Mechanical and electrical oscillations

§3.8,3.9

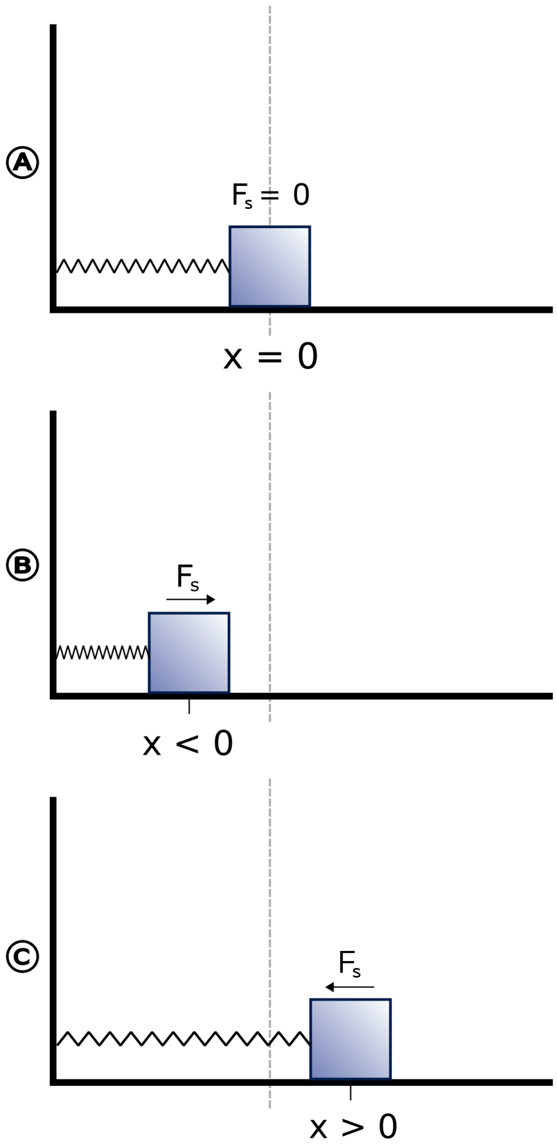

Figure 1: The elastic force is opposite to x: $F_e = -kx$

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This is the mechanical oscillator, with air friction (damping) γ and forcing F.

Electrical analog

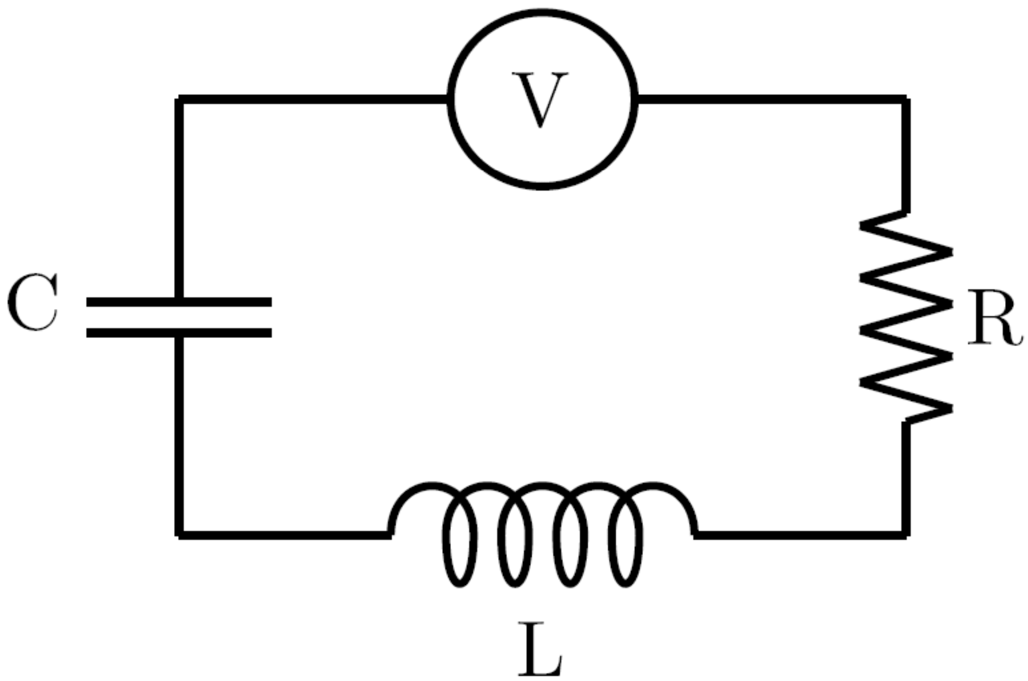

Figure 2: $LQ'' + RQ' + Q/C = V(t)$

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Here, A is called the amplitude of the oscillation, $2\pi/\omega = T$ is the period, θ is the phase.

