§10.7 The wave equation

§10.7 The wave equation

This equation describes the propagation of waves through a medium: in one dimension, such as a vibrating string

$$u_{tt} = a^2 u_{xx}$$

This equation describes the propagation of waves through a medium: in one dimension, such as a vibrating string

$$u_{tt} = a^2 u_{xx}$$

in two dimensions, such as a vibrating membrane:

$$u_{tt} = a^2 (u_{xx} + u_{yy})$$

This equation describes the propagation of waves through a medium: in one dimension, such as a vibrating string

$$u_{tt} = a^2 u_{xx}$$

in two dimensions, such as a vibrating membrane:

$$u_{tt} = a^2 (u_{xx} + u_{yy})$$

in three dimensions, such as vibrating rock in an earthquake:

$$u_{tt} = a^2 (u_{xx} + u_{yy} + u_{zz})$$

This equation describes the propagation of waves through a medium: in one dimension, such as a vibrating string

$$u_{tt} = a^2 u_{xx}$$

in two dimensions, such as a vibrating membrane:

$$u_{tt} = a^2(u_{xx} + u_{yy})$$

in three dimensions, such as vibrating rock in an earthquake:

$$u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz})$$

All three can be solved by separation of variables, but we will only look at one dimension.

This equation describes the propagation of waves through a medium: in one dimension, such as a vibrating string

$$u_{tt} = a^2 u_{xx}$$

in two dimensions, such as a vibrating membrane:

$$u_{tt} = a^2 (u_{xx} + u_{yy})$$

in three dimensions, such as vibrating rock in an earthquake:

$$u_{tt} = a^2 (u_{xx} + u_{yy} + u_{zz})$$

All three can be solved by separation of variables, but we will only look at one dimension. u is the amplitude of the wave.

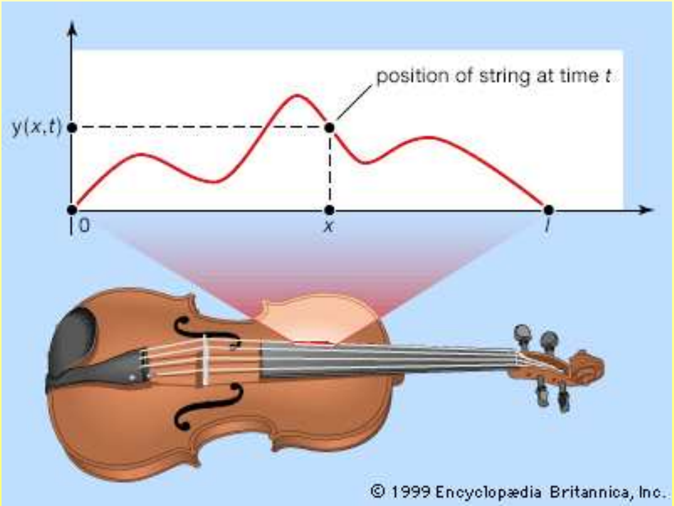
Note: none of the above include damping. We deal with a no-damping approximation, valid for short time.

We need sufficient data as (1) boundary conditions

We need sufficient data as (1) boundary conditions and (2) initial conditions

We need sufficient data as (1) boundary conditions and (2) initial conditions to have a unique solution of the problem.

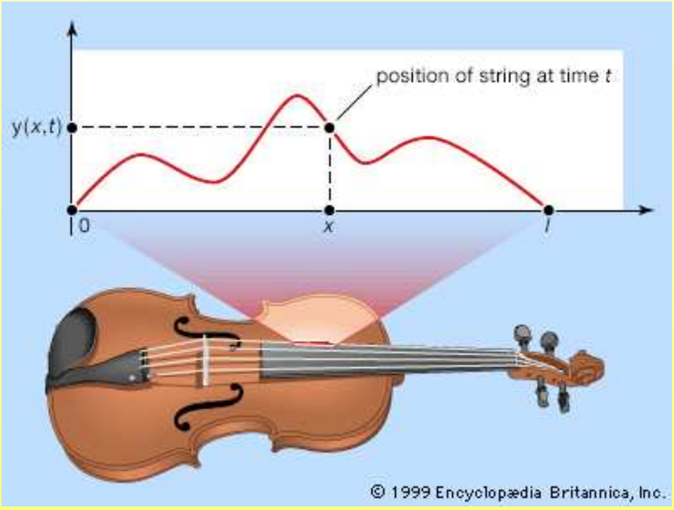
Vibrating string A vibrating string has its endpoints rigidly attached.



