§10.6-10.7 Separation of variables, cont.

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Other Heat Equation settings

Nonhomogeneous boundary conditions. Here, we seek to solve

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that is, we have different temperatures at the endpoints. As in nonhomogeneous ODEs, the solution is essentially **any solution of the nonhomogeneous equation plus the general solution of the homogeneous one**.

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Example:

$u_t = u_{xx}$, $u(0,t) = 20, u(30,t) = 50, u(x,0) = 60 - 2x$ Particular solution:

$$u_0(x) = x(50 - 20)/30 + 20 = x + 20$$

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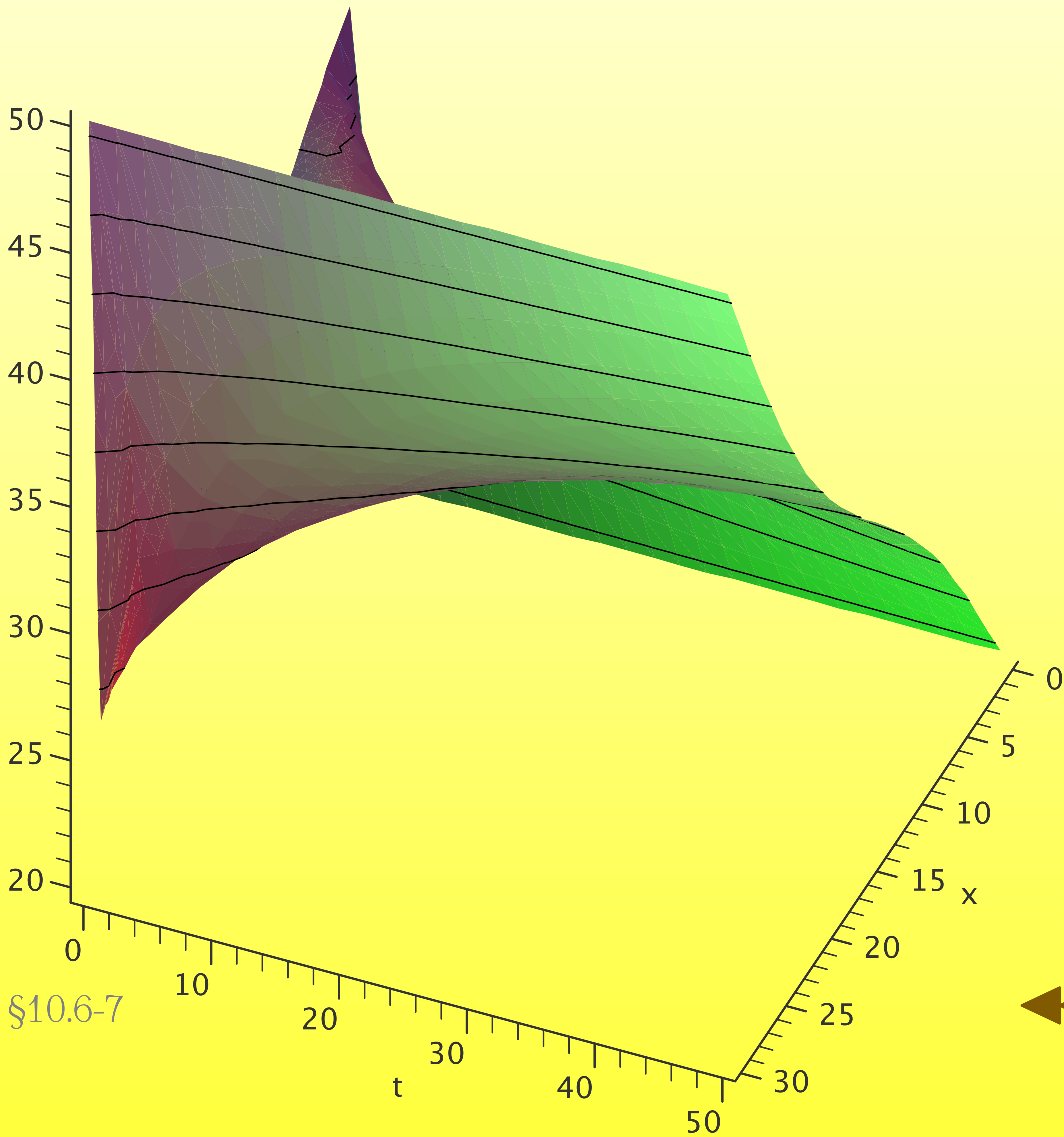
$$c_n = \frac{2}{30}\int_0^{30} \underbrace{(40 - 3x)}_{4} \sin(n\pi x/30)dx = \frac{20(4 + 5(-1)^m)}{m\pi}$$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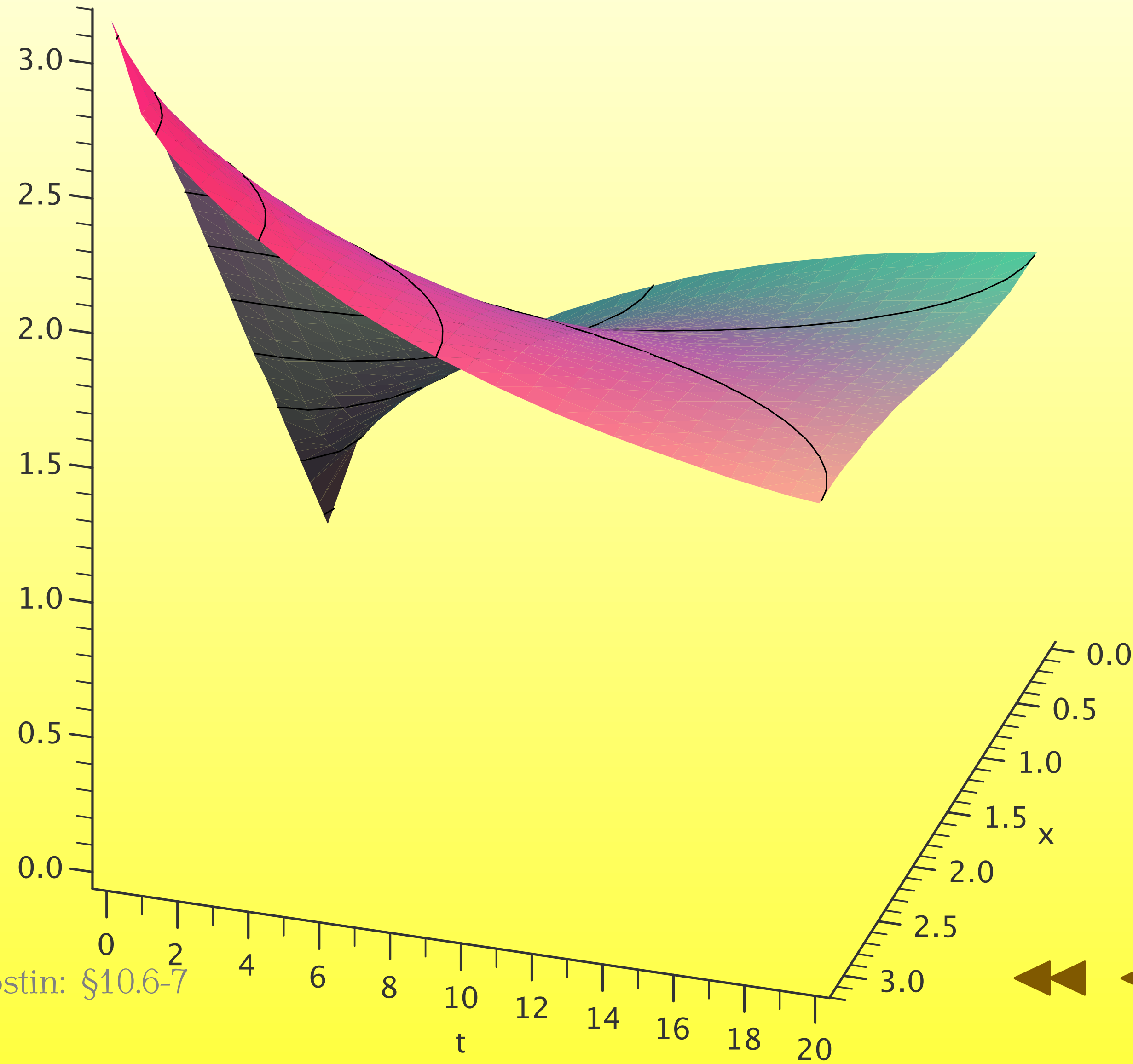
$$c_0 = \pi, c_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx) = -\frac{2}{\pi n^2}(1 - (-1)^n); \quad (n > 1)$$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