eq, boundary conditions but not necessarily the initial condition, then if we write $u = u_0 + v$ we have $(u_0)_t + v_t = \alpha^2 (u_0)_{xx} + v_{xx}$ or $v_t + \underbrace{((u_0)_t - (u_0)_{xx})}_{=0,\text{by construction}} = \alpha^2 v_{xx}$

We need $v(0,t) + u_0(0,t) = T_1$ but $u_0(0,t) = T_1$, by construction, so: $v(0,t) = 0$. Likewise, $v(L,t) = 0$. v satisfies the same problem, with homogeneous boundary values, and initial condition

$$v(x,0) + u_0(x,0) = f(x) \implies v(x,0) = f(x) - u_0(x,0)$$

A particular solution of

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = T_1, u(L,t) = T_2$$

is easy to find. Look, for instance for solutions that don't depend on t. Then

$$u_{xx} = 0,$$

A particular solution of

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = T_1, u(L,t) = T_2$$

is easy to find. Look, for instance for solutions that don't depend on t. Then

$$u_{xx} = 0, \Rightarrow u = Ax + B;$$

A particular solution of

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = T_1, u(L,t) = T_2$$

is easy to find. Look, for instance for solutions that don't depend on t. Then

$$u_{xx} = 0, \Rightarrow u = Ax + B; \;\; A0 + B = T_1, AL + B = T_2$$

A particular solution of

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = T_1, u(L,t) = T_2$$

is easy to find. Look, for instance for solutions that don't depend on t. Then

$$u_{xx} = 0, \Rightarrow u = Ax + B; \quad A0 + B = T_1, AL + B = T_2$$

$$B = T_1, A = (T_2 - T_1)/L;$$

A particular solution of

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = T_1, u(L,t) = T_2$$

is easy to find. Look, for instance for solutions that don't depend on t. Then

$$u_{xx} = 0, \Rightarrow u = Ax + B; \quad A0 + B = T_1, AL + B = T_2$$

$$B = T_1, A = (T_2 - T_1)/L; \; u_0 = x(T_2 - T_1)/L + T_1 \quad (15)$$

Then, the problem for v becomes

$$v_t = \alpha^2 v_{xx}, \quad v(0,t) = 0, v(L,t) = 0, v(x,0) = f(x) - [x(T_2 - T_1)/L + T_1]$$

A particular solution of

$$u_t = \alpha^2 u_{xx}, \qquad u(0,t) = T_1, u(L,t) = T_2$$

is easy to find. Look, for instance for solutions that don't depend on t. Then

$$u_{xx} = 0, \Rightarrow u = Ax + B; \quad A0 + B = T_1, AL + B = T_2$$

$$B = T_1, A = (T_2 - T_1)/L; \; u_0 = x(T_2 - T_1)/L + T_1 \quad (15)$$

Then, the problem for v becomes

$$v_t = \alpha^2 v_{xx}, \quad v(0,t) = 0, v(L,t) = 0, v(x,0) = f(x) - [x(T_2 - T_1)/L + T_1]$$