(In this picture, $L = l, u = y$.)

We need sufficient data as (1) boundary conditions and (2) initial conditions to have a unique solution of the problem.

Vibrating string A vibrating string has its endpoints rigidly attached.



(In this picture, $L = l, u = y$.) Then, we have

$$u(0, t) = 0; \quad u(L, t) = 0$$

How about initial conditions?

How about initial conditions? Now we need two, because the equation is second order in time.

How about initial conditions? Now we need two, because the equation is second order in time. We give: $u(x,0)$ and $u_t(x,0)$. Clearly, both matter: where the string starts (sometimes with zero initial velocity, e.g., guitar), and its initial velocity (impact excitation, e.g., in a piano).

How about initial conditions? Now we need two, because the equation is second order in time. We give: $u(x,0)$ and $u_t(x,0)$. Clearly, both matter: where the string starts (sometimes with zero initial velocity, e.g., guitar), and its initial velocity (impact excitation, e.g., in a piano). In reality, the conditions are some combinations of the above, often not easy to model.

Full problem:

$$u_{tt} = a^2 u_{xx}$$

$$u(0,t) = 0; \quad u(L,t) = 0, u(x,0) = f(x), u_t(x,0) = g(x)$$

How about initial conditions? Now we need two, because the equation is second order in time. We give: $u(x,0)$ and $u_t(x,0)$. Clearly, both matter: where the string starts (sometimes with zero initial velocity, e.g., guitar), and its initial velocity (impact excitation, e.g., in a piano). In reality, the conditions are some combinations of the above, often not easy to model.

Full problem:

$$u_{tt} = a^2 u_{xx}$$

$$u(0,t) = 0; \quad u(L,t) = 0, u(x,0) = f(x), u_t(x,0) = g(x)$$

Here, $a^2 = T/\rho$ depends on the physical setup only: T is the tension (force) in the string, ρ is its density.

Separation of variables in $u_{tt} = a^2 u_{xx}$

$$u(x,t) = X(x)T(t)$$

Separation of variables in $u_{tt} = a^2 u_{xx}$

$$u(x,t) = X(x)T(t)$$

$$X(x)T''(t) = a^2 X''(x)T(t)$$

Separation of variables in $u_{tt} = a^2 u_{xx}$

$$u(x,t) = X(x)T(t)$$

$$X(x)T''(t) = a^2 X''(x)T(t) \tag{1}$$

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}$$

Separation of variables in $u_{tt} = a^2 u_{xx}$

$$u(x,t) = X(x)T(t)$$

$$X(x)T''(t) = a^2 X''(x)T(t) \tag{1}$$

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Thus the pair of ODEs is:

$$X''(x) + \lambda X(x) = 0; \quad X(0) = X(L) = 0$$

Separation of variables in $u_{tt} = a^2 u_{xx}$

$$u(x,t) = X(x)T(t)$$

$$X(x)T''(t) = a^2 X''(x)T(t) \quad (1)$$

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Thus the pair of ODEs is:

$$X''(x) + \lambda X(x) = 0; \quad X(0) = X(L) = 0 \quad (2)$$

(an eigenvalue problem).

$$T''(t) + \lambda a^2 T(t) = 0;$$

Separation of variables in $u_{tt} = a^2 u_{xx}$

$$u(x,t) = X(x)T(t)$$

$$X(x)T''(t) = a^2 X''(x)T(t) \qquad (1)$$

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Thus the pair of ODEs is:

$$X''(x) + \lambda X(x) = 0; \quad X(0) = X(L) = 0 \qquad (2)$$

(an eigenvalue problem).

