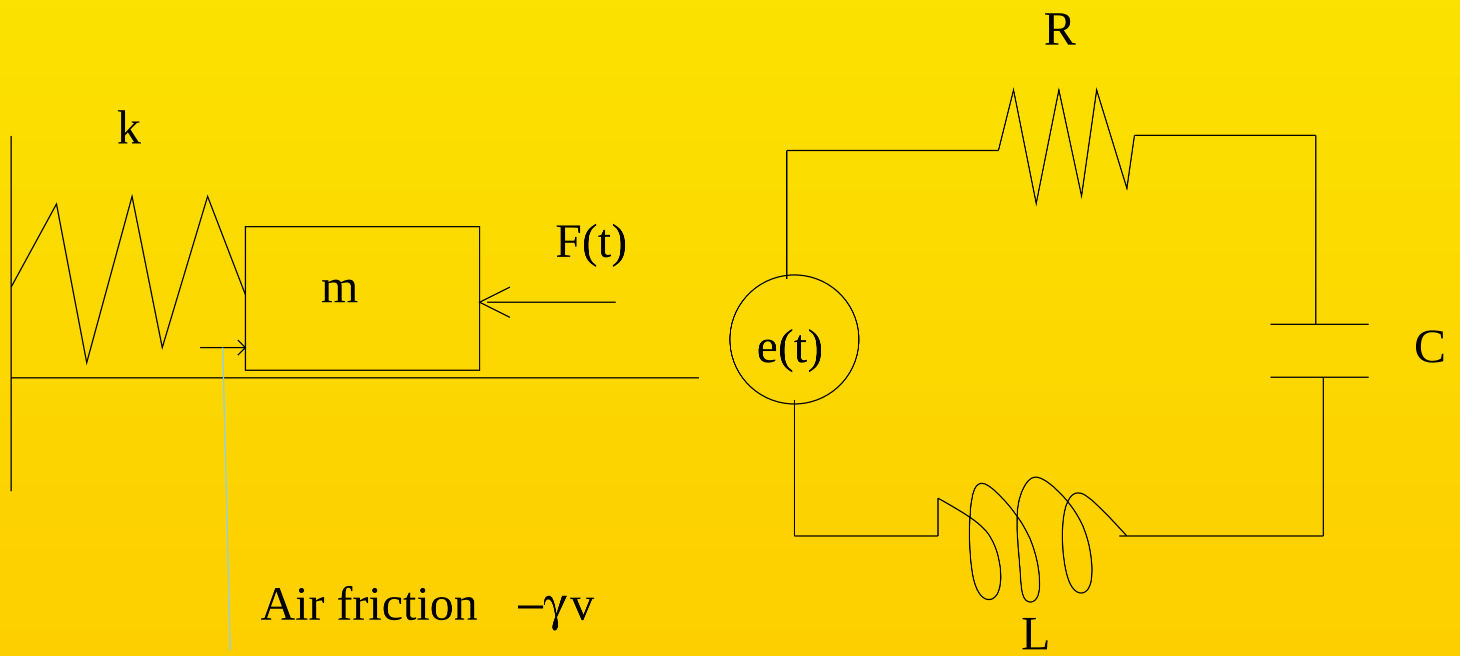
Forced oscillations. Review of power series

