POWERS, FROM FIRST PRINCIPLES

In the following, a will be real and greater than one.

Also, the proofs should not use anything from Chapter 18. The notation ":=" means equality by definition.

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We have already defined a^n and $a^{1/m}$ for positive a and $m, n \in \mathbb{N}$:

$a^0 := 1$. For $n \in \mathbb{N}$, a^n is defined inductively as $a^1 := a$, $a^{n+1} := a^n a$, $a^{-n} := 1/a^n$. Finally, $a^{1/m}$ is the unique positive root of $x^m = a$).

(i) Check by induction that $a^{m+n} = a^m a^n$. If $q = m/n \in \mathbb{Q}$, then a^q is defined by $a^q := (a^{1/m})^n$.

(ii) Check that the equality $a^{p+q} = a^p a^q$ holds if $p, q \in \mathbb{Q}$.

(iii) Check also that if $q > 0$ then $a^q > 1$. As a consequence, a^q is increasing in $q \in \mathbb{Q}$.

(i), (ii) and (iii) above are not part of the exercises, but I would recommend that you check these properties.

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Exercise 1 (15p)**.** Show that

$$\lim_{n\to\infty} a^{1/n} - 1 = 0$$

One way is to note that

$$a - 1 = (a^{1/n})^n - 1 = (a^{1/n} - 1)(1 + a^{\frac{1}{n}} + a^{\frac{2}{n}} + \cdots + a^{\frac{n-1}{n}})$$

By (iii) above, $a^{m/n} > 1$ and thus

$$a^{1/n} - 1 = \frac{a-1}{1 + a^{\frac{1}{n}} + a^{\frac{2}{n}} + \cdots + a^{\frac{n-1}{n}}} < \frac{a-1}{n} \tag{1}$$

Exercise 2 (15p)**.** Use the result in the Exercise 1 to show that if $\{q_n\}_n \subset \mathbb{Q}$ and $q_n \to 0$ as $n \to \infty$, then $a^{q_n} \to 1$. Deduce that if $\{q_n\}_n \subset \mathbb{Q}$ is a convergent sequence with limit x, then $a^{q_n} \to l$ for some l, and for any other sequence of rational numbers $\{Q_n\}$ converging to x, $a^{Q_n} \to l$ for the same l.

Exercise 3 (30p)**.** (a) Show that there is a *unique increasing* function $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(q) = a^q$ for $q \in \mathbb{Q}$.

(b) Show that this f is continuous. It is then natural to write

$$a^x := f(x) \ \forall x \in \mathbb{R}$$

(c) Use Eq. (1) and the monotonicity of f to show that f is differentiable. Hint: Show first that

$$\frac{1}{n}\left(1 + a^{\frac{1}{n}} + a^{\frac{2}{n}} + \cdots + a^{\frac{n-1}{n}}\right) \to \int_0^1 a^s ds \quad \text{as} \quad n \to \infty$$

(d) With the notation $\int_0^1 a^s ds = \frac{a-1}{\ln a}$, check that $f' = \ln a f$.

This would now allow us to "more directly" define $\ln x$ for all $x > 0$. But this is another story...