$$T''(t) + \lambda a^2 T(t) = 0; \ \textbf{no conditions yet}$$

$$X''(x) + \lambda X(x) = 0; \quad X(0) = X(L) = 0$$

$$X''(x) + \lambda X(x) = 0; \quad X(0) = X(L) = 0 \tag{3}$$

We have studied exactly this eigenvalue problem. Its solutions are:

$$\lambda_n = n^2\pi^2/L^2; \quad X_n = c_n \sin\frac{n\pi x}{L}$$

How about T?

$$X''(x) + \lambda X(x) = 0; \quad X(0) = X(L) = 0 \tag{3}$$

We have studied exactly this eigenvalue problem. Its solutions are:

$$\lambda_n = n^2\pi^2/L^2; \quad X_n = c_n \sin\frac{n\pi x}{L}$$

How about T?

$$T''(t) + \lambda_n a^2 T(t) = 0$$

$$X''(x) + \lambda X(x) = 0; \quad X(0) = X(L) = 0 \tag{3}$$

We have studied exactly this eigenvalue problem. Its solutions are:

$$\lambda_n = n^2\pi^2/L^2; \quad X_n = c_n \sin\frac{n\pi x}{L}$$

How about T?

$$T''(t) + \lambda_n a^2 T(t) = 0 \;\; T''(t) + n^2\pi^2 a^2/L^2 T(t) = 0$$

$$T(t) = A_n \sin\frac{n\pi a t}{L} + B_n \cos\frac{n\pi a t}{L}$$

Example: nonzero initial displacement $f(x)$, zero initial velocity ($g(x) = 0$). In this case

$$u_t(x,0) = 0;$$

Example: nonzero initial displacement $f(x)$, zero initial velocity ($g(x) = 0$). In this case

$$u_t(x,0) = 0; \quad \text{thus } T'(0)X(x) = 0;$$

Example: nonzero initial displacement $f(x)$, zero initial velocity $(g(x) = 0)$. In this case

$$u_t(x,0) = 0; \quad \text{thus } T'(0)X(x) = 0; \ \ T'(0) = 0 = A_n$$

Then,

$$X(x)T(t) = c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

Example: nonzero initial displacement $f(x)$, zero initial velocity $(g(x) = 0)$. In this case

$$u_t(x,0) = 0; \quad \text{thus } T'(0)X(x) = 0; \quad T'(0) = 0 = A_n$$

Then,

$$X(x)T(t) = c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

General solution should be

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

Example: nonzero initial displacement $f(x)$, zero initial velocity ($g(x) = 0$). In this case

$$u_t(x,0) = 0; \quad \text{thus } T'(0)X(x) = 0; \quad T'(0) = 0 = A_n$$

Then,

$$X(x)T(t) = c_n \sin\frac{n\pi x}{L} \cos\frac{n\pi a t}{L}$$

General solution should be

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \cos\frac{n\pi a t}{L}$$

The initial condition.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

The initial condition.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \cos\frac{n\pi a t}{L}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$$

The initial condition.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$$

which is again a sine-series.

Thus we have to odd-extend f and then calculate c_n from the usual sine-series formula

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$u(x,t)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \cos\frac{n\pi a t}{L}$$

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$u(x,t)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

$$c_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

u is an infinite sum of terms (**modes**) of the form

$$\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \cos\frac{n\pi a t}{L}$$

$$c_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx$$

u is an infinite sum of terms (**modes**) of the form

$$\sin\frac{n\pi x}{L} \cos\frac{n\pi a t}{L}$$

In t, this is periodic with frequency $\dfrac{n\pi a}{L}$.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

u is an infinite sum of terms (**modes**) of the form

$$\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

In t, this is periodic with frequency $\frac{n\pi a}{L}$. Each such mode has a periodic x behavior too, with space frequency $\frac{n\pi}{L}$.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \cos\frac{n\pi a t}{L}$$