§3.9,5.1



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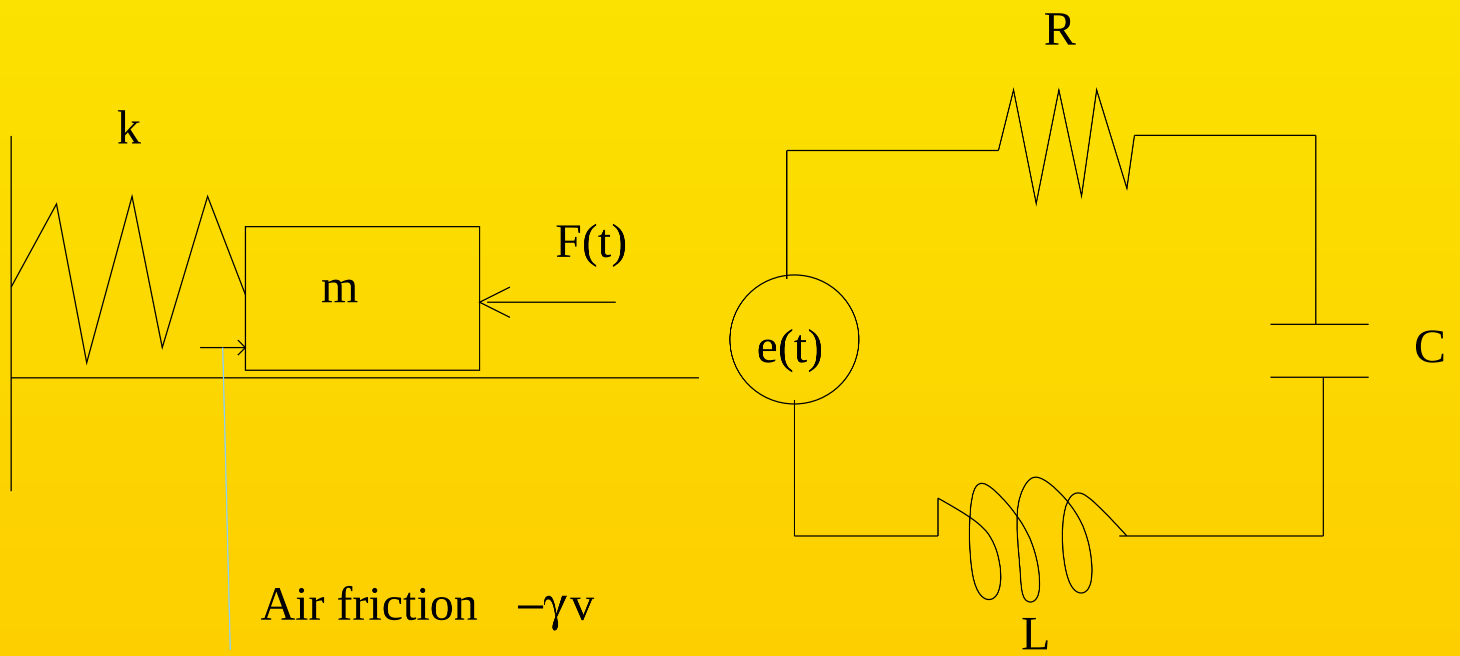

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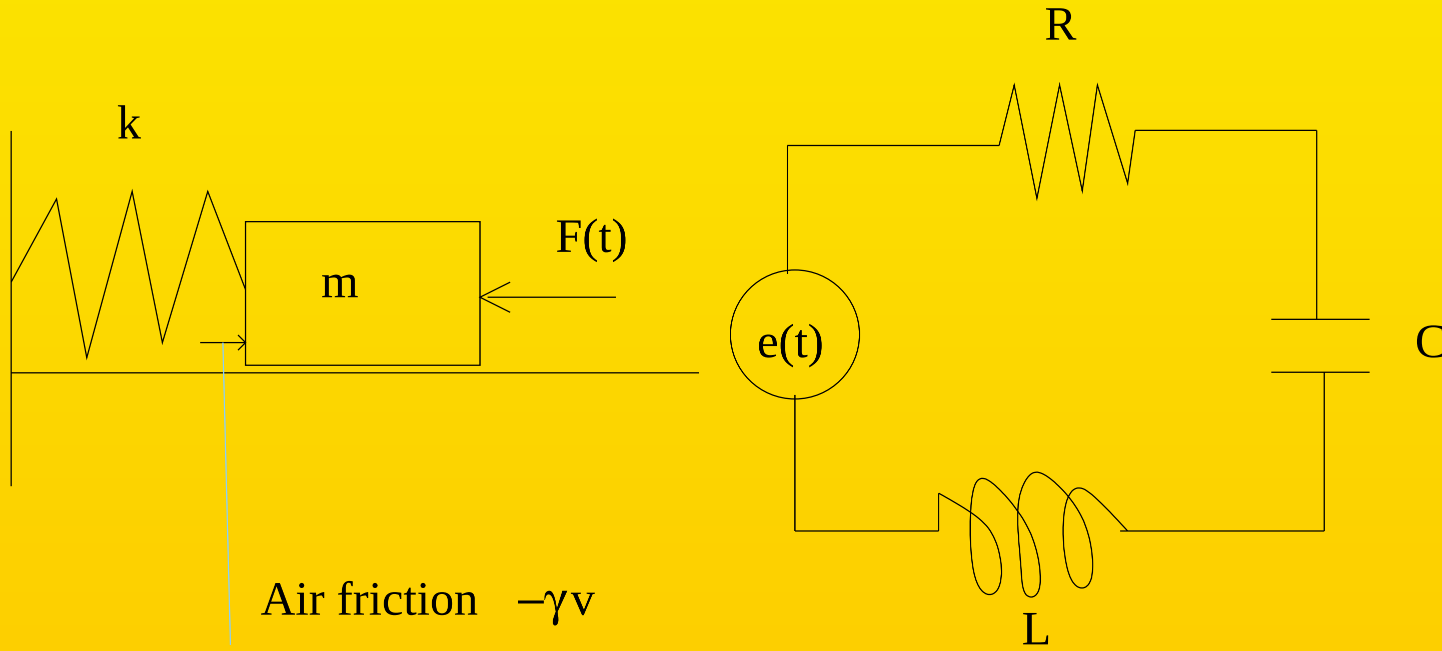