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Solution. The equation of motion is $x'' + 4x = 0$ and thus the characteristic equation is $r^2 + 4 = 0$,$r = \pm 2$ and thus $x = A\sin(2t + \phi)$.

The period is $2\pi/2 = \pi$

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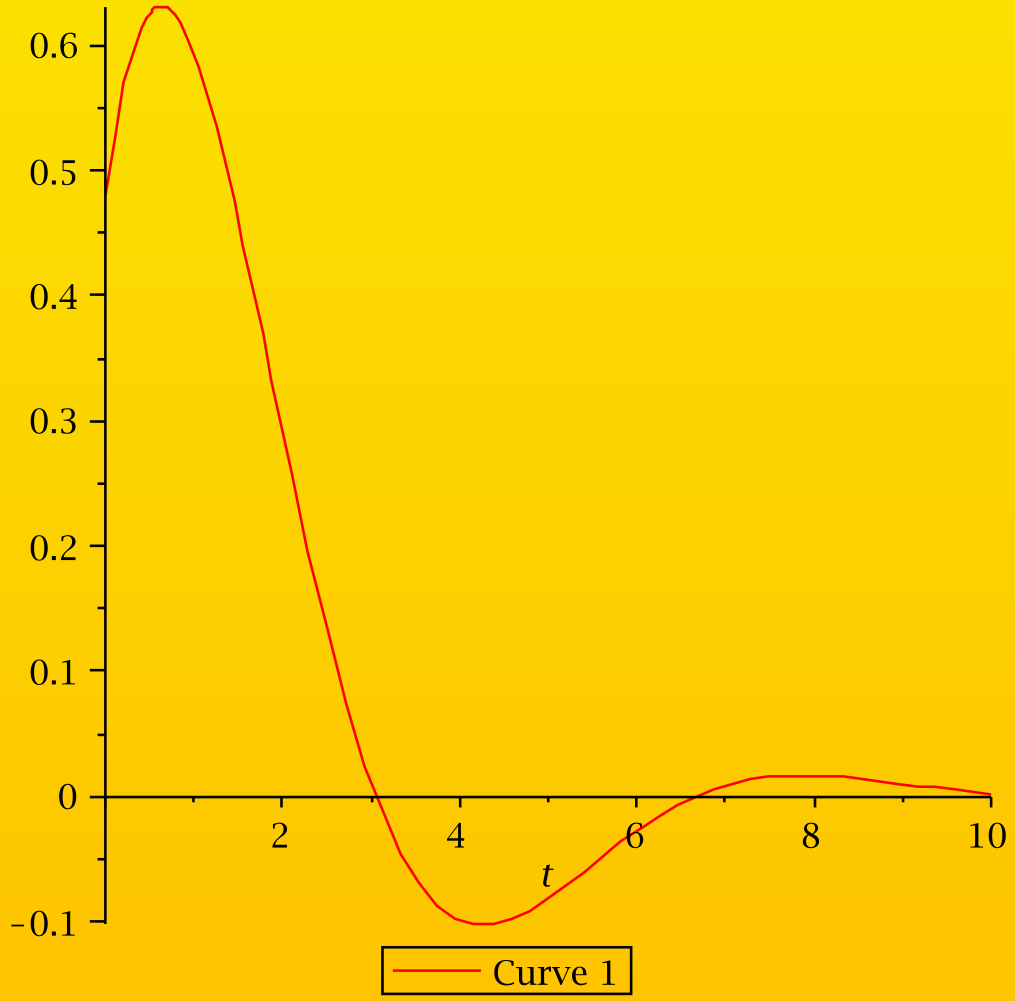