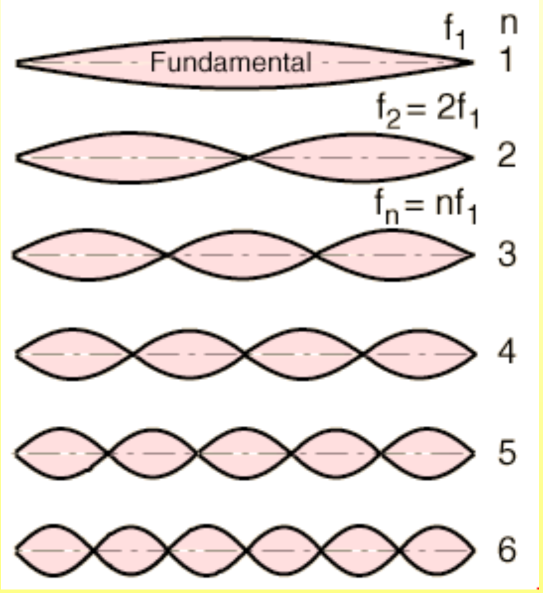
$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

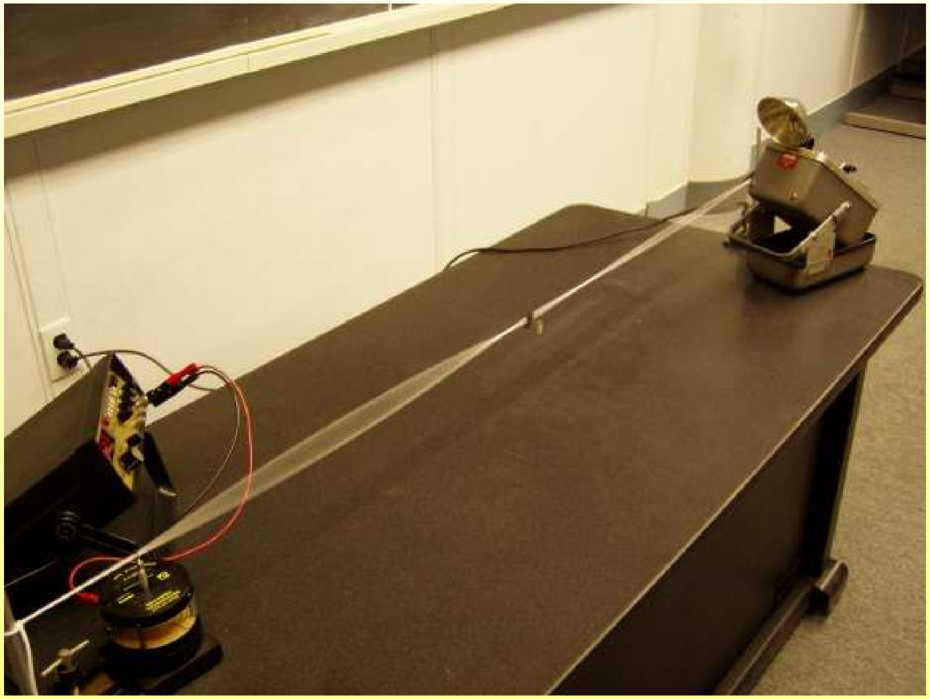
u is an infinite sum of terms (**modes**) of the form

$$\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

In t, this is periodic with frequency $\frac{n\pi a}{L}$. Each such mode has a periodic x behavior too, with space frequency $\frac{n\pi}{L}$. The higher the space frequency, the higher the time frequency. Furthermore, the time frequencies are integer multiples of the first one (that is, the one with $n = 1$), $\dfrac{\pi a}{L}$.