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The characteristic equation is $mr^2 + k = 0$,

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-
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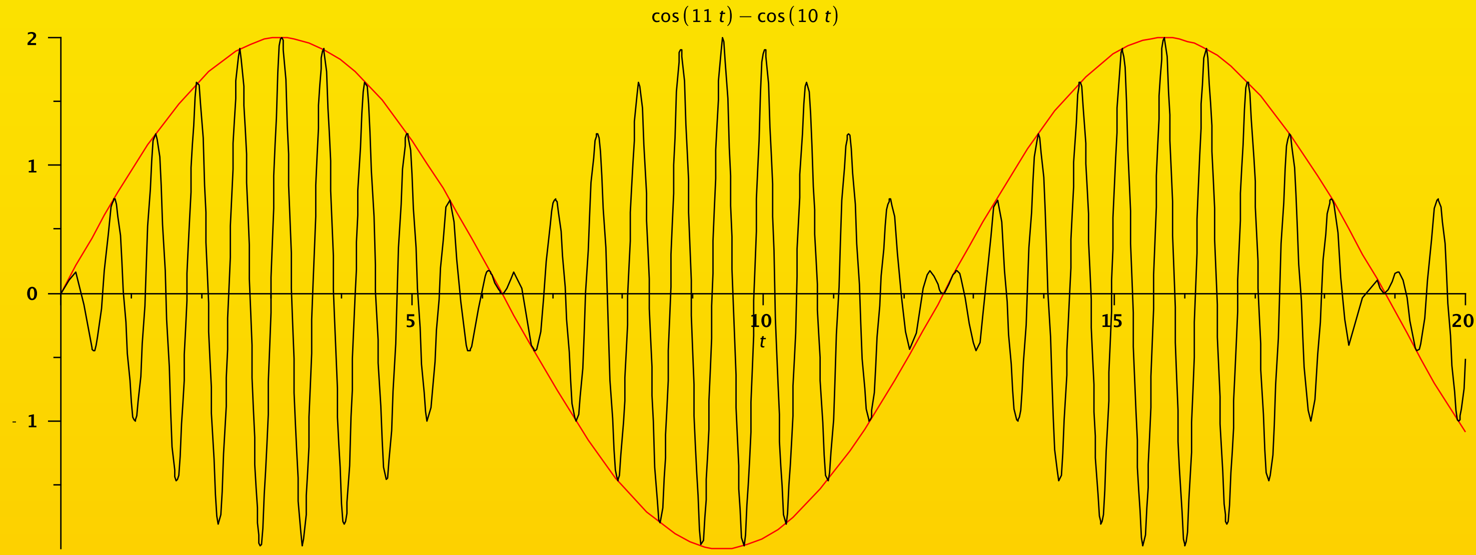
cos(11 t) − cos(10 t)
2
1
0
- 1
5
10
15
20
t

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Substituting we get

$$-Bt\underline{(m\omega_0^2 - k)}\sin\omega t + \underline{2B\omega}\cos(\omega_0 t) - \underline{A_1}\cos(\omega_0 t) = 0)$$

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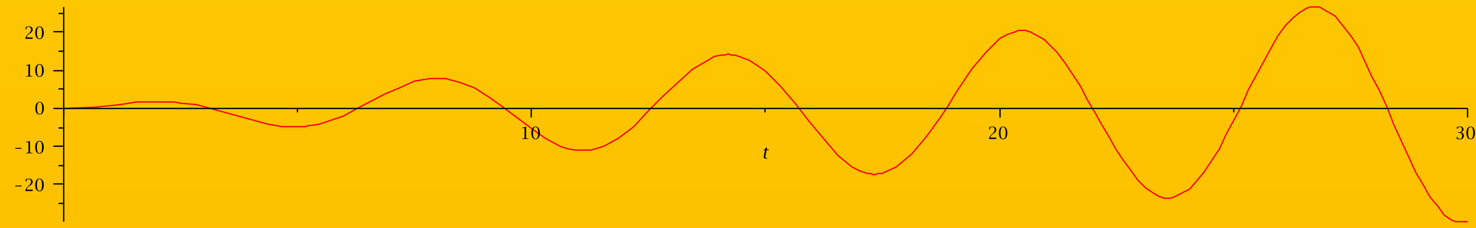
This solution grows without bound.