Figure 3: Typical solution

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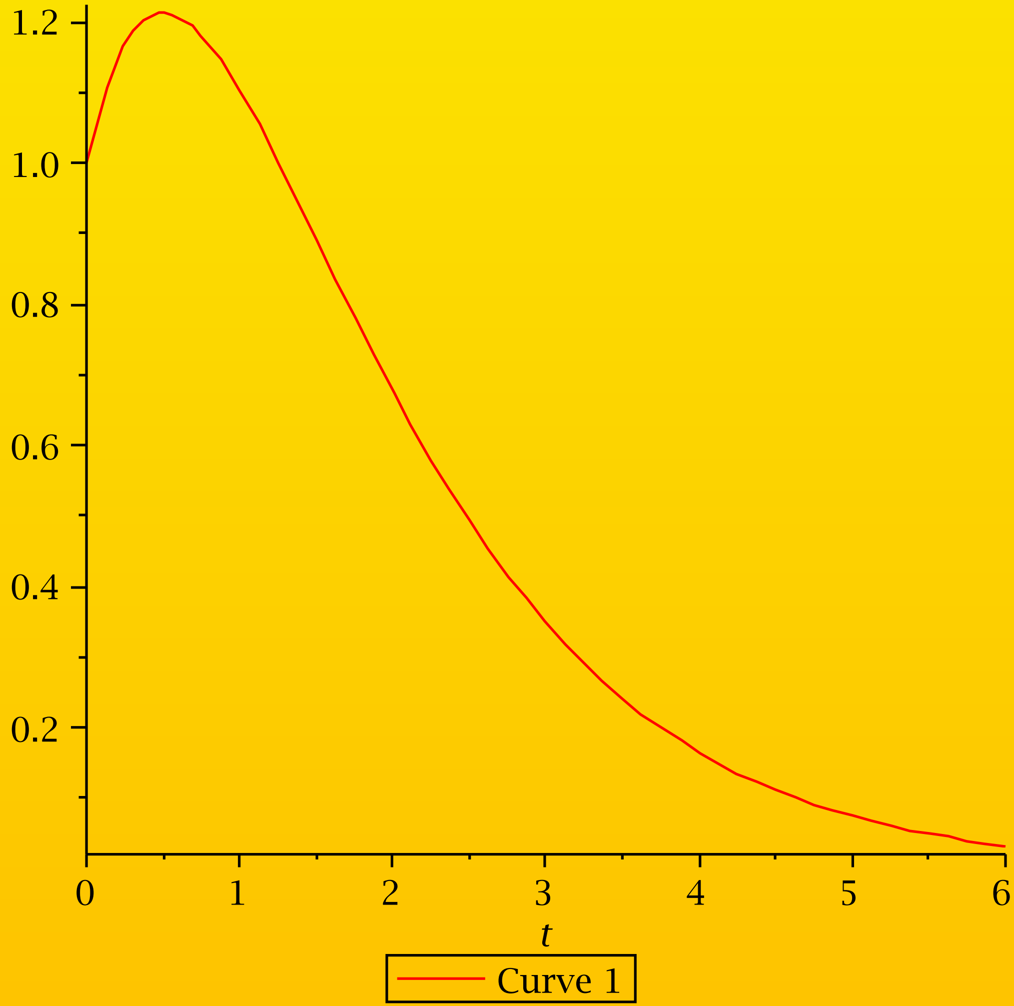