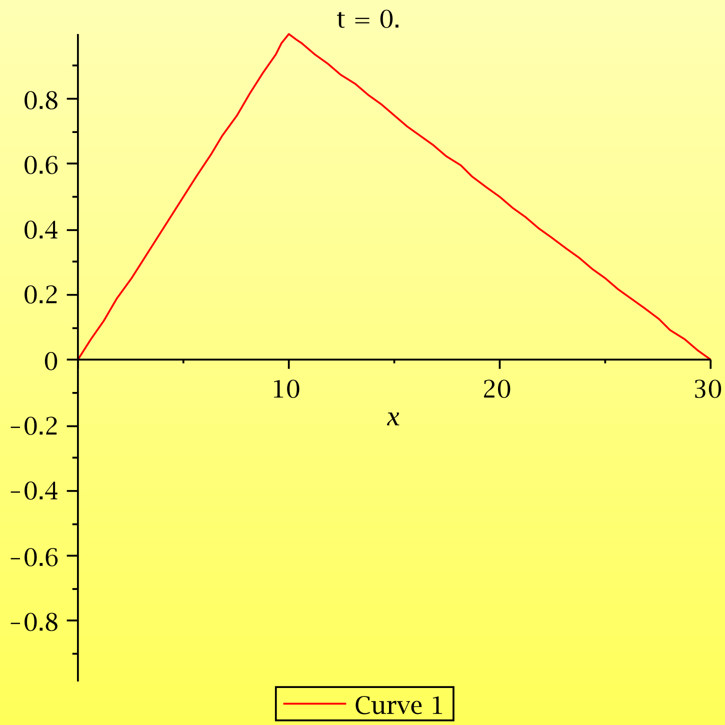
first one (that is, the one with $n = 1$), $\dfrac{\pi a}{L}$. This first one is the fundamental frequency, and the higher ones are harmonics of it.