Tacoma Narrows bridge, 1940

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Read the textbook for the formulas of the other constants, δ etc.

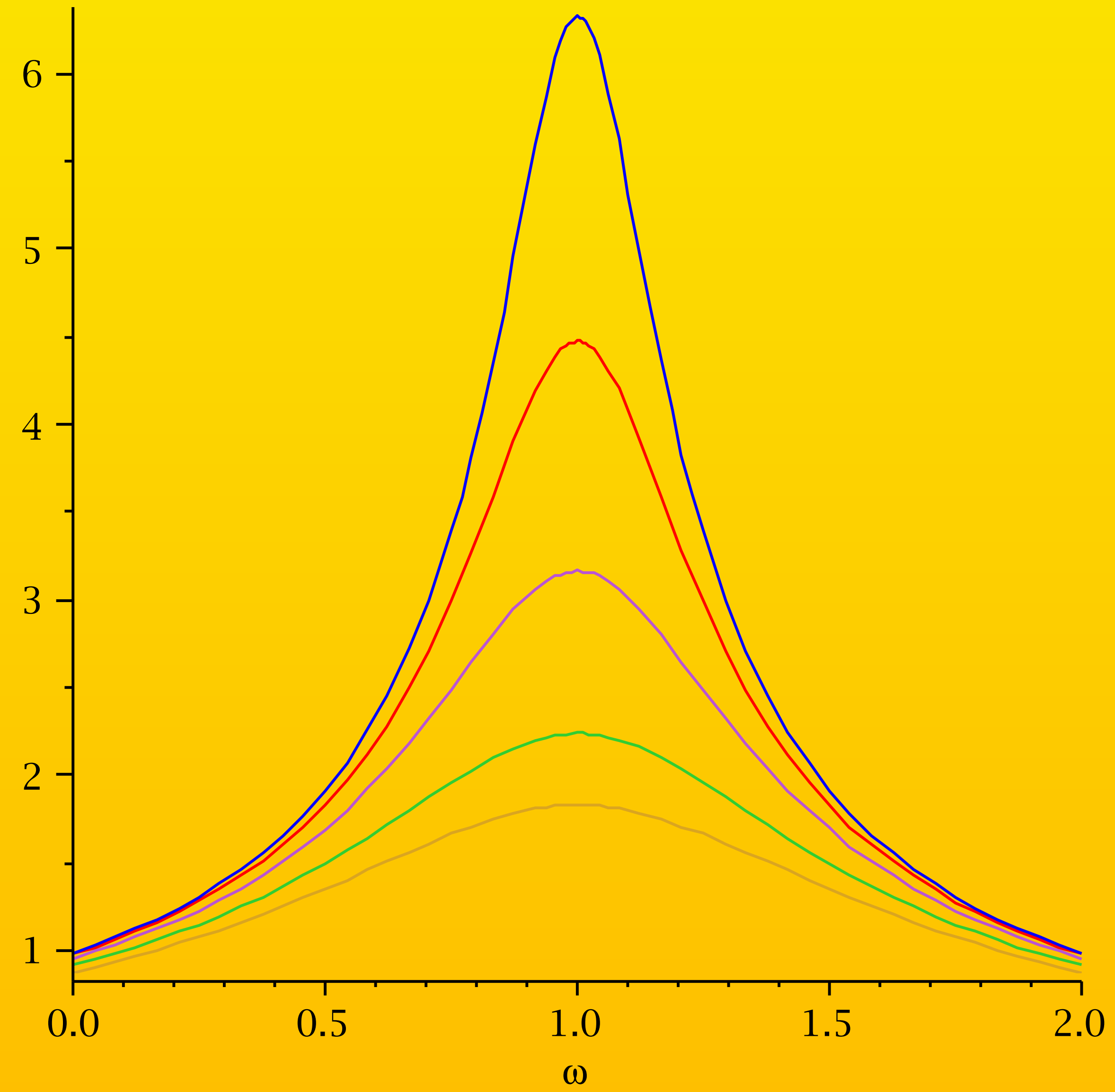

Figure 1: Response vs. frequency.

Series: short review. Please brush up

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