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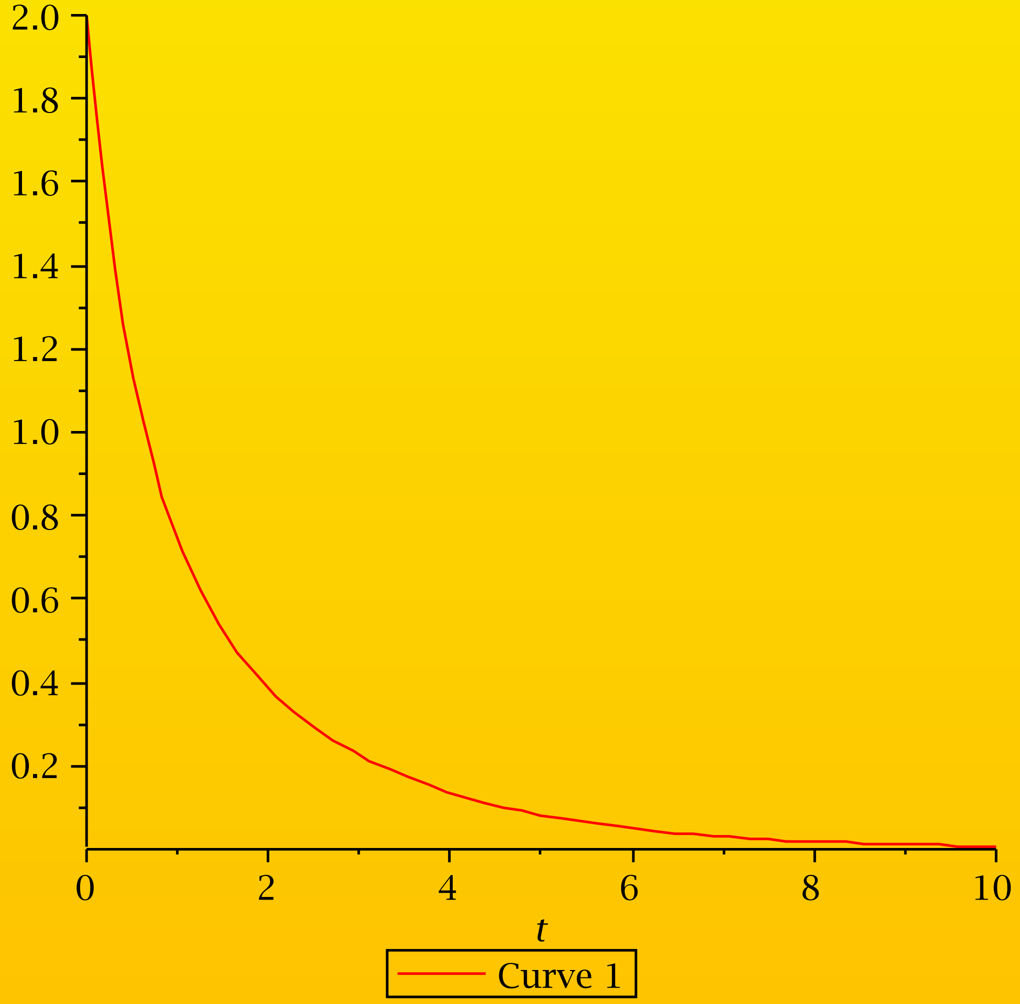
This motion is called **overdamped**.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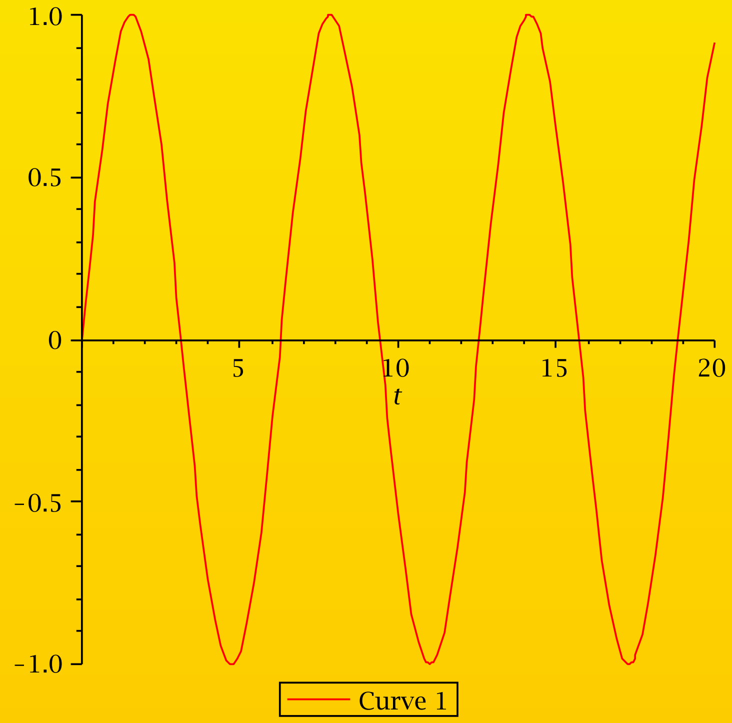
Curve 1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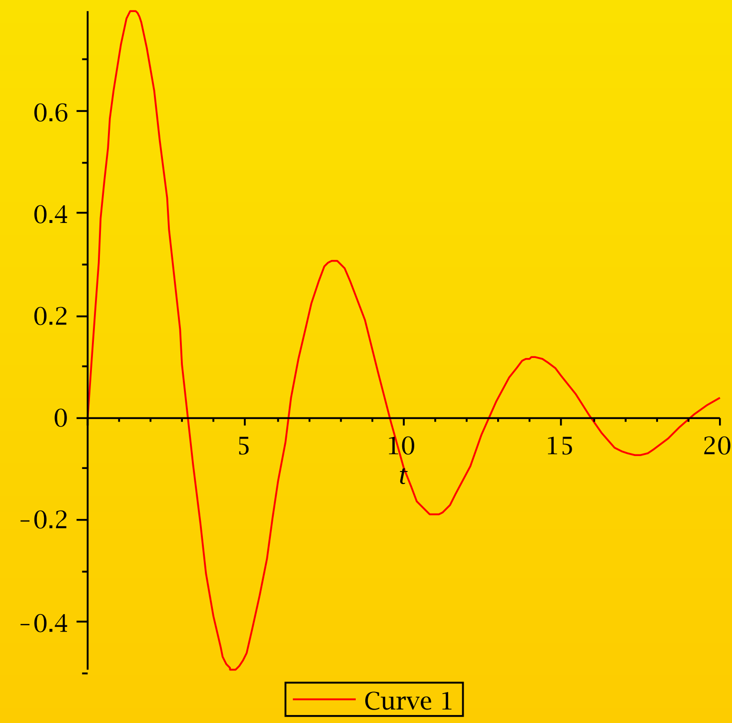