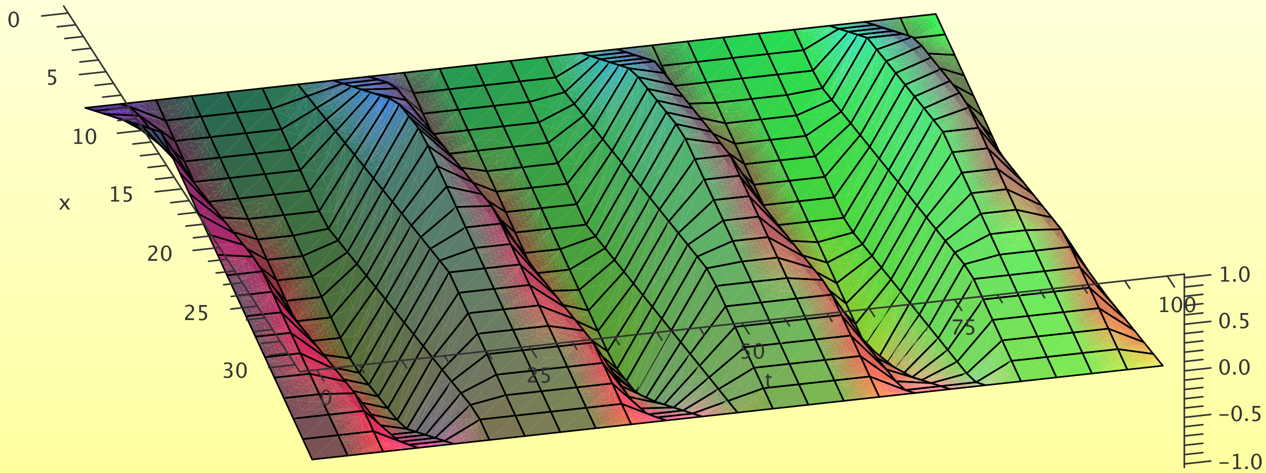


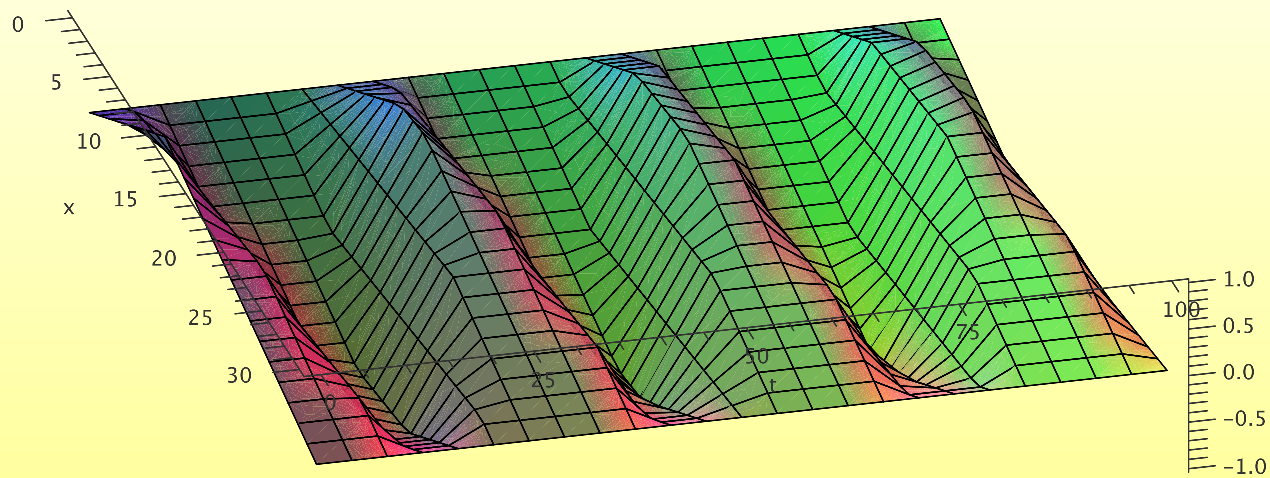


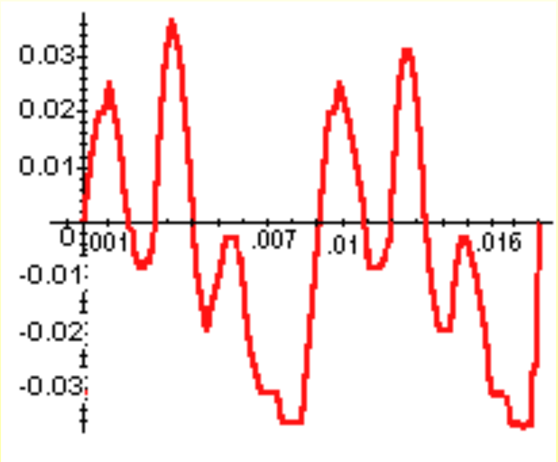
Example:

$$u(x,0) = f(x) = \begin{cases} x/10; & 0 \le x \le 10 \\ (30-x)/20; & 10 < x < 30 \end{cases}$$




0
5
10
x
15
20
25
30
0
25
50
t
75
100
1.0
0.5
0.0
−0.5
−1.0


0
5
10
x
15
20
25
30
0
25
50
t
75
100
1.0
0.5
0.0
−0.5
−1.0



Actual waveform of a guitar string vibration at fixed x

Other initial conditions.

Suppose now we are given

$$u_{tt} = a^2 u_{xx}$$

$$u(0, t) = 0; \quad u(L, t) = 0, u(x, 0) = 0, u_t(x, 0) = g(x)$$

Other initial conditions.

Suppose now we are given

$$u_{tt} = a^2 u_{xx}$$

$$u(0,t) = 0; \quad u(L,t) = 0, u(x,0) = 0, u_t(x,0) = g(x)$$

Such as the string of a piano.

Now the eigenvalue problem is

$$X''(x) + \lambda X(x) = 0; \quad X(0) = X(L) = 0$$

Other initial conditions.

Suppose now we are given

$$u_{tt} = a^2 u_{xx}$$

$$u(0,t) = 0; \quad u(L,t) = 0, u(x,0) = 0, u_t(x,0) = g(x)$$

Such as the string of a piano.

Now the eigenvalue problem is

$$X''(x) + \lambda X(x) = 0; \quad X(0) = X(L) = 0 \tag{4}$$

Thus $X_n(x) = c_n \sin \dfrac{n\pi x}{L}$

$$T''(t) + \lambda a^2 T(t) = 0;$$

Other initial conditions.

Suppose now we are given

$$u_{tt} = a^2 u_{xx}$$

$$u(0,t) = 0; \quad u(L,t) = 0, u(x,0) = 0, u_t(x,0) = g(x)$$

Such as the string of a piano.

Now the eigenvalue problem is

$$X''(x) + \lambda X(x) = 0; \quad X(0) = X(L) = 0 \tag{4}$$

Thus $X_n(x) = c_n \sin \dfrac{n\pi x}{L}$

$$T''(t) + \lambda a^2 T(t) = 0; \; T(0) = 0$$

and then

$$T_n(t) = \sin \frac{n\pi a t}{L}$$

$$X_n T_n = c_n \sin \frac{n\pi x}{L} \sin \frac{n\pi a t}{L}$$

and then

$$T_n(t) = \sin\frac{n\pi a t}{L}$$

$$X_n T_n = c_n \sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L}$$

and then

$$T_n(t) = \sin\frac{n\pi a t}{L}$$

$$X_n T_n = c_n \sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L}$$

$$u_t = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{L}\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

and then

$$T_n(t) = \sin\frac{n\pi a t}{L}$$

$$X_n T_n = c_n \sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L}$$

$$u_t = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{L}\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

$$u_t(0) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{L}\sin\frac{n\pi x}{L}$$

and then

$$T_n(t) = \sin\frac{n\pi a t}{L}$$

$$X_n T_n = c_n \sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L}$$

$$u_t = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{L}\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

$$u_t(0) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{L}\sin\frac{n\pi x}{L}$$

again a sine series.

General initial conditions.

$$u_{tt} = a^2 u_{xx}$$

$$u(0,t) = 0; \quad u(L,t) = 0, u(x,0) = f(x), u_t(x,0) = g(x)$$

General initial conditions.

$$u_{tt} = a^2 u_{xx}$$

$$u(0,t) = 0; \quad u(L,t) = 0, u(x,0) = f(x), u_t(x,0) = g(x)$$

The general solution is $u(x,t) = F(x,t) + G(x,t)$, where

$$F_{tt} = a^2 F_{xx}$$

$$F(0,t) = 0; \quad F(L,t) = 0, F(x,0) = f(x), F_t(x,0) = 0$$

General initial conditions.

$$u_{tt} = a^2 u_{xx}$$

$$u(0,t) = 0; \quad u(L,t) = 0, u(x,0) = f(x), u_t(x,0) = g(x)$$

The general solution is $u(x,t) = F(x,t) + G(x,t)$, where

$$F_{tt} = a^2 F_{xx}$$

$$F(0,t) = 0; \quad F(L,t) = 0, F(x,0) = f(x), F_t(x,0) = 0$$

$$G_{tt} = a^2 G_{xx}$$

$$G(0,t) = 0; \quad G(L,t) = 0, G(x,0) = 0, G_t(x,0) = g(x)$$

General initial conditions.

$$u_{tt} = a^2 u_{xx}$$

$$u(0,t) = 0; \quad u(L,t) = 0, u(x,0) = f(x), u_t(x,0) = g(x)$$

The general solution is $u(x,t) = F(x,t) + G(x,t)$, where

$$F_{tt} = a^2 F_{xx}$$

$$F(0,t) = 0; \quad F(L,t) = 0, F(x,0) = f(x), F_t(x,0) = 0$$

$$G_{tt} = a^2 G_{xx}$$

$$G(0,t) = 0; \quad G(L,t) = 0, G(x,0) = 0, G_t(x,0) = g(x)$$

(check!)