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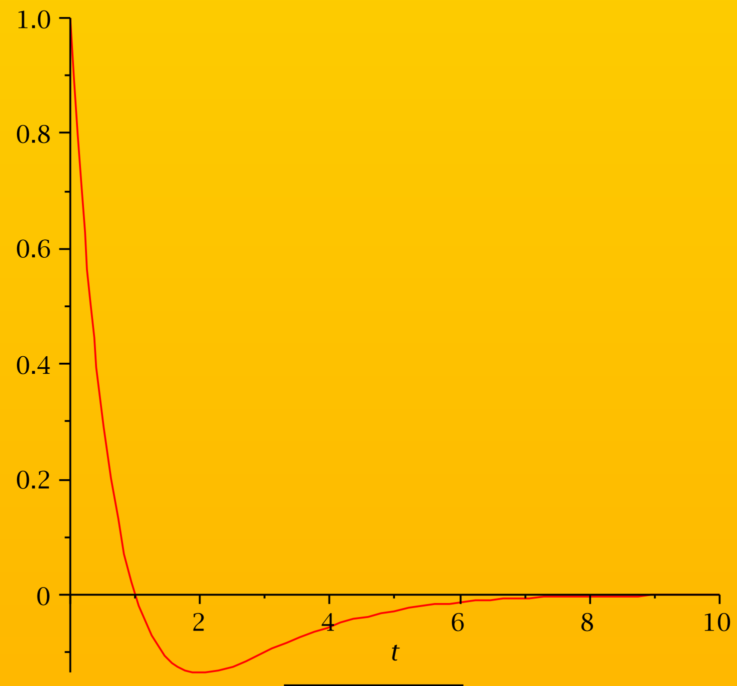
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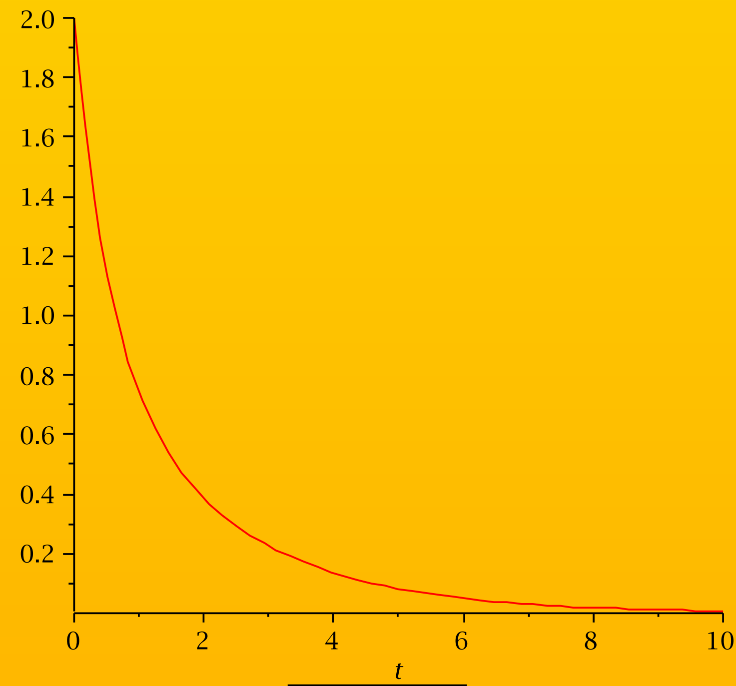
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